Jacobi's four squares identity

Martin Klazar

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C. Jacobi [2] in 1829 proved that for any integer $n \ge 1$,

$$r_4(n) = \#\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid \textstyle\sum_{i=1}^4 x_i^2 = n\} = 8\Big(\displaystyle\sum_{d\,|\,n} d - \sum_{4d\,|\,n} 4d\Big)\ .$$

In other words, *the number of ways to express n as a sum of four squares of integers equals eight times the sum of the divisors of n not divisible by four.* Thus $r_4(n) = 8(1 + \ldots) \ge 8$ for every $n \ge 1$, which gives as a corollary Lagrange's theorem from 1770 that every natural number is a sum of four squares.

We give a complete and purely formal proof of Jacobi's identity by generating functions; the proof is due to Hirschhorn [1] in 1987. Idea: since $r_g(n)$, the number of expressions of an integer $n \ge 0$ as a sum of g squares of integers, equals to the coefficient of x^n in (the formal power series expansion of)

$$\Big(\sum_{n=-\infty}^{+\infty} x^{n^2}\Big)^g\ ,$$

and $\sum_{d\,|\,n} d$ equals to the coefficient of x^n in the Lambert series

$$\sum_{k,n=1}^{\infty} nx^{kn} = \sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n}\ ,$$

it suffices to derive the identity

$$\Big(\sum x^{n^2}\Big)^4 = 1 + 8\Big(\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n} - \sum_{n=1}^{\infty} \frac{4nx^{4n}}{1 - x^{4n}}\Big) \qquad (1)$$

— Jacobi's identity in the GF language. (We write briefly $\sum$ for $\sum_{n=-\infty}^{+\infty}$ and $\prod$ for $\prod_{n=1}^{\infty}$.) We achieve it by a succession of the next seven identities.

1.
$$\sum(-1)^n x^{n^2} = \prod \frac{1-x^n}{1+x^n}\ .$$

2. JTI:
$$\prod(1-x^{2n})(1+zx^{2n-1})(1+z^{-1}x^{2n-1}) = \sum z^n x^{n^2}\ .$$

3.
$$(z-z^{-1})\prod(1-x^n)(1-z^2x^n)(1-z^{-2}x^n) = \sum(-1)^n z^{2n+1}x^{n(n+1)/2}\ .$$

4.
$$\prod(1-x^n)^3 = \tfrac{1}{2}\sum(2n+1)(-1)^n x^{n(n+1)/2}\ .$$

5.
$$\prod(1-x^n)^6 = \tfrac{1}{2}\sum_{r,s}((2r+1)^2-(2s)^2)x^{r^2+r+s^2}\ .$$

6.
$$\prod(1-x^n)^6 = \prod Q_n \cdot \Big[1-8\sum_{n=1}^{\infty}\Big(\frac{(2n-1)x^{2n-1}}{1+x^{2n-1}} - \frac{2nx^{2n}}{1+x^{2n}}\Big)\Big]\ ,$$

where $Q_n = (1-x^{2n})^2(1+x^{2n})^2(1+x^{2n-1})^2$.

7.
$$\prod(1+x^n)^4(1-x^n)^2 = \prod Q_n\ .$$

The key identity, from which we derive everything, is 2, the celebrated *Jacobi's triple product identity* (JTI for short), obtained by C. Jacobi in [2]. We first deduce (1) from the identities, then derive the seven identities assuming the JTI, and at the end we give a proof for the JTI.

We have that

$$\Big(\sum(-1)^n x^{n^2}\Big)^4 \overset{\text{(id.1)}}{=} \prod\Big(\frac{1-x^n}{1+x^n}\Big)^4 = \frac{\text{(the left side of id. 6)}}{\text{(the left side of id. 7)}}$$

equals to the ratio of the right sides

$$1-8\sum_{n=1}^{\infty}\Big(\frac{(2n-1)x^{2n-1}}{1+x^{2n-1}} - \frac{2nx^{2n}}{1+x^{2n}}\Big)\ .$$

