Twelve countings with rooted plane trees

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Abstract

The average number of (1) antichains, (2) maximal antichains, (3) chains, (4) infima closed sets, (5) connected sets, (6) independent sets, (7) maximal independent sets, (8) brooms, (9) matchings, (10) maximal matchings, (11) linear extensions, and (12) drawings in (of) a rooted plane tree on n vertices is investigated. Using generating functions we determine the asymptotics and give some explicit formulae and identities. In conclusion we discuss the extremal values of the above quantities and pose some problems.

1 Rooted plane trees

A *rooted plane tree*, a classical enumerative structure, is a quadruple $T = (r, V, E, L)$ such that

- (V, E) is a nonempty finite directed tree, as usual V is the *vertex set* and E is the *edge set*,
- where all edges are directed away from the *root* $r \in V$,
- and $L = \{(\{w :\ vw \in E\}, <_v) :\ v \in V\}$ is a collection of $|V|$ linear orders.

We call the elements of the set $ch(v) = \{w :\ vw \in E\}$ *children* of v, v is their *parent*. A *leaf* is a vertex with no child. Rooted plane trees will be called shortly *trees*. A tree T is visualized by embedding it in the plane (see Figure 1) so that the root is at the lowest position, all edges are straight segments directed up, and the orders $<_v$ coincide with the natural left-right order.

By $\mathcal{T}$ we denote the collection of all substantially different trees and by $\mathcal{T}_n$ the collection of those having n vertices. The aim of the paper is, given a *weight* $w : \mathcal{T} \to \{0, 1, 2, \ldots\}$, to count the total weight $w(n) = \sum_{T\in\mathcal{T}_n} w(T)$ of trees on n vertices. We consider twelve combinatorial weights w and for the first ten of them we determine the *generating function*

$$F_w(x) = \sum_{\mathcal{T}} w(T)x^{|V(T)|} = \sum_{n\geq 1} w(n)x^n.$$

For the eleventh and twelfth weight n stands for $|E|$ and the *exponential generating function* will be determined.

For instance, setting $w(T) = 1$ for all T one gets the celebrated *Catalan function*

$$C = C(x) = \sum_{n\geq 1} |\mathcal{T}_n|x^n = \sum_{n\geq 1} c_{n-1}x^n = \frac{1}{2}\left(1 - \sqrt{1-4x}\,\right) = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + \ldots$$

counting the number of trees on n vertices. $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth *Catalan number*. Catalan function satisfies the quadratic equation $C^2 - C + x = 0$.

What are the weights? Mostly the numbers of subsets of V or E with special properties. The first four of them appear by understanding a tree T as a poset. The standard partial ordering $(V, \leq)$ is defined by $u \leq v$ iff u lies on the path joining r and v. A *chain* in T is then a subset $X \subset V$ of pairwise comparable vertices. On the contrary an *antichain* X consists of mutually incomparable vertices. A tree with n vertices may have as many as $2^n - 1$ nonempty chains and as few as $2n - 1$. As for the antichains, there may be as few as n and as many as 2^{n-1} of them. These are extremes but what is going on in average? One would expect that in average antichains are much more numerous than chains, is this really the case? How fast the average numbers grow? Seeking answers to this sort of questions and led by the joy of counting by generating functions we investigated twelve weights of this kind. Our arguments are more or less standard but, except for w_8 and w_{11} which we discuss later, we failed to find any reference to results of this type in [5], [8], and [12], or to localize the sequences $\{w(n)\}_{n\geq 1}$ in [11].

We need to review some more definitions. We say that $X \subset V$ is *infima closed* (in a tree T) if X contains with any two vertices $u, v \in X$ also the merging point of the paths joining r and u, and r and v (i.e., the infimum $u \wedge v$). Six weights arise from graph-theoretical considerations. A set $X \subset V$ is *independent* if $uv \in E$ for no two $u, v \in X$. A set $X \subset V$ is *connected* if any two vertices of X can be joined by an undirected path lying completely in X. A *matching* $X \subset E$ is a set of pairwise disjoint edges. A *broom* $X \subset E$ is a set of pairwise intersecting edges, all directed up. Single vertex is also a broom. Two more weights arise from the concept of drawing trees. Suppose $T = (r, V, E, L)$ is a tree. A *simple drawing* of T is a permutation of edges $(e_1, e_2, \ldots, e_{|E|})$ of T such that $r \in e_1$ and, for any $i = 2, \ldots, |E|$, e_i intersects some of the edges $e_1, e_2, \ldots, e_{i-1}$. A *drawing* of T is a sequence of trees $(T_1, T_2, \ldots, T_n)$, $n = |V|$, such that $T_n = T$ and T_{i-1} arises from T_i by deleting a leaf of T_i.

Now we list the weights. Maximality is meant to inclusion and maximal sets are nonempty by definition. For a given tree T, $w_1(T)$ is the number of nonempty antichains in T, $w_2(T)$ is the number of maximal antichains, $w_3(T)$ is the number of nonempty chains, $w_4(T)$ counts the number of nonempty infima closed sets, $w_5(T)$ counts nonempty connected sets, $w_6(T)$ counts all independent sets (including $\emptyset$), $w_7(T)$ counts maximal independent sets, $w_8(T)$ counts the number of brooms in T, $w_9(T)$ counts matchings (including $\emptyset$), $w_{10}(T)$ counts maximal matchings, $w_{11}(T)$ is the number of simple drawings of T, and $w_{12}(T)$ is the number of drawings of T.

The paper is organized as follows. In the next section we summarize the results — explicit formulae or equations for generating functions, asymptotics — for the first ten weights. In Section 3 we give proofs or sketches of proofs to these results. Applications of the Lagrange inversion formula to the weights w_6, w_7, and w_9 are given in Section 4. In particular, we derive a closed formula for $w_6(n)$. Weights w_{11} and w_{12} are handled in Section 5. In Section 6 we give some concluding comments and open problems, and we determine $\max_{T\in\mathcal{T}_n} w_2(T)$.

2 Subset countings — results

First we list the closed formulae for the generating functions F_1, F_2, F_3, F_4, F_5, and F_8, $F_i(x) = \sum_{n\geq 1} w_i(n)x^n$.

$$F_1(x) = \frac{1 + \sqrt{1-4x} - \sqrt{2}\sqrt{\sqrt{1-4x} + 1 - 10x}}{4} \tag{1}$$

$$F_2(x) = \frac{3 - 2x - \sqrt{1-4x} - \sqrt{2}\sqrt{(1+2x)\sqrt{1-4x} + 1 - 8x + 2x^2}}{4} \tag{2}$$

$$F_3(x) = \frac{x\left(1 + 3\sqrt{1-4x}\right)}{4\left(1 - \frac{9}{2}x\right)} \quad F_4(x) = \frac{\left(1 + \sqrt{1-4x}\right)\left(1 - \sqrt{3 - 2/\sqrt{1-4x}}\right)}{4} \tag{3}$$

$$F_5(x) = \frac{x}{x - C^2(x)} F_1(x) = \frac{1}{8}\left(1 + \frac{1}{\sqrt{1-4x}}\right)\left(1 + \sqrt{1-4x} - \sqrt{2}\sqrt{1 + \sqrt{1-4x} - 10x}\right) \tag{4}$$

$$F_8(x) = \frac{x}{2(1-4x)} + \frac{x}{2\sqrt{1-4x}} \tag{5}$$

The four functions F_6, F_7, F_9, and F_{10} satisfy the following algebraic equations.

