Matematické struktury tutorial 2 on February 27, 2017: hat problems, paradoxical decompositions and circle-squaring

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It is worth to note that in Theorem 2 in the last lecture the set $X \subset \mathbb{R}$ of exceptions where oracle errs is not only at most countable but also (unlike, say, $\mathbb{Q}$) *nowhere dense* — for every nonempty open interval $(a, b) \subset \mathbb{R}$ there is a nonempty open interval $(c, d) \subset (a, b)$ such that $(c, d) \cap X = \emptyset$. This was in fact proven by the proof because every well-ordered subset Y of $\mathbb{R}$ is nowhere dense: if $(a, b) \cap Y = \emptyset$ we take $(c, d) = (a, b)$, and else if $y \in (a, b) \cap Y$ we take $(c, d) = (y, z)$ where z is the successor of y in Y if it exists and lies in (a, b) and $z = b$ else.

Theorem 2 is due to Hardin and Taylor in

- Ch. S. Hardin and A. D. Taylor, A peculiar connection between the axiom of choice and predicting the future, *Amer. Math. Monthly* **115** (2008), 91–96.

They wrote on the topic a book

- Ch. S. Hardin and A. D. Taylor, *The mathematics of coordinated inference. A study of generalized hat problems*, Developments in Mathematics, 33. Springer, Cham, 2013

(a preliminary version is available on-line). What is a "hat problem"? We borrow one from the book introduction:

> Two prisoners are brought to the prison director, are seated and each is put red or green hat on his head. No prisoner can see his hat but can see the hat of his prison-mate. They have to guess simultaneously what is the color of the hat they have on head. They will be released if and only if at least one their two guesses is correct. During the session they of course cannot communicate but they can meet before and agree on a common strategy. Is there a strategy ensuring that the prisoners will be always released?

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Exercise. *Solve the previous hat problem.*
Solution: neo esassum atth het wot rscolo rea alequ nda het eroth atth eyth erdiff (to decode, shift each word cyclicly two letters back).

Could you prove Proposition 3 from the last lecture that one can break the unit circle $C \subset \mathbb{R}^2$ into countably many pieces in such a way that they can be reassembled into two disjoint copies $C \cup D$ of C?

Proof. We prove Proposition 3. Using the notation from the proof of Theorem 1 we enumerate $\mathcal{R}$ as $\mathcal{R} = \{\alpha_1, \alpha_2, \dots\}$, for $n \in \mathbb{N}$ set $X_n = \varphi_{\alpha_n}(X)$ and denote by ψ the translation by vector $(3, 0)$ that moves C to D and by $\varphi_{m,n}$ the rotation of C around the origin that moves X_m to X_n. Then we have partitions

$$C = \bigcup_{n=1}^{\infty} \varphi_{2n,n}(X_{2n}) \quad \text{and} \quad D = \bigcup_{n=1}^{\infty} \psi(\varphi_{2n-1,n}(X_{2n-1})) \ .$$

Hence the decomposition works with the pieces $A_n = X_n$ and the rigid motions (isometries) $\psi_n = \varphi_{n,n/2}$ for even n and $\psi_n = \psi \circ \varphi_{n,(n+1)/2}$ for odd n. $\square$

The idea behind the decomposition is simple: if $\mathbb{N} = E \cup O$ is a partition into even numbers E and odd numbers O, then $E/2 = (O + 1)/2 = \mathbb{N}$.

Could one have Proposition 3 with only finitely many pieces? S. Banach proved that in one or two dimensions this is impossible (finitely equidecomposable and measurable subsets of $\mathbb{R}$ and $\mathbb{R}^2$ must have equal measures). But in three dimensions it turns out to be possible! We have the famous *Banach–Tarski paradox.*

Theorem 1 (Banach and Tarski, 1924) *Let*

$$\begin{aligned} B_1 &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\} \quad \textit{and} \\ B_2 &= \{(x, y, z) \in \mathbb{R}^3 \mid (x - 3)^2 + y^2 + z^2 \leq 1\} \end{aligned}$$

be two (disjoint) copies of the closed unit ball in three dimensions. Then there exist five sets $X_1, \dots, X_5 \subset \mathbb{R}^3$ and five isometries (translations combined with rotations around some axes) $\psi_1, \dots, \psi_5 : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$B_1 = \bigcup_{i=1}^{5} X_i \quad \textit{and} \quad B_1 \cup B_2 = \bigcup_{i=1}^{5} \psi_i(X_i)$$

are partitions (the sets in the two unions are pairwise disjoint).