Replacing x with $-x$, we obtain the identity

$$\Big(\sum x^{n^2}\Big)^4 = 1 + 8\sum_{n=1}^{\infty}\Big(\frac{(2n-1)x^{2n-1}}{1-x^{2n-1}} + \frac{2nx^{2n}}{1+x^{2n}}\Big)\,,$$

which is similar to (1). We rewrite the last sum as

$$\sum_{n=1}^{\infty}\Big(\frac{(2n-1)x^{2n-1}}{1-x^{2n-1}} + \frac{2nx^{2n}}{1-x^{2n}}\Big) - \sum_{n=1}^{\infty}\Big(\frac{2nx^{2n}}{1-x^{2n}} - \frac{2nx^{2n}}{1+x^{2n}}\Big)\,.$$

The first sum equals $\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}$ by regrouping, and the summand in the second sum equals, by subtracting the two fractions, to $\frac{4nx^{4n}}{1-x^{4n}}$. Thus the obtained identity is not just similar to but indeed identical to (1).

We start from identity 2, the JTI, and derive from it the rest.

$2 \rightsquigarrow 1$. We set $z = -1$ in the JTI and get that $\sum(-1)^n x^{n^2}$ equals to

$$\prod(1-x^{2n})(1-x^{2n-1})^2\,.$$

Since $\prod(1-x^{2n})(1-x^{2n-1}) = \prod(1-x^n)$ and $(1-x^n) = (1-x^{2n})/(1+x^n)$, regrouping yields the right side of 1.

$2 \rightsquigarrow 3$. We set $z = -xz^2$ in the JTI, then $x = x^{1/2}$ and multiply the result by z.

$3 \rightsquigarrow 4$. We differentiate 3 by z, set $z = 1$ and multiply the result by $\frac{1}{2}$.

$4 \rightsquigarrow 5$. Squaring 4 gives

$$\prod(1-x^n)^6 = \tfrac{1}{4}\sum_{m,n}(2m+1)(2n+1)(-1)^{m+n}x^{(m^2+m+n^2+n)/2}\,.$$

We split the sum in two subsums, the first with even $m+n$ and the second with odd $m+n$. In the first subsum we change the variables to $r = (m+n)/2, s = (m-n)/2$, and in the second to $r = (m-n-1)/2, s = (m+n+1)/2$. It is easy to check that in both cases m^2+m+n^2+n turns into $2(r^2+r+s^2)$ and $(-1)^{m+n}(2m+1)(2n+1)$ into $(2r+1)^2-(2s)^2$. Hence the two subsums coincide and the sign disappears; we get the right side of 5.

$5, 2 \rightsquigarrow 6$. We write the right side of 5 as a difference of two sums, by the difference $(2r+1)^2-(2s)^2$, and separate in each sum the variables r and s. Then we express the coefficients by differentiation and obtain

$$\tfrac{1}{2}\Big(\sum_s x^{s^2}\cdot(1+4x\tfrac{d}{dx})\sum_r x^{r^2+r} - \sum_r x^{r^2+r}\cdot 4x\tfrac{d}{dx}\sum_s x^{s^2}\Big)\,.$$

We replace each of the four $\sum$s by a $\prod$ using the JTI, setting $z = 1$ in $\sum_s$ and $z = x$ in $\sum_r$ ($\frac{1}{2}$ gets cancelled by $1 + x^0$):

$$\prod(1 - x^{2n})(1 + x^{2n-1})^2 \cdot (1 + 4x\tfrac{d}{dx})\prod(1 - x^{2n})(1 + x^{2n})^2$$

minus

$$\prod(1 - x^{2n})(1 + x^{2n})^2 \cdot 4x\tfrac{d}{dx}\prod(1 - x^{2n})(1 + x^{2n-1})^2 \ .$$

We differentiate the infinite products by means of the identity $(\prod f_n)' = \prod f_n \cdot \sum_{n=1}^{\infty} f_n'/f_n$ and, denoting $a = 1 + x^{2n}$, $b = 1 + x^{2n-1}$, $c = 1 - x^{2n}$ and taking out the factor $Q_n = (abc)^2$, get the expression

$$\prod Q_n \cdot \Big[1 + 4x\sum_{n=1}^{\infty}\Big(\frac{(a^2c)'}{a^2c} - \frac{(b^2c)'}{b^2c}\Big)\Big] \ .$$

The last summand simplifies after an easy calculation to $2(a'/a - b'/b)$. We get the right side of 6.