$$F_6^3 - 2F_6^2 + (1+2x)F_6 + x^2 - 2x = 0 \tag{6}$$

$$F_7^4 - 3F_7^3 + (3+x)F_7^2 - (1+x)^2 F_7 - x^3 + x^2 + x = 0 \tag{7}$$

$$F_9^4 - 3F_9^3 + (3+x)F_9^2 - (1+2x)F_9 + x^2 + x = 0 \tag{8}$$

$$\begin{aligned} F_{10}^7 - (6+x)F_{10}^6 + (15+6x)F_{10}^5 + (x^2 - 15x - 20)F_{10}^4 - (2x^2 - 20x - 15)F_{10}^3 - \\ -(15x+6)F_{10}^2 + (2x^2 + 6x + 1)F_{10} + x^4 - x^2 - x = 0 \end{aligned} \tag{9}$$

In the first order asymptotics we use the notation $f(n) \sim g(n)$ for $\lim_{n\to\infty} f(n)/g(n) = 1$.

$$w_1(n) \sim \frac{1}{\sqrt{15\pi}} \frac{1}{n\sqrt{n}} \left(\frac{25}{4}\right)^n \quad w_2(n) \sim 0.16584\, n^{-3/2}\, (4.80261)^n \quad w_3(n) \sim \frac{1}{9}\left(\frac{9}{2}\right)^n \tag{10}$$

$$w_4(n) \sim \frac{5}{16}\sqrt{\frac{5}{6\pi}} \frac{1}{n\sqrt{n}} \left(\frac{36}{5}\right)^n \quad w_5(n) \sim \frac{4}{3} w_1(n) \sim \frac{4}{3\sqrt{15\pi}} \frac{1}{n\sqrt{n}} \left(\frac{25}{4}\right)^n \tag{11}$$

$$w_6(n) \sim \frac{4}{9\sqrt{3\pi}} \frac{1}{n\sqrt{n}} \left(\frac{27}{4}\right)^n \quad w_7(n) \sim \frac{\sqrt{5731 - 4635/\sqrt{17}}}{256\sqrt{\pi}} \frac{1}{n\sqrt{n}} \left(\frac{107 + 51\sqrt{17}}{64}\right)^n \tag{12}$$

$$w_8(n) \sim \frac{1}{8}\, 4^n \tag{13}$$

$$w_9(n) \sim \frac{\sqrt{5 - 1/\sqrt{13}}}{4\sqrt{6\pi}} \frac{1}{n\sqrt{n}} \left(\frac{70 + 26\sqrt{13}}{27}\right)^n \quad w_{10}(n) \sim 0.12075\, n^{-3/2}\, (5.22159)^n \tag{14}$$

The constants in the asymptotics of w_2 and w_{10} are just approximations but, as we shall see in the next section, in principle we can give closed algebraic expressions for them as well. Numerically the asymptotics read as follows. $w_1(n) \sim 0.14567\, n^{-3/2}\, 6.25^n$, $w_2(n) \sim 0.16584\, n^{-3/2}\, 4.80261^n$, $w_3(n) \sim 0.11111\, 4.5^n$, $w_4(n) \sim 0.16095\, n^{-3/2}\, 7.2^n$, $w_5(n) \sim 0.19423\, n^{-3/2}\, 6.25^n$, $w_6(n) \sim 0.14477\, n^{-3/2}\, 6.75^n$,

$w_7(n) \sim 0.14958\ n^{-3/2}\ 4.95747^n$, $w_8(n) \sim 0.125\ 4^n$, $w_9(n) \sim 0.12514\ n^{-3/2}\ 6.06460^n$, $w_{10}(n) \sim 0.12075\ n^{-3/2}\ 5.22159^n$. A remarkable fact is that all the ten linear constants lie in the interval $(0.1, 0.2)$.

In the left table below we list the first eight values $w_i(n), n = 1, 2, \ldots, 8$, for each $i = 1, 2, \ldots, 10$. For this and other heavy calculations we used MATHEMATICA and MAPLE. For $i = 1, 2, 3, 4, 5, 8$ we took directly the generating function. For $i = 6, 7, 9, 10$ we started with $F_i(0) = 0$ and then, differentiating the equation, we applied the relations $w_i(n) = F_i^{(n)}(0)/n!$. For $i = 6, 7, 9$ one can apply alternatively the Lagrange inversion formula — see Section 4. In the right table we sort the weights by their exponential growth rates.

w_1	1	2	7	19	131	625	3099	15818
w_2	1	2	5	15	50	178	663	2553
w_3	1	3	12	51	222	978	4338	19323
w_4	1	3	13	63	326	1769	9964	57843
w_5	1	3	12	52	236	1109	5366	26639
w_6	2	3	10	42	198	1001	5304	29070
w_7	1	2	4	13	44	164	636	2559
w_8	1	3	11	42	163	638	2510	9908
w_9	1	2	6	23	98	447	2134	10530
w_{10}	1	1	4	12	44	175	718	3052

w_8	brooms	4^n
w_3	chains	4.5^n
w_2	max. antichains	4.80261^n
w_7	max. ind. sets	4.95747^n
w_{10}	max. matchings	5.22159^n
w_9	matchings	6.06460^n
w_1	antichains	6.25^n
w_5	connected sets	6.25^n
w_6	independent sets	6.75^n
w_4	infima closed sets	7.2^n

We conclude the section with a few comments. Note the relation between w_1 and w_5. From (5) it follows at once a closed formula for $w_8(n)$, see (24). In Section 4 we derive a closed formula (29) for $w_6(n)$ and a nice recurrent formula (30) for $w_7(n)$. Expressions and equations (1)–(9) yield effective procedures calculating for a given n the numbers $w_i(n)$, $1 \leq i \leq 10$. A natural question is whether one can calculate effectively, given a tree T, the numbers $w_i(T)$. This turns out to be possible for each of the weights, in the next section we give the corresponding recurrent relations.

Thus, indeed, the average tree has asymptoticly much more antichains than chains in spite the tendency shown by the first nine values. For $n \geq 10$ we have, in accordance with the asymptotics, $w_1(n) > w_3(n)$. Even maximal antichains beat asymptoticly chains but now $w_2(n) < w_3(n)$ for $n = 2, 3, \ldots, 99$. Only from 100 vertices on the asymptotics prevails and the average tree starts to have more maximal antichains than chains.

