The Füredi–Hajnal Conjecture Implies the Stanley–Wilf Conjecture*

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Abstract. We show that the Stanley–Wilf enumerative conjecture on permutations follows easily from the Füredi–Hajnal extremal conjecture on 0-1 matrices. We apply the method, discovered by Alon and Friedgut, that derives an (almost) exponential bound on the number of some objects from a (almost) linear bound on their sizes. They proved by it a weaker form of the Stanley–Wilf conjecture. Using bipartite graphs, we give a simpler proof of their result.

Покажем, что гипотеза Стэнли и Вилфа о числе перестановок вытекает простым образом из экстремальной гипотезы Фиреды и Хайнала о 0-1 матрицах. Применяем метод вывода (почти) экспоненциальной оценки числа объектов из (почти) линейной оценки их величин открытый Алоном и Фридгутом. Этим методом они доказали гипотезу Стэнли и Вилфа в ослабленной форме. С помощью двудольных графов получим более простое доказательство их результата.

The Stanley–Wilf conjecture asserts that the number of n-permutations not containing a given permutation is exponential in n. Alon and Friedgut [1] proved that it is true provided we have a linear upper bound on lengths of certain words over an ordered alphabet. They also proved a weaker version of it with an almost exponential upper bound. In the present note we want to inform the reader about this interesting development by reproving the latter result in a simpler way. We use bipartite graphs instead of words. We point out that in 1992 Füredi and Hajnal almost made an extremal conjecture on 0-1 matrices that now can easily be seen to imply the Stanley–Wilf conjecture. We prove that both extremal conjectures are logically equivalent.

We use $\mathbf{N}$ to denote the set $\{1, 2, \ldots\}$ and $[n]$ to denote the set $\{1, 2, \ldots, n\}$. The sets of all finite sequences (words) over $\mathbf{N}$ and $[n]$ are denoted $\mathbf{N}^*$ and $[n]^*$. If $u \in \mathbf{N}^*$, $|u|$ is the length of u. A sequence $v = b_1 b_2 \ldots b_l \in \mathbf{N}^*$ is *k-sparse* if $b_j = b_i, j > i$, implies $j - i \geq k$. In other words, in each interval in v of length

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$\leq k$ all terms are distinct. Permutations are represented as elements of $\mathbf{N}^*$ in the standard way, e.g., $p = 32154$. If $|p| = k$, we speak of a *k-permutation.*

Let $u = a_1 a_2 \dots a_k$ and $v = b_1 b_2 \dots b_l$ be two sequences from $\mathbf{N}^*$. If $k = l$ and, for every i and j, $a_i < a_j$ if and only if $b_i < b_j$, we say that u and v are *isomorphic*. Thus, in particular, $a_i = a_j$ if and only if $b_i = b_j$. The isomorphism of u and v means that, using only comparisons by $<$, we cannot tell apart u and v. If $u, v \in \mathbf{N}^*$ are two sequences and v has a (not necessarily consecutive) subsequence isomorphic to u, we say that v *contains* u and write $v \supset_< u$. Then, clearly, $|v| \geq |u|$.

We define analogous notions for the equality relation. Two sequences $u = a_1 a_2 \dots a_k$ and $v = b_1 b_2 \dots b_l$ are *weakly isomorphic* if $k = l$ and, for every i and j, $a_i = a_j$ if and only if $b_i = b_j$. A sequence v *weakly contains* another sequence u, denoted $v \supset_= u$, if v has a subsequence weakly isomorphic to u. Then, again, $|v| \geq |u|$.

In summary, $v \supset_< u$ means that a subsequence of v induces the same "bigger-smaller" pattern as u, while $v \supset_= u$ requires only a subsequence that shares with u just the equality pattern. Thus $v \supset_< u$ implies $v \supset_= u$ but the opposite implication usually does not hold. As an example consider the sequences 31225345 and 2121. We have $31225345 \supset_= 2121$ (because of the subsequence 3535) but $31225345 \not\supset_< 2121$.