$\emptyset \rightsquigarrow 7$. We have

$$\prod Q_n = \prod(1 - x^{2n})^2(1 + x^{2n})^2(1 + x^{2n-1})^2 \ .$$

Since $1 - x^{2n} = (1 - x^n)(1 + x^n)$ and $\prod(1 + x^{2n})(1 + x^{2n-1}) = \prod(1 + x^n)$, regrouping gives the left side of 7.

It remains to prove the JTI

$$\prod(1 - x^{2n})(1 + zx^{2n-1})(1 + z^{-1}x^{2n-1}) = \sum z^n x^{n^2} \ .$$

We give a purely formal proof. Let $S(z, x)$ be the product on the left side. It follows that

$$zx \cdot S(zx^2, x) = S(z, x) \ ,$$

because the substitution $z \mapsto zx^2$ almost preserves the two arithmetic progressions of odd positive integers in the two exponents of x: it just adds to $S(z, x)$ the factor $(1 + z^{-1}x^{-1})/(1 + zx) = 1/zx$. Therefore if we expand $S(z, x)$ in the integral powers of z as

$$S(z, x) = \sum a_n z^n = \sum a_n(x) z^n$$

(n runs through the whole $\mathbb{Z}$) with the power series coefficients $a_n \in \mathbb{C}[[x]]$, comparison of the coefficients of z^n on both sides of the functional equation gives the relation

$$x^{2n-1}a_{n-1} = a_n, \ n \in \mathbb{Z} \ .$$

Thus $a_1 = xa_0 = a_{-1}$ $(n = 1, 0)$, $a_2 = x^3 a_1 = x^3 a_{-1} = a_{-2}$ $(n = 2, -1)$ and $a_2 = x^4 a_0 = a_{-2}$. In general,

$$a_n = a_{-n} = x^{n^2} a_0, n = 1, 2, \ldots\ ,$$

since $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. (The equality $a_n = a_{-n}$ is immediate also from the symmetry $S(z, x) = S(z^{-1}, x)$.) So we have deduced that

$$S(z, x) = \prod (1 - x^{2n})(1 + zx^{2n-1})(1 + z^{-1}x^{2n-1}) = a_0(x) \sum z^n x^{n^2}$$

and are almost done — it 'only' remains to be shown that $a_0 = 1$. This we prove by three specializations of the last identity (the JTI with the undetermined coefficient $a_0(x)$): (i) $z = x = 0$, (ii) $z = i$ and (iii) $z = -1, x = x^4$. Then (i) gives that $a_0(0) = 1$, (ii) gives that

$$\prod (1 - x^{2n})(1 + x^{4n-2}) = a_0(x) \sum (-1)^n x^{(2n)^2}$$

and (iii) gives that

$$\prod (1 - x^{8n})(1 - x^{8n-4})^2 = a_0(x^4) \sum (-1)^n x^{4n^2}\ .$$

Clearly, the sums on the right sides are equal. So are the products on the left sides, despite appearance: since $\prod(1 - x^{8n})(1 - x^{8n-4}) = \prod(1 - x^{4n})$, $1 - x^{8n-4} = (1 - x^{4n-2})(1 + x^{4n-2})$ and $\prod(1 - x^{4n})(1 - x^{4n-2}) = \prod(1 - x^{2n})$, the product of (iii) equals to that of (ii). Hence the remaining factors have to be equal too, $a_0(x) = a_0(x^4)$, which forces that $a_0(x) = a_0(0) = 1$, as we needed to show. This completes the proof of the JTI and of Jacobi's four squares identity. $\square$

Exercise. Justify rigorously formal manipulations in the above proof. In particular, clarify in what sense does the formal equality

$$\prod (1 - x^{2n})(1 + zx^{2n-1})(1 + z^{-1}x^{2n-1}) = \sum a_n(x) z^n$$

hold and why it is preserved by the substitution $z \mapsto zx^2$.

References

[1] M. D. Hirschhorn, A simple proof of Jacobi's four-square theorem, Proc. Amer. Math. Soc. 101 (1987), 436–438.

[2] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum,* Sumtibus fratrum, 1829.