3 Subset countings — proofs

Let $T = (r, V, E, L)$ be a tree and $v \in V$ be a vertex. A *subtree* T_v of T rooted in v is the subtree spanned by the upset $\{x \in V :\ x \geq v\}$. A *degree* $deg(v)$ of v is the number $|ch(v)|$ of children of v. A *principal subtree* of T is a subtree T_v such that $v \in ch(r)$. T is determined uniquely by the list $ps(T) = (T_v :\ v \in ch(r))$ of its principal subtrees. A *singleton* s is the trivial one vertex tree. Let us remind the Catalan function C satisfying $C^2 - C + x = 0$, see Section 1.

To determine the generating function F_w we use arguments of two kinds. In the *recurrence argument* we take the decomposition $ps(T) = (T_1, T_2, \ldots, T_k)$ and find, for a weight w, the recurrent relation that transforms the list $(w(T_1), w(T_2), \ldots, w(T_k))$ into the number $w(T)$. The relation can be often translated to an equation for F_w. This way we obtain both the *individual count* (the recurrence for $w(T)$) and the *collective count* (the function F_w that counts $w(n)$). An alternative approach via another decomposition is indicated in the concluding section.

The *extension argument* is basically counting in two ways. We count the number of extensions of a fixed set $X \subset V$ with a special property to a tree. See Figure 1. Draw a tree $T = (r, V, E, L)$ in the plane. The *gaps* of $v \in V$ are the wedge-shaped areas into which the edges incident with v split v's neighborhood. Thus v has $deg(v) + 1$ gaps. All gaps of all vertices form the set $g(T)$ with $2|V| - 1$ elements. In the *gap extension* we take a tree $T \in \mathcal{T}_m$ and into each gap $g \in g(T)$ we insert a tree T_g. The root $r(T_g)$ and the vertex of g are identified. A moment of thought reveals that the number of choices for which a tree from $\mathcal{T}_n$ arises is the coefficient at x^n in $x^m(C(x)/x)^{2m-1}$. In the *edge extension* we mark on a fixed oriented edge $e \in \mathcal{T}_2$ from top to bottom $k \geq 0$ points $p_1, \ldots, p_k$ and we put a tree T_i to the left and a tree U_i to the right of p_i, identifying p_i with the roots $r(T_i)$ and $r(U_i)$. A tree from $\mathcal{T}_n$ (we do not count the endpoints of e) is obtained for $[x^n]\ \sum_{k\geq 0}(C^2/x)^k = [x^n]\ x/(x - C^2)$ choices. Here and further on $[x^n]\ f$ denotes the coefficient at x^n in the power series f. In the l *edges extension* we extend this way independently l edges. While saying nothing about the individual count this method is usually more elegant than the recurrence argument.

gap extension edge extension

Figure 1: Extensions.

1 Antichains by extension. Consider an antichain $X \subset V(T)$ and the tree T^* spanned by the downset $\{v \in V(T) :\ v \leq x \in X\}$. Obviously T is a gap extension of T^* and therefore

$$F_1(x) = \sum_{m\geq 1} c_{m-1} x^m \left(\frac{C(x)}{x}\right)^{2m-1} = \frac{x}{C(x)} \sum_{m\geq 1} c_{m-1} \left(\frac{C^2(x)}{x}\right)^m = \frac{C(C^2(x)/x)}{C(x)/x}.$$

The rest is a matter of simplifications.

Antichains by recurrence. For singleton we have $w_1(s) = 1$. For a nonsingleton T with $ps(T) = (T_1, T_2, \ldots, T_k)$ we have the recurrence

$$w_1(T) = \prod_{i=1}^{k}(1 + w_1(T_i)) \tag{15}$$

whose proof is immediate. It translates to $F_1 = x\sum_{k\geq 0}(F_1 + C)^k = x/(1 - F_1 - C)$ which simplifies to $F_1^2 + (C - 1)F_1 + x = 0$. Solving this we get again the formula (1).

2 Maximal antichains by recurrence. Similarly to (15) we get $w_2(s) = 1$ and, $ps(T) = (T_1, T_2, \ldots, T_k)$,

$$w_2(T) = 1 + \prod_{i=1}^{k} w_2(T_i). \tag{16}$$

This translates to $F_2 = C + x\sum_{k\geq 1} F_2^k = C + xF_2/(1-F_2)$, i.e. to $F_2^2 + (x - C - 1)F_2 + C = 0$. The quadratic formula yields (2).

3 Chains by extension. Consider a chain $X = (x_1, \ldots, x_m) \subset V$ in T and think of the x_{i-1}–x_i path as an edge, $i = 1, \ldots, m$, $x_0 = r$. Then T is a gap extension and m edges extension of X. Hence

$$F_3(x) = \sum_{m\geq 1} x^m \left(\frac{C(x)}{x}\right)^{2m-1} \left(\frac{x}{x - C^2(x)}\right)^m = \frac{xC(x)}{x - 2C^2(x)}.$$

After further simplifications we obtain the formula for F_3 in (3).

Chains by recurrence. The recurrence for chains is $w_3(s) = 1$, $ps(T) = (T_1, T_2, \ldots, T_k)$,

$$w_3(T) = 1 + 2\sum_{i=1}^{k} w_3(T_i). \tag{17}$$

Consider the generating function

$$G(x, y) = \sum_{T} x^{w_3(T)} y^{|V(T)|}.$$

Then (17) reads as

$$G(x, y) = xy \sum_{k\geq 0} G(x^2, y)^k = \frac{xy}{1 - G(x^2, y)}.$$

Clearly $G(1, y) = C(y)$ and $F_3(y) = G_x(1, y)$. Taking the partial derivative by x of the equation for G and evaluating it at $(1, y)$ we find

$$F_3(y) = y\frac{1 - C(y) + 2F_3(y)}{(1 - C(y))^2} \quad \text{that solves as } F_3(y) = y\frac{1 - C(y)}{(1 - C(y))^2 - 2y}.$$

Simplifications lead again to the formula in (3).

4 Infima closed sets by extension. Consider a nonempty infima closed set $X \subset V(T)$, $|X| = m$. By replacing all u–v paths, $u, v \in X$, not containing other vertices of X by an edge we produce a tree T^* on m vertices. Clearly T is a gap and m edges extension of T^*, in the same way as for chains. Only now we are extending all trees on m vertices, not only the path. Thus

$$F_4(x) = \sum_{m\geq 1} c_{m-1} x^m \left(\frac{C(x)}{x}\right)^{2m-1} \left(\frac{x}{x - C^2(x)}\right)^m = \frac{x}{C(x)}\, C\left(C^2(x)/(x - C^2(x))\right).$$

Simplifications lead to the formula in (3).

For the sake of completeness we mention the recurrent formula. Let $ps(T) = (T_1, \ldots, T_k)$. Then $w_4(s) = 1$,

$$w_4(T) = \sum_{i=1}^{k} w_4(T_i) + \prod_{i=1}^{k}(1 + w_4(T_i)). \tag{18}$$

5 Connected sets by extension. Consider a connected set $X \subset V$. It is easy to see that T is a gap and (one) edge extension of X. The edge corresponds to the path $r(T)$–$r(X)$. Thus the additional factor $x/(x - C^2(x))$ in (4) compared to antichains.

