Matematické struktury
tutorial 6 on March 27, 2017: characterization of modular lattices and characterization of distributive lattices

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Characterization of modular lattices

We suppose that the lattice L is not modular and find in it a copy of C_5. Thus there are $u, v, w \in L$ such that

$$u \le w \;\text{ and }\; u \vee (v \wedge w) < (u \vee v) \wedge w\ .$$

We set

$$a = v,\ b = v \wedge w,\ c = v \vee u,\ x = u \vee (v \wedge w) \;\text{ and }\; y = (u \vee v) \wedge w\ .$$

We show that these five elements are distinct and induce a copy of C_5. Note that we changed the definition of c compared to the last lecture — there is a misprint at the end of the proof of Theorem 2.2 in the lecture notes

- A. Pultr, *Matematické struktury*, 2005, 155 pp., available at `http://kam.mff.cuni.cz/~pultr/`

I emphasize that the results on applications of fixed points theorems and on modular and distributive lattices I take from these lecture notes.

Clearly, $x < y$. We show that a is incomparable to both x and y. Suppose not, then $a \le y$ or $a \ge x$. In the former case we have $v \le w$, thus $u \vee v \le w$ (recall that $u \le w$) and

$$x = u \vee (v \wedge w) = u \vee v = (u \vee v) \wedge w = y\ ,$$

a contradiction. In the latter case similarly $v \ge u$, thus $v \wedge w \ge u$ and

$$x = u \vee (v \wedge w) = v \wedge w = (u \vee v) \wedge w = y\ ,$$

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a contradiction.

It remains to show that $a \vee y = c$ and $a \wedge x = b$, that is

$$v \vee [(u \vee v) \wedge w] = v \vee u \quad \text{and} \quad v \wedge [u \vee (v \wedge w)] = v \wedge w \ .$$

Consider the first equation. Since $(u \vee v) \wedge w \geq u$ (as $w \geq u$), the left side is at least the right side. On the other hand, the right side majorizes the left side because it majorizes each of the two arguments of $\vee$ on the left. Thus the two sides are equal. Very similar (dual) argument proves the second equation. By the incomparability of a to both x and y we get that $c \neq a, y$ and $b \neq a, x$ (and of course $b \neq c$). Thus a, b, c, x, y are all distinct and form a copy of C_5 in L. □

Characterization of distributive lattices

A lattice L is *distributive* if for every $a, b, c \in L$ we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

— $\wedge$ distributes over $\vee$ in the same way as in arithmetics $\cdot$ distributes over $+$. Unlike in arithmetics, in lattices this remains true when $\wedge$ and $\vee$ are swapped.

Proposition 1 *A lattice L is distributive if and only if for every $a, b, c \in L$ we have*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \ .$$

Proof. Suppose L is distributive and $a, b, c \in L$. Then

$$\begin{aligned} \underline{(a \vee c)} \wedge (a \vee b) &=_D \underbrace{((a \vee c) \wedge a)}_{=a} \vee ((a \vee c) \wedge \underline{b}) \\ &=_D \underbrace{a \vee (b \wedge a)}_{=a} \vee (b \wedge c) \\ &= a \vee (b \wedge c) \ . \end{aligned}$$

We used distributivity by "multiplying" with the underlined elements. We also used associativity of $\vee$ and commutativity of $\wedge$.

Exercise. *Prove the opposite implication.* □

Since $a \leq c$ implies $a \vee c = c$, the last proposition shows that each distributive lattice is modular. For the characterization of distributive lattices by forbidden configurations we need the next identity.

Proposition 2 *A modular lattice L is distributive if and only if for every $a, b, c \in L$ we have the* symmetric identity

$$(a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c) \ .$$

Proof. It is easy to see that we always have $\le$ (each of the three arguments of $\vee$ on the left is $\le$ than each of the three arguments of $\wedge$ on the right). Suppose that L is distributive. Then, starting with the right side of the symmetric identity,

$$\begin{aligned}(a\vee b)\wedge(a\vee c)\wedge(b\vee c) &=_d [\overline{a}\vee(b\wedge(a\vee c))]\wedge\underline{(b\vee c)}\\ &=_D [\underline{a}\wedge(b\vee c)]\vee[\underline{b}\wedge(a\vee c)]\\ &=_D (a\wedge b)\vee(a\wedge c)\vee(b\wedge a)\vee(b\wedge c)\ .\end{aligned}$$

In the first line we used the dual distributivity of Proposition 1 and "multiplied" by the overlined element. Usual distributivity and "multiplication" by the underlined term gets us on the second line and similarly we get on the third line. We may omit $b\wedge a$ and have the left side of the symmetric identity.

