Shifted squares n^2+1 have in average $\frac{3}{\pi}\log n$ divisors

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For $n \in \mathbb{N} = \{1, 2, \dots\}$, let $\tau(n)$ be the number of solutions of the equation $n = ab$, that is, the number of divisors of n — in this text, variables $a, b, d, i, j, m, n, \dots$ run in $\mathbb{N}$, and $x > 0$ denotes a real number, usually going to $+\infty$. Prove that

$$\sum_{n \le x} \tau(n^2+1) = (3/\pi)x \log x + O(x) \ . \tag{1}$$

This asymptotics is an exercise in the book of Iwaniec and Kowalski [2, p. 23], an example in the book of Friedlander and Iwaniec [1, p. 15] and was discussed in the blog of Tao [3], with further references to related articles given. Since in these three references justification absents or is very sketchy, and we like the result and want to know why it is true, we derive (1) in detail.

The first step is the transformation

$$\sum_{n \le x} \tau(n^2+1) = 2x \sum_{d \le x} \rho(d)/d + O(\textstyle\sum_{d \le x} \rho(d)) \ , \tag{2}$$

where $\rho(d)$ is the number of solutions of the congruence $a^2 + 1 \equiv 0$ modulo d. Indeed, if $db = n^2 + 1$ and $n \ge 2$, then $d < n < b$ or $b < n < d$, but it is not true for $n = 1$. We get, for $x > 1$,

$$\sum_{n \le x} \tau(n^2+1) = 2 + 2 \sum_{d < x} \sum_{n \in (d,x],\ d \,|\, n^2+1} 1 = 2 + 2 \sum_{d < x} \Big(\frac{x}{d} + \delta(d)\Big)\rho(d) \ ,$$

with $\delta(d) \in (-2, 0]$, by partitioning the interval of integers $(d, x]$ into the intervals $[id+1, id+d]$, $1 \le i \le \lfloor x/d \rfloor - 1$ and $[\lfloor x/d \rfloor d + 1, x]$. Thus equality (2) follows.

We need to handle sums involving $\rho(d)$. We begin with the arithmetic function $\nu(n)$, defined by $\nu(n) = \mu(m)$ if $n = m^2$, and $\nu(n) = 0$ if n is not a square, where $\mu(m)$ is the Möbius function: $\mu(m) = (-1)^r$ if m is a product of r distinct primes, and $\mu(m) = 0$ else. Then

$$\sum_{n \le x} \frac{\nu(n)}{n} = \sum_{m \ge 1} \frac{\mu(m)}{m^2} - \sum_{m^2 > x} \frac{\mu(m)}{m^2} = \frac{1}{\zeta(2)} + O(x^{-1/2}) \ ,$$

where $\zeta(s) = \sum_{n \ge 1} 1/n^s$. This follows from an integral estimate, and the fact that the formal identity $1/\zeta(s) = 1/\prod_p (1 - p^{-s})^{-1} = \prod_p (1 - p^{-s}) =$

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$\sum_{n\geq 1} \mu(n)/n^s$ holds analytically for every real $s > 1$. Using $\nu(n)$, we can express the characteristic function of square-free numbers $\mu(n)^2$:

$$\mu(n)^2 = \sum_{ab=n} \nu(a) \ .$$

Indeed, formally,

$$\sum_{n\geq 1} \frac{\mu(n)^2}{n^s} = \prod_p (1+p^{-s}) = \prod_p (1-p^{-2s}) \prod_p (1-p^{-s})^{-1} = \sum_{n\geq 1} \frac{\nu(n)}{n^s} \sum_{n\geq 1} \frac{1}{n^s} \ .$$

From this representation of $\mu(n)^2$ and the above asymptotic estimate it follows that

$$\sum_{n\leq x} \frac{\mu(n)^2}{n} = \frac{\log x}{\zeta(2)} + O(1) \ . \qquad (3)$$

Indeed,

$$\sum_{n\leq x} \frac{\mu(n)^2}{n} = \sum_{ab\leq x} \frac{\nu(a)}{a} \frac{1}{b} = \sum_{b\leq x} \frac{1}{b} \sum_{a\leq x/b} \frac{\nu(a)}{a} = \sum_{b\leq x} \frac{1}{b} (\zeta(2)^{-1} + O(\sqrt{b/x})) \ ,$$

which equals

$$\frac{1}{\zeta(2)} \sum_{b\leq x} \frac{1}{b} + \frac{1}{\sqrt{x}} \sum_{b\leq x} \frac{O(1)}{\sqrt{b}} = \frac{\log x}{\zeta(2)} + O(1) \ .$$