Again, given a T, we can effectively calculate $w_5(T)$:

$$w_5(T) = \sum_{v \in V} w_1(T_v) \tag{19}$$

where T_v is the subtree rooted in v.

6 Independent sets by recurrence. We need an auxiliary weight $z(T)$ counting $\emptyset$ and the independent sets in T not containing r. Let $ps(T) = (T_1, \ldots, T_k)$. A moment of thought reveals that $z(s) = 1$, $w_6(s) = 2$,

$$z(T) = \prod_{i=1}^{k} w_6(T_i) \text{ and } w_6(T) = \prod_{i=1}^{k} w_6(T_i) + \prod_{i=1}^{k} z(T_i). \tag{20}$$

Translated to generating functions,

$$F_z = x \sum_{k \geq 0} F_6^k = \frac{x}{1 - F_6} \text{ and } F_6 = x \sum_{k \geq 0} F_z^k + x \sum_{k \geq 0} F_6^k = \frac{x}{1 - F_z} + \frac{x}{1 - F_6}. \tag{21}$$

Eliminating F_z from the system we get the cubic equation (6).

7 Maximal independent sets by recurrence. So far we always calculated the number at a vertex from the numbers at its children, now we need to consider also the numbers at grandchildren. We define two auxiliary weights t and q. Let $t(T) = \#$ of ind. sets in T not containing r which are maximal or extendable only by the root r. Further $q(s) = 1$, and $q(T) = t(T_1)t(T_2)\ldots t(T_k)$ where $ps(T) = (T_1, \ldots, T_k)$. Then $w_7(s) = t(s) = q(s) = 1$ and, $ps(T) = (T_1, \ldots, T_k)$,

$$t(T) = \prod_{i=1}^{k} w_7(T_i) \text{ and } w_7(T) = \prod_{i=1}^{k} t(T_i) + \prod_{i=1}^{k} w_7(T_i) - \prod_{i=1}^{k} (w_7(T_i) - q(T_i)). \tag{22}$$

The first equality is easy — to take an r-free ind. set in T extendable at most by r is the same as to take a max. ind. set in each T_i. In the second equality in (22) we count first by the product $\prod t(T_i)$ the number of max. ind. sets containing the root. To take a max. ind. set in T not containing r is the same as to take a max. ind. set in each T_i, not all of them avoiding $r(T_i)$. There are $q(T_i)$ max. ind. sets in T_i containing $r(T_i)$. This gives the rest of the second equation. (22) expressed in generating functions is

$$F_t = \frac{x}{1 - F_7} \text{ and } F_7 = \frac{x}{1 - F_t} + \frac{x}{1 - F_7} - \frac{x}{1 - F_7 + x/(1 - F_t)} \tag{23}$$

because the generating function corresponding to q is $x/(1 - F_t)$. The elimination of F_t yields the quartic (7).

8 Brooms by extension. Fix a broom B with m vertices in a tree T. T is a gap extension and one edge extension (as for connected sets) of B and therefore

$$F_8(x) = \frac{x}{x - C^2(x)} \sum_{m \geq 1} x^m \left(\frac{C(x)}{x} \right)^{2m-1} = \frac{x^2}{C.(x - C^2)} \frac{C^2/x}{1 - C^2/x} = \frac{x^2 C}{(x - C^2)^2} = \frac{x^2 C}{(2x - C)^2} =$$

$$= \frac{1}{1 - 4x} \frac{x^2 C}{C - x} = \frac{x}{1 - 4x} \frac{x}{C} = \frac{x}{1 - 4x} \frac{1 + \sqrt{1 - 4x}}{2} = \frac{x}{2(1 - 4x)} + \frac{x}{2\sqrt{1 - 4x}}.$$

It is easy to extract the coefficient by the binomial formula. On the other hand clearly $w_8(T) = \sum_{v \in V} 2^{deg(v)}$ and we have the identity

$$w_8(n) = \sum_{T \in \mathcal{T}_n} \sum_{v \in V(T)} 2^{deg(v)} = \frac{4^{n-1} + \binom{2n-2}{n-1}}{2}. \tag{24}$$

In our derivation we used only that for any $m \geq 1$ there is exactly one broom on m vertices. Thus more generally:

Theorem 3.1 *Suppose $\mathcal{S} \subset \mathcal{T}$ is a family of trees such that $|\mathcal{S} \cap \mathcal{T}_n| = 1$ for any $n \geq 1$. Let $w(T)$ count the total number of ways to embed a member of $\mathcal{S}$ into T. Then $w(n) = \sum_{T \in \mathcal{T}_n} w(T) = w_8(n) = (4^{n-1} + \binom{2n-2}{n-1})/2$.*

If $\mathcal{S}$ is the family of all paths we obtain the identity

$$\sum_{T \in \mathcal{T}_n} |\{(u, v) \in V(T) \times V(T) : \ u \text{ and } v \text{ are comparable in } T\}| = 4^{n-1}$$

because the left hand side is $2w_8(n) - nc_{n-1}$. We remark that a quantity similar to w_8, namely the average vertex altitude, was counted by D. E. Knuth, see [8].

9 Matchings by recurrence. We set $z(T)$ to be the number of matchings in T not covering the root, the empty set included. Let $ps(T) = (T_1, \ldots, T_k)$. Then $z(s) = w_9(s) = 1$,

$$z(T) = \prod_{i=1}^{k} w_9(T_i) \text{ and } w_9(T) = \prod_{i=1}^{k} w_9(T_i). \left(1 + \sum_{i=1}^{k} \frac{z(T_i)}{w_9(T_i)}\right). \tag{25}$$

The first relation follows from the fact that a matching in T avoiding r arises simply by taking in each T_i either a matching or the empty set. In the second relation we add the numbers of matchings using the edge $r(T)r(T_i)$. To translate this to generating functions we use the identity $\sum_{k \geq 0}(k+1)x^k = 1/(1-x)^2$. Thus

$$F_z = \frac{x}{1 - F_9} \text{ and } F_9 = \frac{x}{1 - F_9} + \frac{xF_z}{(1 - F_9)^2}.$$

Eliminating F_z we obtain the quartic equation (8).

10 Maximal matchings by recurrence. From technical reasons we set $w_{10}(s) = 1$. Consider two auxiliary weights z and q. $z(s) = 0$ and $z(T)$ counts the number of max. matchings in T covering the root, $q(s) = 1$ and $q(T) = w_{10}(T_1)w_{10}(T_2) \ldots w_{10}(T_k)$ where $ps(T) = (T_1, \ldots, T_k)$. Then $z(s) = 0$ and $q(s) = w_{10}(s) = 1$,

$$z(T) = \prod_{i=1}^{k} w_{10}(T_i). \sum_{i=1}^{k} \frac{q(T_i)}{w_{10}(T_i)} \text{ and } w_{10}(T) = z(T) + \prod_{i=1}^{k} z(T_i). \tag{26}$$

In the first relation we count the number of max. matchings using the edge $r(T)r(T_i)$. Those arise by taking a max. matching in each $T_j, j \neq i$, (or $\emptyset$ if $T_j = s$, that's why we set $w_{10}(s) = 1$) and an $r(T_i)$-free matching in T_i (or $\emptyset$ if $T_i = s$) extendable eventually only by some edge going up from $r(T_i)$. Such matchings are counted by $q(T_i)$. In the second relation we add to $z(T)$ the number of max. matchings avoiding $r(T)$. Algebraicly,

$$F_z = \frac{xF_q}{(1 - F_{10})^2} \text{ and } F_{10} = F_z + \frac{x}{1 - F_z} \text{ where } F_q = \frac{x}{1 - F_{10}}.$$

From this one obtains the relation $F_{10} = x^2/(1 - F_{10})^3 + x/(1 - x^2/(1 - F_{10})^3)$ which simplifies to the equation of degree 7 in (9).