Their definition of the sets X_i uses the axiom of choice and non-measurable sets appear. What is the difference between $\mathbb{R}^2$ and $\mathbb{R}^3$ that allows such paradoxical decomposition in the latter space but not in the former? The group of isometries is commutative for $\mathbb{R}^2$ but not for $\mathbb{R}^3$. In the plane, in any combination of translations and rotations around some points order does not matter: the result is the same if we perform them in any order. Not so in three dimensions! Lay a book in front of you on the table and rotate it by $\pi/4$ (a quarter turn counterclockwise) first around the vertical axis and then around the horizontal axis going away from you, and then do the same (with the book in the identical starting position, of course) in the opposite order. The two actions bring the book to two different positions. And exactly this non-commutativity of motions in $\mathbb{R}^3$ lies behind the possibility of the B.–T. paradox. For details how to prove Theorem 1 (and many other paradoxes and jokes) google the very nice and thorough article

- L. Pick, Hrášek a sluníčko, *PMFA* **55 č. 3** (2010), 190–214.

Exercise. *Prove the following proposition.*

Proposition 2 (Banach–Tarski for babies) *Let $A, B \subset \mathbb{R}^d$ be two bounded sets with nonempty interiors (i.e. each contains a ball with positive radius) in the Euclidean space of any dimension. Then there exist finitely many sets $X_1, X_2, \ldots, X_n \subset \mathbb{R}^d$ and translations $\psi_1, \psi_2, \ldots, \psi_n : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$A = \bigcup_{i=1}^{n} X_i \quad \text{and} \quad B = \bigcup_{i=1}^{n} \psi_i(X_i)$$

— the sets in the two unions now may intersect.

Exercise. *Prove that this is not possible if A is bounded but B is unbounded. Investigate other examples of impossibility of such equicovering of A and B when the assumptions of boundedness and nonemptyness of interiors are dropped.*

It is well known that *circle-squaring* is impossible: no construction using compass and ruler exists that could transform a square in $\mathbb{R}^2$ into a circle with the same area. In 1925, A. Tarski asked the following question.

Problem 3 (Tarski's circle-squaring problem) *Suppose that S and D are a square and a disc (circle) in the plane with equal areas. Do there exist finitely many sets $X_1, X_2, \ldots, X_n \subset \mathbb{R}^2$ and isometries $\psi_1, \psi_2, \ldots, \psi_n : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$S = \bigcup_{i=1}^{n} X_i \quad \text{and} \quad D = \bigcup_{i=1}^{n} \psi_i(X_i)$$

are partitions?

In other words, is there a puzzle with finitely many plane pieces X_i that can be in one way assembled in a square S and in another way in a circle D? Since S and D have equal area, the earlier mentioned result of Banach does not preclude possibility of such "circle-squaring".

In 1963, Dubins, Hirsch and Karush proved in

- L. Dubins, M. W. Hirsch and J. Karush, Scissor congruence, *Israel J. Math.* **1** (1963), 239–247 that Tarski's circle-squaring is impossible with pieces X_i that are topological discs. We say that $A \subset \mathbb{R}^2$ is a topological disc if $A = F(D)$ where $F:\ \mathbb{R}^2 \to \mathbb{R}^2$ is a bijection such that F and F^{-1} are continuous maps and

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

is the closed unit disc in the plane. The result of Dubins, Hirsch and Karush is actually a little stronger than just negating Tarski's circle-squaring: there do not exist two partitions as above even if we require only the interiors of X_i and $\psi_i(X_i)$ be disjoint and their boundaries may intersect. We allow it also in the next two exercises — technical term for this kind of equidecomposability is "scissor congruence".

Exercise. *Show that Tarski's circle-squaring is impossible with pieces X_i that are (possibly non-convex) polygons.*

Too trivial? Then try the following harder problem.

Exercise. *Show that Tarski's circle-squaring is impossible with pieces X_i that are topological discs whose boundaries are closed piecewise smooth curves (this means that continuously varying tangent line exists at every point of each boundary, possibly with the exception of finitely many points).*
For solution see the beginning of the article of Dubins, Hirsch and Karush.

It was a big surprise when in 1990 Hungarian mathematician M. Laczkovich showed in

- M. Laczkovich, Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem, *J. Reine Angew. Math.* **404** (1990), 77–117

that Tarski's circle-squaring in fact *is* possible. The number of pieces, which are defined by AC, is astronomical but all $\psi_1, \dots, \psi_n$ are just translations. Laczkovich's "puzzle" that finitely equidecomposes circle and square of the same area by translations was recently much improved in

- Ł. Grabowski, A. Máthé and O. Pikhurko, Measurable circle squaring, ArXiv:1501.06122, 40 pages, 2015 (4th version in 2016) and
- A. S. Marks and S. T. Unger, Borel Circle Squaring, ArXiv:1612.05833, 20 pages, 2016.

The former preprint shows that Tarski's circle-squaring can be done with measurable pieces and the latter that even Borel pieces suffice (again, all ψ_i are translations). *Borel sets* in $\mathbb{R}^d$ is the smallest family of subsets that contains all open balls and is closed under countable unions and complements to $\mathbb{R}^d$.