The following conjecture is about ten years old and is well known among enumerative combinatorialists.

The Stanley–Wilf conjecture. For each permutation p there is a constant $c = c(p) > 1$ such that the number $S_n(p)$ of n-permutations q, $q \not\supset_< p$, satisfies $S_n(p) < c^n$.

One of the evidences for it is the result of Bóna [2] that it holds for all permutations p of the form $p = I_1 I_2 \dots I_r$ where all terms of I_i are smaller than all terms of I_{i+1} and each sequence I_i is decreasing.

The Alon–Friedgut conjecture. For each k-permutation p there is a constant $c = c(p) > 0$ such that if $v \in [n]^*$ is k-sparse and $v \not\supset_< p$ then $|v| < cn$.

In [1] it is shown that if the AFC is true then so is the SWC. To be precise, the AFC is put there in the form of a question rather than a conjecture.

Suppose $M = (m_{ij})$ and $N = (n_{ij})$ are $a \times b$ and $c \times d$ matrices with entries in $\{0, 1\}$. M *contains* N if there are indices $1 \leq i_1 < \cdots < i_c \leq a$ and $1 \leq j_1 < \cdots < j_d \leq b$ such that, for all $r \in [c]$ and $s \in [d]$, $m_{i_r, j_s} = 1$ whenever $n_{r,s} = 1$. In other words, M has a (not necessarily consecutive) submatrix of N's size that has 1s on all the places where N has them and maybe on some others.

Füredi and Hajnal [3] investigated the extremal function $f(m, n; N)$ that is defined as the maximum number of 1s in an $m \times n$ 0-1 matrix M that does not contain N; $f(n; N) = f(n, n; N)$. They looked systematically at all 37 substantially distinct Ns with four 1s and no zero row or column. (We leave to the interested reader to prove as an exercise that if N has at most three 1s then

$f(m,n;N)=O(m+n)$. For four 1s the situation is much more complicated.) In the concluding section of [3] they ask if for each permutation matrix N we have $f(n;N)=O(n)$: "Is it true that the complexity of all permutation configurations are linear?" We take the liberty to formulate it as a conjecture.

The Füredi–Hajnal conjecture. For each permutation matrix N we have the estimate $f(n;N)=O(n)$.

We show, using the ideas of [1], that the FHC implies the SWC. We prefer to think of the matter in terms of bipartite graphs. Let G and H be two simple bipartite graphs with the parts $[a],[b]'$ and $[c],[d]'$. Here $[b]'=\{1',2',\ldots,b'\}$ and the parts are linearly ordered in the natural way. We say that G *contains* H if H is a (ordered!) subgraph of G, that is, there exist *increasing* injections $f:[c]\to[a]$ and $g:[d]'\to[b]'$ such that if $\{i,j'\}$ is an edge of H then $\{f(i),g(j')\}$ is an edge of G. Besides being a sequence and a 0-1 matrix, each k-permutation $p=a_1a_2\ldots a_k$ is also a bipartite graph G_p with the parts $[k],[k]'$ and the edges $\{a_i,i'\},i\in[k]$. The FHC then says that each bipartite graph on $[n],[n]'$ not containing G_p has only $O(n)$ edges.

Theorem 1. *If the FHC is true then so is the SWC.*

Proof. Let p be a permutation and $M(n)$ the set of simple bipartite graphs on $[n],[n]'$ that do not contain G_p. We assume the FHC — there is a constant c such that $|E(G)|<cn$ for each $G\in M(n)$. Let $n>1$. For each $G\in M(n)$ we define a bipartite graph G_1 on $[m],[m]'$, where $m=\lceil n/2\rceil$, by

$$\{i,j'\}\in E(G_1)\Longleftrightarrow \exists\, e\in E(G):\ e\cap\{2i-1,2i\}\neq\emptyset\,\&\,e\cap\{2j'-1,2j'\}\neq\emptyset\ .$$