The asymptotics of the numbers $\mathbf{w_1(n), \ldots, w_{10}(n)}$. We start with the simple cases and proceed to more complicated ones. Catalan numbers have the asymptotics

$$c_n \sim \frac{4^n}{n\sqrt{\pi n}}. \tag{27}$$

This follows by Stirling formula.

$\mathbf{w_8(n)}$. The asymptotics (13) for $w_8(n)$ is immediate from (24).

When F_i is given by square roots the next theorem of Bender, p. 496 in [2], is useful. We need also binomial and Stirling formulae and basic concepts of analytic functions.

Theorem 3.2 *Let $A(x) = \sum a_n x^n$, $B(x) = \sum b_n x^n$, and $C(x) = A(x)B(x) = \sum d_n x^n$ be three power series, and let A and B have radii of convergence $\alpha > \beta \geq 0$. Suppose $b_{n-1}/b_n \to \beta$ as $n \to \infty$, and $A(\beta) \neq 0$. Then*

$$d_n \sim A(\beta) b_n.$$

$\mathbf{w_3(n)}$. For $F_3(x)$ we use Theorem 3.2 with $A(x) = x(1 + 3\sqrt{1-4x})/4$, $B(x) = 1/(1 - 9x/2)$, $\alpha = 1/4$, $\beta = 2/9$, and $A(2/9) = 1/9$. The asymptotics (10) for $w_3(n)$ follows.

$\mathbf{w_4(n)}$. To obtain the asymptotics (11) for $w_4(n)$ we write $F_4(x) = (1 + \sqrt{1-4x})/4 - A(x)B(x)$ where

$$A(x) = \frac{\sqrt{5}}{4} \frac{1 + \sqrt{1-4x}}{\sqrt{(3\sqrt{1-4x}+2)\sqrt{1-4x}}} \text{ and } B(x) = \sqrt{1 - \frac{36x}{5}}.$$

Theorem 3.2 is applied with $\alpha = 1/4$, $\beta = 5/36$, and $A(5/36) = (5/8)\sqrt{5/6}$. The coefficient b_n in $B(x) = \sum b_n x^n = (1 - 36x/5)^{1/2}$ can be estimated by means of binomial and Stirling formulae.

$\mathbf{w_1(n)}$. We observe that the expresion under the big radical in (1) determines a function that is analytic in the 1/4 circle and that is nonzero there except for the simple zero 4/25. Thus we can write $F_1(x) = (1 + \sqrt{1-4x})/4 - A(x)B(x)$ with $B(x) = \sqrt{1 - 25x/4}$ and $A(x)$ a function analytic in the 1/4 circle. Further, $A(4/25) = 2/\sqrt{15}$. Theorem 3.2 implies the first asymptotics in (10).

$\mathbf{w_5(n)}$. Here $A(x) = (1/2)(1 + 1/\sqrt{1-4x})$, $B(x) = F_1(x)$, $\alpha = 1/4$, $\beta = 4/25$, and $A(4/25) = 4/3$. The second asymptotics in (11) follows.

$\mathbf{w_2(n)}$. The expression under the big radical in (2) is analytic in the 1/4 circle and is nonzero there except for the simple zero $\beta = 0.20821\ldots$ (the only real root of $x^3 - 4x^2 + 20x - 4$). Thus we have again $F_5(x) = (3 - 2x - \sqrt{1-4x})/4 - A(x)B(x)$ with $B(x) = \sqrt{1 - x/\beta}$ and $A(x)$ a function analytic in the 1/4 circle. One can calculate that

$$A(\beta) = \sqrt{\frac{\beta}{2}} \sqrt{\frac{3\beta}{\sqrt{1-4\beta}} - \beta + 2}.$$

The second asymptotics in (10) is obtained.

To resolve the remaining cases when F_i satisfies an equation of degree > 2 we use the following result, found on p. 502 in [2].

Theorem 3.3 *A power series $f(x) = \sum a_n x^n$ with nonnegative coefficients satisfying $F(x, f(x)) = 0$ and two real numbers $\alpha > 0$ and $\beta > a_0$ are given. Suppose that*

(a) *for some $\delta > 0$, $F(x, y)$ is analytic whenever $|x| < \alpha + \delta$, $|y| < \beta + \delta$,*

(b) $F(\alpha, \beta) = F_y(\alpha, \beta) = 0$,

(c) $F_x(\alpha, \beta) \neq 0$ *and* $F_{yy}(\alpha, \beta) \neq 0$, *and*

(d) *if (κ, λ) is another solution of the system in (b) then $|\kappa| > \alpha$ or $|\lambda| > \beta$.*

Then

$$a_n \sim \sqrt{\frac{\alpha F_x(\alpha, \beta)}{2\pi F_{yy}(\alpha, \beta)}} \frac{1}{n\sqrt{n}} \left(\frac{1}{\alpha}\right)^n. \tag{28}$$

This is exactly what we need but the difficulty is that the theorem is incorrect, as pointed out by Canfield [3]. However, the conclusion (28) still holds if we can present positive reals (α, β), $f(\alpha) = \beta$, such that (01) (α, β) lies inside the analyticity domain of F (i.e., (a) holds), (02) the condition (c) holds, (03) α is the radius of convergence of $f(x)$, and (04) $f(x)$ has no other singularity on the boundary than α.

We know, by implicit function theorem, that the pair (α, β) we look for (as well as and any other singularity on the boundary) is hidden among the solutions of the simultaneous equations (b). In general it may be difficult to determine which solution is the right one or even to find all solutions. Therefore several conditions for F making (α, β) unique or localizing it among the solutions were proposed, see [9] and [10], p. 1162–3.

For the four functions F_6, F_7, F_9, and F_{10} we can always find (α, β) meeting the conditions (01)–(04). Indeed, $F(x, y)$ is a bivariate polynomial, thus analytic everywhere, and it is not too difficult to find all solutions of the algebraic system (b). Notice that $c_{n-1} \leq w_i(n) \leq 2^n c_{n-1}$. By (27) we know that the radius of convergence of any $F_i(x)$, $i = 1 \ldots 10$, lies in $[1/8, 1/4]$. In all four cases there is only one (complex) solution (α, β) such that $1/8 \leq |\alpha| \leq 1/4$. Thus (01)–(04) holds and (28) is true.

