Hello, my name is Karel. Requirements for getting credit ”zapocet”. What are you interested in? How you should study…

0. Do you need me to recapitulate something? You should know: how to solve a system of linear equations using Gaussian elimination (Gauss-Jordan), how to multiply matrices, how to tell coordinates. We will focus on linear maps when we need to.

   Can you code? In a programming language of your choice and nothing complicated.

1. Let B be a basis of $\mathbb{R}^4$ with vectors:

   $(1/2, 1/2, 1/2, 1/2)^T, (1/2, -1/2, -1/2, 1/2)^T, (-1/2, 1/2, -1/2, 1/2)^T, (-1/2, -1/2, 1/2, 1/2)^T$

   Find the matrix corresponding to change of basis from the canonical basis to B (that is find the matrix ${}_B[id]_K$ such that $[u]_B = {}_B[id]_K[u]_K$. Find coordinates of the vector $(3, 1, 4, 1)^T$ in basis B. Did you noticed something about matrix ${}_B[id]_K$?

2. Let V be a vector space over $\mathbb{R}$, we define a *dot product* (or a *scalar product*) as a binary operation $\langle \cdot \mid \cdot \rangle \colon V^2 \to \mathbb{R}$, such that for each $u, v, w \in V$ a $c \in \mathbb{R}$ we have:

   (a) $\langle u \mid u \rangle \geq 0$ and equality holds only for $u = \vec{0}$
   (b) $\langle u + v \mid w \rangle = \langle u \mid w \rangle + \langle v \mid w \rangle$
   (c) $\langle cu \mid v \rangle = c\langle u \mid v \rangle$
   (d) $\langle u \mid v \rangle = \langle v \mid u \rangle$ (respectively $\langle u \mid v \rangle = \overline{\langle v \mid u \rangle}$ for complex numbers).

   We say that u, v are *orthogonal* if $\langle u \mid v \rangle = 0$.

   We may define a *norm* using a dot product: $\|u\| = \sqrt{\langle u \mid u \rangle}$. Intuitively a norm gives you the length of a vector. Note that a norm can be defined in a more general way but this definition is extremely useful.

   Geometric interpretation of the standard dot product in $\mathbb{R}^n$ is $\langle u \mid v \rangle = \|u\|\|v\| \cos(\varphi)$, where φ is the angle between vectors u, v (compare with the definition of orthogonality).

   Moreover orthogonality of vectors implies linear independence.

3. Show that the following are dot products.

   (a) (Standard dot product) In $\mathbb{R}^n$ we define $\langle u \mid v \rangle = u^T v = \sum_{i=1}^n u_i v_i$
   (b) In the space $\mathcal{C}_{[a,b]}$ of all continuous functions on the interval $[a, b]$ we define a dot product $\langle f \mid g \rangle = \int_a^b f(x)g(x)dx$.

4. Compute standard dot products of given vectors: $(1, 2, 3)^T, (0, 0, 1)^T, (1, -2, 1)^T$. Which ones are orthogonal? What is the length of the first vector? How far apart are the first and third vector?

5. Let us denote the rows of a matrix A as $v_1, \ldots, v_m$ and columns of a matrix B by $w_1, \ldots, w_p$. What are the entries of the matrix AB?

   Prove that the row space of a matrix A and the kernel of the matrix A are orthogonal.

6. For the dot product $\langle f|g \rangle = \int_{-1}^1 f(x)g(x)\,dx$ show that functions $3x^2 - 1$ a $5x^3 - 3x$ are orthogonal.

7. Let A be a symmetric real matrix such that $u^T Au > 0$ for each non-zero vector $u \in \mathbb{R}^n$ we call such matrices positive definite. Let us define a dot product as $\langle u|v \rangle = u^T Av$. Show that this is indeed a dot product if and only if A is positive definite.

   Given a dot product $\langle u|v \rangle$ as a black-box find a way how to find the corresponding positive definite matrix A which defines it.

   Show that a sum of two positive definite matrices is a positive definite matrix. Show that a positive multiple of a positive definite matrix is a positive definite matrix.