Assume that L is modular and the symmetric identity holds. Then

$$\begin{aligned}(a\vee b)\wedge\underline{c} &= [(a\vee b)\wedge\underline{(a\vee c)\wedge(b\vee c)]\wedge c}\\ &=_{SI} [(a\wedge b)\vee\underline{(a\wedge c)\vee(b\wedge c)]}\wedge c\\ &=_M \underline{(a\wedge c)\vee(b\wedge c)}\vee\underline{(a\wedge b\wedge c)}=(a\wedge c)\vee(b\wedge c)\ .\end{aligned}$$

In the first line the underlined terms are equal and we get on the second line by replacing the term in the square brackets by the symmetric identity. Here the underlined term is at most c and modularity gives the third line. Here the first underlined term majorizes the second one and we get the result. Thus L is distributive. ☐

Finally we state and prove a characterization of distributivity by forbidden configurations.

Theorem 3 *A lattice L is distributive if and only if it does not contain sublattices C_5 and D_3 where in*

$$D_3=(\{a,b,c,x,y\},\le)$$

(five distinct elements) we have $b<a,x,y<c$ and a,x,y are pairwise incomparable.

Proof. If L contains C_5 then it is not modular and hence not distributive. If L contains D_3 then

$$a\wedge(x\vee y)=a\wedge c=a\ne b=b\vee b=(a\wedge x)\vee(a\wedge y)$$

and L is not distributive.

In the other way, suppose that L is not distributive. If it is not modular then, by the previous result, $L\supseteq C_5$. Thus we assume that L is modular but not distributive. By Proposition 2, there exist $a,b,c\in L$ such that

$$d=(a\wedge b)\vee(a\wedge c)\vee(b\wedge c)<h=(a\vee b)\wedge(a\vee c)\wedge(b\vee c)\ .$$

We define

$$\begin{aligned} u &= (a \vee (b \wedge c)) \wedge (b \vee c) \\ v &= (b \vee (a \wedge c)) \wedge (a \vee c) \\ w &= (c \vee (a \wedge b)) \wedge (a \vee b) \end{aligned}$$

and show that

$$u \wedge v = u \wedge w = v \wedge w = d \quad \text{and} \quad u \vee v = u \vee w = v \vee w = h\,.$$

This by itself implies that u, v, w are pairwise incomparable and $d < u, v, w < h$ form a copy of D_3 in L. If two of u, v, w were comparable, then one would be d and the other h and also the remaining third element would be d or h but then exactly one of the six equalities would not hold.

It suffices to prove just one equality, say $u \wedge v = d$, the other five follow from it by symmetry. Permuting a, b, c does not change d (and h) but permutes u, v, w. Thus we get $u \wedge w = v \wedge w = d$. Swapping $\wedge$ and $\vee$ interchanges d and h but leaves each of u, v, w invariant, because for example

$$u = (\underline{(b \wedge c)} \vee a) \wedge \underline{(b \vee c)} =_M (b \wedge c) \vee (a \wedge (b \vee c))$$

by modularity as the first underlined term is at most the second one. Similarly for v and w. This swapping therefore changes the first three equations to $u \vee v = u \vee w = v \vee w = h$.

Let us prove that $u \wedge v = d$:

$$\begin{aligned} u \wedge v &= \underline{(a \vee (b \wedge c))} \wedge \overline{(b \vee c)} \wedge \overline{(b \vee (a \wedge c))} \wedge \underline{(a \vee c)} \\ &= (a \vee \underline{(b \wedge c)}) \wedge \underline{(b \vee (a \wedge c))} \\ &=_M (\underline{a} \wedge (b \vee \underline{(a \wedge c)})) \vee (b \wedge c) \\ &=_M (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = d\,. \end{aligned}$$

On the first line, the first underlined term is $\leq$ than the second one, and the second overlined term is $\leq$ than the first one, so the fourfold meet is determined by only these two terms. On the second line the first underlined term is $\leq$ than the second one and modularity gives the third line. Here the second underlined term is $\leq$ than the first one and modularity gives the result. □