$\mathbf{w_6(n)}$. $F_6(x)$ satisfies the cubic equation (6). The system (b) has four solutions: $(0, 1)$ (with multiplicity 3) and $(\alpha, \beta) = (4/27, 5/9)$. Plugging in the formula (28) we obtain the first bound in (12).

$\mathbf{w_7(n)}$. The equation for $F_7(x)$ is given by (7). The solutions of (b) are: $(0, 1)$ (with multiplicity 4), $((-51\sqrt{17} - 107)/512, (33 - 7\sqrt{17})/128)$, and $(\alpha, \beta) = ((51\sqrt{17} - 107)/512, (33 + 7\sqrt{17})/128)$. The second bound in (12) follows.

$\mathbf{w_9(n)}$. The equation for $F_9(x)$ is (8). The solutions of (b) are: $(0, 1)$ (multiplicity 2), $((-13\sqrt{13} - 35)/72, (1 - \sqrt{13})/12)$, and $(\alpha, \beta) = ((13\sqrt{13} - 35)/72, (1 + \sqrt{13})/12)$. The first bound in (14) follows.

$\mathbf{w_{10}(n)}$. $F_{10}(x)$ satisfies (9). The system (b) has 12 solutions: $(0, 1)$ (multiplicity 8), $(-0.26689 \pm 0.51782i, 0.01231 \pm 0.40950i)$, $(11.67188, 8.47407)$, and $(\alpha, \beta) = (0.19151, 0.38840)$. The four y solutions different from 1 are roots of the quartic $248y^4 - 2204y^3 + 912y^2 - 389y + 137$. x appears in $F_{yy}(x, y) = 0$ only in the second degree. Thus α and β still express in radicals. The second bound in (14) follows.

4 Applications of the LIF

The generating functions F_6, F_7, F_9, and F_{10} satisfy an algebraic equation of degree > 2. Such an equation is often very hard, if not impossible, to solve explicitly. Nevertheless, sometimes we can find easily the inverse to the solution. Then the *Lagrange inversion formula* applies.

Theorem 4.1 (LIF) *Suppose $f(x)$ is a power series with $[x^0]f = 0$ and $[x^1]f \neq 0$. Then*

$$[x^n]f(x)^{<-1>} = n^{-1}[x^{n-1}](f(x)/x)^{-n}.$$

For more details see [14], [10] (p. 1106), and [7] (p. 1032).

Theorem 4.2 *Let $n \geq 1$. Recall that $w_6(n)$ is the total number of all independent sets in all $T \in \mathcal{T}_n$ (the empty set counted) and $z(n)$ is the number of those avoiding the root. Then*

$$w_6(n) = \frac{1}{n-1}\binom{3n-3}{n} \text{ and } z(n) = \frac{1}{n}\binom{3n-2}{n-1}. \tag{29}$$

Proof. We start with $z(n)$. Eliminating F_6 from (21) we obtain $F_z(1 - F_z)^2 = x$. Thus $F_z(x)^{<-1>} = x(1 - x)^2$. The formula for $z(n)$ follows readily by the LIF.

To determine $w_6(n)$ we observe that

$$3xF_6' - 2F_6 - 4xF_z' + 2F_z = 0.$$

This is not difficult to check by means of the relations (21). We leave the straightforward calculations to the reader as an exercise. In terms of coefficients:

$$(3n-2)w_6(n) = (4n-2)z(n).$$

Substituting the formula for $z(n)$ we finish the proof. □

Theorem 4.3 *Let $n \geq 1$. Recall that $w_7(n)$ is the total number of all maximal independent sets in all $T \in \mathcal{T}_n$ and $t(n)$ is the number of independent sets avoiding the root and extendable at most by it. Then*

$$t(n) = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k \binom{n+k-1}{k}\binom{3n-k-2}{n-k-1} = \frac{1}{n}\sum_{k=0}^{\lfloor (n-1)/2 \rfloor}\binom{2n-2-2k}{n-1-2k}\binom{n+k-1}{k}$$

and

$$w_7(n) = t(n+1) - \sum_{k=2}^{n} t(k).w_7(n-k+1). \tag{30}$$

Proof. Eliminating F_7 from (23) we find that $F_t(1-F_t)(1-F_t^2) = F_t(1+F_t)(1-F_t)^2 = x$. Thus $F_t(x)^{<-1>} = x(1-x)(1-x^2) = x(1+x)(1-x)^2$. The LIF yields the formula for $t(n)$. The recurrence for $w_7(n)$ follows from the relation $F_t(1-F_7) = x$. □

As to the values of w_9, the LIF helps here too. $F_9(x)^{<-1>}$ is easily found by solving (8) for x. We obtain a more comfortable way to calculate $w_9(n)$ (instead of taking derivatives) but no nice explicit formula seems to arise here. The details are omitted. We did not succed in applying the LIF to w_{10}.

5 Drawing countings

The calculations for the weights w_{11} and w_{12} are more elegant when the main parameter n is $|E|$ rather than $|V|$. We use exponential instead of ordinary generating function. We determine

$$F_i(x) = \sum_{n\geq 0} \frac{w_i(n)}{n!}\, x^n$$

where $i = 11, 12$ and in $w_i(n) = \sum_T w_i(T)$ we sum over the trees with n edges.

A simple drawing $(e_1, e_2, \ldots, e_n)$ of a tree T with n edges is a way of planting T from the root. To look on it differently consider the vertices $(v_1, v_2, \ldots, v_n)$ where v_i is the endpoint of e_i. Obviously, $(r, v_1, \ldots, v_n)$ is a linear extension of the tree as a poset. And vice versa, any linear extension determines a simple drawing of T. Thus $w_{11}(T)$ is the number of linear extensions of T. This notion and the results below (Theorems 5.1 and 5.2) seem be frequently rediscovered, as we learned after proving the theorems.

Theorem 5.2 is close in statement and proof to Lemma 2.1 in [1]. Theorem 5.1 is proved, in a more complicated manner, in [13]. Another proof of Theorem 5.1, much the same as the one below, can be found in [6]. There the authors point to the thesis [4] as to an older reference for this result and mention that R. P. Stanley proved it before as well. We join in and include, for the readers convenience, our (independent) proofs. As to the notation, $(2n-1)!!$ stands, as usual, for $1.3.5\ldots(2n-1)$. For triple and quadruple factorials see [6]!!

Theorem 5.1 *Let $n > 0$. Then*

$$w_{11}(0) = 1,\ w_{11}(n) = (2n-1)!! \text{ and } F_{11}(x) = \frac{1}{\sqrt{1-2x}}. \tag{31}$$

Proof. So $w_{11}(T)$ counts the labelings of vertices by $0, 1, \ldots, n$ such that the label of u is smaller than that of v whenever $u < v$. Thus r is always labeled by 0. Clearly $w_{11}(0) = 1$. For $T \in \mathcal{T}_{n+1}$, $n \geq 1$, in any of the labelings n sits at a leaf l and deleting l we get a proper labeling of a $T^* \in \mathcal{T}_n$. From each labeled T^* we can get, adding l back, exactly $2n-1$ different labeled T's since each T^* has $2n-1$ gaps to place l. Hence $w_{11}(n) = (2n-1).w_{11}(n-1)$ and we obtain the first formula in (31). The second formula follows from the first one after rewriting $(2n-1)!!$ as $n!\binom{2n}{n}/2^n$. □

The asymptotics

$$w_{11}(n) \sim \sqrt{2}\,\left(\frac{2n}{e}\right)^n$$

follows by Stirling formula.