Back to $\rho(d)$. We claim that

$$\rho(d) = \sum_{ab=d} \mu(a)^2 \chi(b) \ , \qquad (4)$$

where $\chi: \mathbb{Z} \to \{-1,0,1\}$ is given by $\chi(2n) = 0$, $\chi(4n+1) = 1$ and $\chi(4n+3) = -1$; note that χ is completely multiplicative. To see equation (4), note that, trivially, $\rho(1) = \rho(2) = 1$, $\rho(2^k) = 0$ for $k \geq 2$, and by quadratic residues, $\rho(p^k) = 0$ if $p = 4n+3$, and $\rho(p^k) = 2$ if $p = 4n+1$. Also, $\rho(d)$ is multiplicative by Chinese remainder theorem. Now (4) follows from a formal calculation:

$$\sum_{n\geq 1} \frac{\rho(n)}{n^s} = \prod_p \left(1 + \sum_{k\geq 1} \frac{\rho(p^k)}{p^{ks}}\right) = (1+2^{-s}) \prod_{p=4n+1} \left(1 + \sum_{k\geq 1} \frac{2}{p^{ks}}\right) ,$$

which equals

$$\prod_p \frac{1+p^{-s}}{1-\chi(p)p^{-s}} = \prod_p (1+p^{-s}) \prod_p (1-\chi(p)p^{-s})^{-1} = \sum_{n\geq 1} \frac{\mu(n)^2}{n^s} \sum_{n\geq 1} \frac{\chi(n)}{n^s} \ .$$

To handle $\sum_{d\leq x} \rho(d)/d$, we need a variant of (3) with $\mu(n)^2$ replaced by $\chi(n)$. But this is straightforward:

$$\sum_{n\leq x} \frac{\chi(n)}{n} = L(1,\chi) + O(x^{-1}) \ , \qquad (5)$$

where, by the classical sum,

$$L(1,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}\ .$$

Of course, as proved first by Euler,

$$\zeta(2)=\sum_{n\geq 1}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}\ .$$

Now estimate (5) follows by applying to the remainder of the infinite series $L(1,\chi)$ Abel's inequality: if $a_i\in\mathbb{C}, b_i\in\mathbb{R}$, $i=1,2,\ldots,n$, with $b_1\geq b_2\geq\cdots\geq b_n\geq 0$, then

$$\left|\sum_{i=1}^{n}a_ib_i\right|\leq\max_{j=1,2,\ldots,n}\left|\sum_{i=1}^{j}a_i\right|\cdot b_1$$

(write $a_j=\sum_{i=1}^{j}a_i-\sum_{i=1}^{j-1}a_i$, regroup and use triangle inequality). Indeed, setting $a_i=\chi(i)$, $b_i=1/i$ and using that $|\sum_{n\in I}\chi(n)|\leq 1$ for any finite interval I of integers, we get that in absolute value the error term in (5) does not exceed $1/x$.

Now we prove that

$$\sum_{d\leq x}\frac{\rho(d)}{d}=\frac{L(1,\chi)}{\zeta(2)}\log x+O(1)=\frac{3}{2\pi}\log x+O(1)\ . \tag{6}$$

Using (4), (5) and (3), we get that the sum in (6) equals

$$\sum_{ab\leq x}\frac{\mu(a)^2}{a}\frac{\chi(b)}{b}=\sum_{a\leq x}\frac{\mu(a)^2}{a}\sum_{b\leq x/a}\frac{\chi(b)}{b}=\sum_{a\leq x}\frac{\mu(a)^2}{a}(L(1,\chi)+O(a/x))=$$

$$=L(1,\chi)\sum_{a\leq x}\frac{\mu(a)^2}{a}+\frac{1}{x}\sum_{a\leq x}\mu(a)^2O(1)=\frac{L(1,\chi)}{\zeta(2)}\log x+O(1)\ .$$

Finally, by (4) and the bound $|\sum_{n\in I}\chi(n)|\leq 1$,

$$\sum_{d\leq x}\rho(d)=\sum_{a\leq x}\mu(a)^2\sum_{b\leq x/a}\chi(b)=\sum_{a\leq x}\mu(a)^2O(1)=O(x)\ .$$

In view of this last estimate, (6) and (2), the desired asymptotics (1) follows.

Concluding remark. At last this derivation turns out to be easier than I thought and the hints in [2] and [3] seem to suggest: neither hyperbola method nor estimates of coefficients of Dirichlet series are needed.

References

[1] J. Friedlander and H. Iwaniec, *Opera de Cribro*, AMS, 2010.

[2] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS, 2004.

[3] T. Tao, Counting the number of solutions to the Erdös-Straus equation on unit fractions, entry of July 31, 2011, available at `http://terrytao.wordpress.com/`