Schur's asymptotics for $p_A(n)$

Martin Klazar[1]

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For a k-tuple $A = (a_1, \ldots, a_k)$ of (not necessarily distinct) numbers $a_i \in \mathbb{N}$ and $n \in \mathbb{N}$, we denote by $p_A(n)$ the number of partitions of n into the parts a_i, that is, the number of solutions $(m_1, \ldots, m_k) \in \mathbb{N}_0^k$ of the equation

$$m_1 a_1 + m_2 a_2 + \cdots + m_k a_k = n \ .$$

In [4] Issai Schur (1875–1941) found the asymptotics for $p_A(n)$:

Theorem (Schur, 1926). *Suppose that* $\gcd(a_1, \ldots, a_k) = 1$. *Then for* $n = 1, 2, \ldots$ *we have*

$$p_A(n) = \frac{n^{k-1}}{a_1 a_2 \ldots a_k \cdot (k-1)!} + O(n^{k-2}) \ .$$

This appears also as problem 27 in part 1 of the book [3] of Pólya and Szegö, first published in 1925. In the solution they refer to collected papers of Laguerre, published in 1898. Of course, it is nothing that could not have been possibly calculated by L. Euler in the 18th century (but apparently it was not — such asymptotic approach was not in the spirit of the time).

One can prove the theorem by induction on k — see Nathanson [1, Chapter 15.2] and [2]. In the last lecture (January 9, 2014) of the course Introduction to Number Theory I gave the standard algebraic proof using decomposition of the generating function into partial fractions, and for the benefit of the students and myself I write it up here.

We start from scratch and derive decomposition into partial fractions itself. The ring $\mathbb{C}[x]$ is Euclidean (i.e., has the division algorithm), thus Bachet's identity holds in it: if $p, q \in \mathbb{C}[x]$ are coprime polynomials then for some $r, s \in \mathbb{C}[x]$ we have

$$rp + sq = 1, \quad \text{or} \quad \frac{1}{pq} = \frac{s}{p} + \frac{r}{q} \ .$$

Iterating this we get that if the polynomials $p_i \in \mathbb{C}[x]$ are pairwise coprime then for some $q_i \in \mathbb{C}[x]$ we have

$$\frac{1}{p_1 p_2 \ldots p_k} = \frac{q_1}{p_1} + \frac{q_2}{p_2} + \cdots + \frac{q_k}{p_k} \ .$$

Suppose that, moreover, each p_i is a power of a linear polynomial, $p_i = r_i^{m_i}$ with $\deg r_i = 1$ and $m_i \in \mathbb{N}$. Then we express each q_i as a $\mathbb{C}$-linear combination

[1] klazar@kam.mff.cuni.cz

of the powers $1, r_i, r_i^2, \ldots$ and get the partial fractions decomposition

$$\frac{1}{p_1p_2\ldots p_k} = q + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\beta_{i,j}}{r_i^j},\ \beta_{i,j}\in\mathbb{C}\ \text{ and }\ q\in\mathbb{C}[x]\ .$$

But $x\to+\infty$ shows that $q=0$.

For $\alpha\in\mathbb{C}$ and $m\in\mathbb{N}$ recall the expansion

$$\frac{1}{(1-\alpha x)^m} = \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}\alpha^n x^n = \sum_{n=0}^{\infty}(n^{m-1}/(m-1)! + O(n^{m-2}))\alpha^n x^n$$

where the implicit constant in O depends only on m. It is easy to see that the generating function of the numbers $p_A(n)$ is given by

$$\sum_{n=0}^{\infty}p_A(n)x^n = \frac{1}{(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_k})}$$

(a formal power series equality; no need for $|x|<1$ and the like).

For $a\in\mathbb{N}$ a variant of the usual factorization in $\mathbb{C}[x]$ is

$$1-x^a = \prod_{j=0}^{a-1}(1-\exp(2\pi i\cdot j/a)x)\ .$$

Thus, for the given k-tuple A,

$$\begin{aligned}\prod_{l=1}^{k}(1-x^{a_l}) &= \prod_{l=1}^{k}\prod_{j=0}^{a_l-1}(1-\exp(2\pi i\cdot j/a_l)x)\\ &= \prod_{d,\ d\,|\,a_l}\ \prod_{0\le e<d,\ (e,d)=1}(1-\exp(2\pi i\cdot e/d)x)^{m_d}\\ &=: \prod_{d,\ d\,|\,a_l}\ \prod_{0\le e<d,\ (e,d)=1} p_{d,e}^{m_d},\ \deg p_{d,e}=1\ ,\end{aligned}$$

where $m_d\in\mathbb{N}$ is the number of the a_ls divisible by d; so d runs over the numbers that divide some a_l and e is coprime with d. We obtained this expression simply by reducing the fractions j/a_l to the least terms e/d. The linear polynomials $p_{d,e}$ are pairwise coprime (the roots of unity $\exp(2\pi i\cdot e/d)$ are pairwise distinct) and the above partial fractions decomposition gives

$$\frac{1}{(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_k})} = \sum_{d,e}\sum_{j=1}^{m_d}\frac{\beta_{d,e,j}}{p_{d,e}^j},\ \beta_{d,e,j}\in\mathbb{C}\ ,$$

with d,e and m_d as before. But $p_{d,e}^j = (1-\alpha x)^j$ where $\alpha\in\mathbb{C}$ is a d-th root of 1 and $1\le j\le m_d$, $m_1=k$ and, by the assumption in the theorem, $m_d<k$ for

$d > 1$. Hence the above expansion shows that the coefficient $p_A(n)$ of x^n in the power series expansion of the left side has form

$$p_A(n) = \beta_{1,0,k} n^{k-1}/(k-1)! + O(n^{k-2}) .$$

Since $1 - x^a = (1-x)(1 + x + x^2 + \cdots + x^{a-1})$, $x \to 1$ shows that $\beta_{1,0,k} = 1/(a_1 a_2 \ldots a_k)$ and we are done.

References

[1] M. B. Nathanson, *Elementary Methods in Number Theory*, Springer, 2000.

[2] M. B. Nathanson, Partitions with parts in a finite set, *Proc. Amer. Math. Soc.* **128** (2000), 1269–1273.

[3] G. Pólya and G. Szegö, *Problems and Theorems in Analysis. I. Series, Integral Calculus, Theory of Functions*, Springer, 1978. [First German edition by Springer in 1925.]

[4] I. Schur, Zur Additiven Zahlentheorie, *S.-B. Preuss. Akad. Wiss. Phys. Math. Klasse* (1926), pp. 488–495.