We show now how to perform for w_{11} the individual count.

Theorem 5.2 *Recall that T_v stands for the subtree of T rooted in $v \in V$. We abbreaviate $|V(T_v)|$ by $|T_v|$. Then, for a tree T with $|V| = n+1$ vertices,*

$$w_{11}(T) = \frac{(n+1)!}{\prod_{v\in V}|T_v|} = \frac{n!}{\prod_{v\in V, v\neq r}|T_v|}. \tag{32}$$

Proof. By induction on the height of T. Clearly $w_{11}(s) = 1$. For a nonsingleton tree T with $ps(T) = (T_1, T_2, \ldots, T_k)$ we have

$$w_{11}(T) = \binom{n}{|T_1|\ |T_2| \ldots |T_k|} \prod_{i=1}^{k} w_{11}(T_i)$$

because for each of the choices $\{1, 2, \ldots, n\} = X_1 \cup X_2 \cup \ldots \cup X_k$, $|X_i| = |T_i|$, X_i mutually disjoint, of the sets of labels for vertices $V(T_i)$ (r is labeled by 0) we have exactly $\prod w_{11}(T_i)$ labelings. Plugging in the formulae for $w_{11}(T_i)$ and canceling the factorials we get (32). □

The counting of $w_{12}(n)$ is more interesting. Note that $w_{12}(T)$ counts different ways to plant T from its root too but "different" has other meaning compared to w_{11}. For instance, if T_0 is the V-shaped tree on 5 vertices then $w_{11}(T_0) = 6$ but $w_{12}(T_0) = 4$. The key fact is that the insertion of a new leaf in T in different gaps may produce the same tree. More precisely:

Lemma 5.3 *Suppose T has $n \geq 1$ edges and l leaves. Adding the new leaf in all $2n+1$ gaps yields $2n+1-l$ new different trees with $n+1$ edges, l of them have l leaves and $2n+1-2l$ have $l+1$ leaves.*

Proof. Consider the trees $X = \{T_g :\ g \in g(T)\}$ where T_g arises by adding the new leaf in the gap g. T_g and T_h coincide iff g and h share the same vertex v and all edges between g and h going up from v lead to leaves. Thus $|X| = 2n+1-c$ where c is the number of gaps whose left edge leads to a leaf. Clearly $c = l$. The number of leaves does not change iff we add the new leaf to a leaf and then we produce l new trees. Otherwise the number of leaves increases by one. □

Theorem 5.4

$$F_{12}(x) = \sum_{\mathcal{T}} \frac{w_{12}(T)}{|E(T)|!} x^{|E(T)|} = \sum_{n \geq 0} \frac{w_{12}(n)}{n!} x^n = \frac{1}{\sqrt{2e^{-x} - 1}}. \tag{33}$$

Proof. Consider the bivariate exp. gen. function ($l(T)$ is the number of leaves of T)

$$F^*(x, y) = \sum_{T \in \mathcal{T}} \frac{w_{12}(T)}{|E(T)|!} x^{|E(T)|} y^{l(T)} = 1 + xy + \frac{x^2 y}{2} + \frac{x^2 y^2}{2} + \ldots$$

Lemma 5.3 translates to generating functions as

$$\int_x (y \frac{\partial}{\partial y} + 2xy \frac{\partial}{\partial x} + y - 2y^2 \frac{\partial}{\partial y}) F^* = F^* - 1.$$

This yields the partial differential equation

$$(\frac{1}{y} - 2x) \frac{\partial F^*}{\partial x} + (2y - 1) \frac{\partial F^*}{\partial y} = F^*. \tag{34}$$

(34) is of the type $a(x, y)F_x + b(x, y)F_y = f(x, y, F)$ that reduces to two ordinary diff. equations. We review briefly the standard resolution and apply it to (34). First one solves the equation

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)} \tag{35}$$

which gives the system of *characteristic curves* $\{y_c(x) : c \in D\}$ (D is a set of real parameters). Along each of the curves F turns into a univariate function $F_c(x) = F(x, y_c(x))$ that satisfies

$$\frac{dF_c}{dx} = \frac{f(x, y_c(x), F_c(x))}{a(x, y_c(x))} \tag{36}$$

(this follows by the chain rule for partial derivatives). The value of F at a point $p = (x_0, y_0)$ is then $F_c(x_0)$ where $c = c(p)$ is chosen so that y_c goes through p.

The equation (35) becomes for (34)

$$\frac{dy}{dx} = \frac{2y - 1}{1/y - 2x}$$

which is an exact equation $(1/y - 2x)dy + (1 - 2y)dx = 0$. Solving it in a standard way we get the following equation for characteristic curves:

$$y\, e^{(1-2y)x} = c. \tag{37}$$

(36) turns into a separated variables equation

$$\frac{dF_c^*}{dx} = \frac{y_c'}{2y_c - 1} F_c^*$$

whose solution is $F_c^*(x) = d(c).\sqrt{2y_c(x) - 1}$. From (37) we have $y_c(0) = c$ and from $F_c^*(0) = 1$ we get $d(c) = 1/\sqrt{2c - 1}$. Thus $F_c^*(x) = \sqrt{2y_c(x) - 1}/\sqrt{2c - 1}$ and, using (37),

$$F^*(x, y) = \sqrt{\frac{2y - 1}{2y.e^{x(1-2y)} - 1}}\ .$$

Specializing $y = 1$ we obtain (33). □

Setting in (34) $y = 1/2$ we get for $g(x) = F^*(x, 1/2)$ the ord. diff. equation $2(1-x)g' = g$, thus ($g(0) = 1$) $g(x) = 1/\sqrt{1-x}$. Hence

$$2^n \sum_{T \in \mathcal{T}_{n+1}} w_{12}(T) \left(\frac{1}{2}\right)^{l(T)} = (2n-1)!!. \tag{38}$$

Let $k(T)$ stand for the number of nonleaves of T. By (38) the sum $\sum w_{12}(T).2^{k(T)-1}$ over all trees with n edges gives the same result as the sum $\sum w_{11}(T)$.

The function $F_{12}(x)$ satisfies $F_{12}(x)'.(2-e^x) = F_{12}(x)$. This provides us with the simple recurrence $w_{12}(0) = 1$,

$$w_{12}(n+1) = w_{12}(n) + \sum_{i=1}^{n} w_{12}(i).\binom{n}{i-1}. \tag{39}$$

The first few numbers are

$$\{w_{12}(n)\}_{n \geq 0} = \{1,\ 1,\ 2,\ 7,\ 35,\ 226,\ 1787,\ 16717,\ 180560,\ 2211181,\ \ldots\}.$$

To determine the asymptotics we proceed as in Section 3. The function $2e^{-x} - 1$ is entire and nonzero, except for the simple zeros $\log 2 + 2k\pi i$. Thus we write $F_{12}(x) = (1 - x/\log 2)^{-1/2} A(x)$ where $A(x)$ is analytic in the $((\log 2)^2 + 4\pi^2)^{1/2}$ circle and $A(\log 2) = 1/\sqrt{\log 2}$. By Theorem 3.2

$$w_{12}(n) = n!\ [x^n] F_{12}(x) \sim n!\ \frac{1}{\sqrt{\pi n \log 2}} \left(\frac{1}{\log 2}\right)^n \sim \sqrt{\frac{2}{\log 2}} \left(\frac{n}{e \log 2}\right)^n. \tag{40}$$

6 Concluding remarks

1 An alternative decomposition. In all recurrence arguments we used the decomposition $ps(T) = (T_1, T_2, \ldots, T_k)$. However, one can use the decomposition $T = (T_1, T^*)$ where T_1 is the subtree rooted in the leftmost child of r and T^* is the rest. In some cases this leads to easier derivations of equations for generating functions. On the other hand this decomposition is not well suited to do the indiviual count.

We advice the reader to try some individual counts by the formulae (15)–(20), (22), (25), (26), and (32). For instance, to calculate $w_1(T)$ one writes 1 to each leaf of T and then, by (15), recursively assignes to each vertex v the product of by 1 increased numbers assigned to v's children. Then $w_1(T)$ is the number assigned to r. By such calculations we were motivated to some of the problems stated below.

2 The weight $\mathbf{w_{12}}$. The individual count for the weights w_i, $i = 1, 2, \ldots, 11$ can be done by the (recurrent) formulae (15)–(20), (22), (25), (26), and (32) ($w_8(T)$ can be easily calculated from the definition). The question is how to calculate efficiently for any given T the number $w_{12}(T)$. It would be also interesting to give direct combinatorial proofs and interpretations to (39) and (38).

3 Extremal weight values. We define, for $i = 1, 2, \ldots, 12$,

$$m_i(n) = \min w_i(T) \text{ and } M_i(n) = \max w_i(T)$$

where for $i = 1, 2, \ldots, 10$ the extremum is taken over $\mathcal{T}_n$ and for $i = 11, 12$ over $\mathcal{T}_{n+1}$. In many cases it is easy to determine the extremal value. It is trivial that $m_1(n) = n$ (path), $M_1(n) = 2^{n-1}$ (broom), $m_2(n) = 2$ (broom), $m_3(n) = 2n - 1$ (broom), $M_3(n) = 2^n - 1$ (path), $M_4(n) = 2^n - 1$

(path), $m_7(n) = 2$ (broom), $m_8(n) = 2n - 1$ (path), $M_8(n) = 2^{n-1}$ (broom), $m_9(n) = n - 1$ (broom), $m_{11}(n) = 1$ (path), $M_{11}(n) = n!$ (broom), and $m_{12}(n) = 1$ (path).

It is not difficult to show that $m_5(n) = \binom{n}{2} + n$ (path), $M_5(n) = 2^{n-1} + n - 1$ (broom), $M_6(n) = 2^{n-1}$ (broom), and $(n \geq n_0)$ $m_{10}(n) = n - 1$ (broom). Now we determine $M_2(n)$.

Theorem 6.1 *Let $n = 1 + 3m + i > 2$, $i \in \{0, 1, 2\}$. Denote by $\mathcal{U}_n \subset \mathcal{T}_n$ the set of trees whose nonroot vertices have only the degrees 1 or 0 and which have only the branches with 3 edges and either 0, 1 or 2 branches with 2 edges or 1 branch with 4 edges. Then*

$$w_2(T) = M_2(n) \text{ for any } T \in \mathcal{U}_n \text{ and } w_2(T) < M_2(n) \text{ for any } T \in \mathcal{T}_n \backslash \mathcal{U}_n \text{ where}$$

$$M_2(n) = 1 + 3^m \text{ for } i = 0, \ = 1 + 3^m + 3^{m-1} \text{ for } i = 1, \quad \text{and } \ = 1 + 2.3^m \text{ for } i = 2.$$

Proof. Suppose T has a nonroot vertex v with $deg(v) = l \geq 2$. Denote by u the parent of v and by x_i the children of v. The tree T^* arises from T by cutting the edge joining v and x_l and joining x_l to u. We write a_i for $w_2(T_{x_i})$, a for the product of a_i's, and b for the product $\prod w_2(T_t)$ where t runs through the children of u different from v ($b = 1$ if there is no such child). By (16)

$$w_2(T_u) = 1 + (1 + a)b = 1 + b + ab \leq 1 + a_l b + ab = 1 + (1 + a_1 \dots a_{l-1}) a_l b = w_2(T_u^*).$$

Thus $w_2(T) \leq w_2(T^*)$, the equality holds iff x_l is a leaf. Applying repeatedly the transformation we change T into a tree U with the same number of vertices, with no nonroot vertex of degree > 1, and with w_2 at least as large. Let $d_1, d_2, \dots, d_k$ stand for the number of edges of the branches of U. It holds $w_2(U) = 1 + d_1 d_2 \dots d_k$ and $d_1 + d_2 + \dots + d_k = |V(T)| - 1$. We reduced our problem to a well known riddle asking what is the maximum product of a collection of positive integers with fixed sum. The answer follows by easy splitting arguments and is described above — the maximum is achieved exactly when all d_i's equal to 2 or 3 and there is as many 3's as possible, two 2's may be traded for one 4. The trees U with such d_i's form the set $\mathcal{U}_n$. We see that $w_2(T) = w_2(U)$ implies $T = U$ or $d_i = 1$ for some i. But $d_i = 1$ implies that the maximum product is not attained. Therefore the inequality is strict for the trees outside $\mathcal{U}_n$. □

The problem is to determine the remaining extremal values $m_4(n), m_6(n), M_7(n), M_9(n), M_{10}(n)$, and $M_{12}(n)$ or to give some bounds on them. To single some of them out: what is $m_4(n)$ and what are the trees with few infima closed sets? What is $M_{12}(n)$ and what are the trees with many drawings? For $\varepsilon > 0$ fixed and n large we have the bounds

$$\frac{1 - \varepsilon}{4\sqrt{\log 2}} \frac{1}{n} \left(\frac{1}{\log 16} \right)^n n! < M_{12}(n) \leq n!$$

The upper bound is trivial and the lower bound follows by the averaging argument from (27) and (40). The problem is how to improve these bounds. The remaining undetermined extremal values can be estimated in a similar way.

4 Two more problems. Is there any tree T different from s for which $w_1(T) = w_3(T)$, i.e., has the same number of chains and antichains? Are there infinitely many of them? We define the *height* of a positive integer m as the minimum height of a tree T such that $w_1(T) = m$. Are there numbers with arbitrary large height? Similarly for w_2.

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