Clearly, $G_1\in M(m)$. Also, one G_1 arises from at most $15^{|E(G_1)|}<15^{cm}$ graphs G because there are 15 possibilities for a nonempty restriction of G to $\{2i-1,2i\},\{2j'-1,2j'\}$. Hence,

$$|M(n)|<15^{c\lceil n/2\rceil}\cdot|M(\lceil n/2\rceil)|\ .$$

Iterating the inequality until $|M(1)|=2$, we obtain an upper bound on $|M(n)|$ that is exponential in n. But $S_n(p)\le|M(n)|$ because for each n-permutation $q,q\not\supset_< p$, we have $G_q\in M(n)$. Thus

$$S_n(p)\le|M(n)|<15^{2cn}\ .$$

□

Alon and Friedgut prove the weaker form of the SWC with an almost exponential bound by means of the following result due to Klazar [5]. Suppose $u\in[k]^*$ is given. If $v\in[n]^*$ is k-sparse and $v\not\supset_= u$ — notice that now we use the weak containment, then

$$|v|<nc^{\alpha(n)^d} \tag{1}$$

where $c, d > 1$ are moderate constants depending only on u and $\alpha(n)$ is the inverse of the Ackermann function $A(n)$ known from recursion theory.

We remind the reader the definitions of $A(n)$ and $\alpha(n)$. If $F_1(n) = 2n$, $F_2(n) = 2^n$, and $F_{i+1}(n) = F_i(F_i(\ldots F_i(1)\ldots))$ with n iterations of F_i, then $A(n) = F_n(n)$ and $\alpha(n) = \min\{m : A(m) \geq n\}$. Although $\alpha(n) \to \infty$, in practice $\alpha(n)$ is bounded:

$$\alpha(n) \leq 4 \text{ for } n \leq 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$$

where the height of the tower is $2^{16} = 65536$.

Theorem 2. *For each fixed permutation p,*

$$S_n(p) \leq |M(n)| < 225^{n\beta(n)} \quad \textit{with} \quad \beta(n) = c^{\alpha(n)^d}$$

where $c, d > 1$ are constants depending only on $|p|$ and $\alpha(n)$ is the inverse of the Ackermann function.

Proof. We use the notation of the previous proof and set $k = |p|$. If instead of $|E(G_1)| < cm$ the bound $|E(G_1)| < m\beta(m)$ with an increasing function $\beta(m)$ is used, we get

$$S_n(p) \leq |M(n)| < 15^{2n\beta(n)} = 225^{n\beta(n)} \ .$$

Let $G \in M(n)$. It remains to derive a good bound $|E(G)| < n\beta(n)$. Consider the sequence $v = L_1 L_2 \ldots L_n \in [n]^*$, where L_i is the list of the neighbours of i' in G, in the increasing order. The sequence v is in general not k-sparse but it is easy to see that by deleting $\leq (k-1)(n-1)$ appropriate elements, $\leq k-1$ from the beginning of each of $L_2, \ldots, L_n$, we can obtain a k-sparse subsequence w. It is also not difficult to see that if $v \supset_= u(k)$, where $u(k) = 12\ldots k12\ldots k\ldots 12\ldots k$ has $2k$ segments $12\ldots k$, then G contains G_p. (The repetitions in the weak $u(k)$-copy in v force a subsequence isomorphic to p whose terms lie in k distinct L_is.) Thus $w \not\supset_= u(k)$ and we can apply the aforementioned result:

$$|E(G)| = |v| \leq (k-1)(n-1) + |w| < (k-1)(n-1) + nc^{\alpha(n)^d} \ .$$

□

Our bound is weaker compared to the bound of Alon and Friedgut in [1]. They use a more complicated induction step in which they decrease n more than to ours $\lceil n/2 \rceil$. As we mentioned, they do not work with graphs but in $(\mathbf{N}^*, \supset_<)$ and $(\mathbf{N}^*, \supset_=)$.

We show that both extremal conjectures are equivalent.

Theorem 3. *The AFC and the FHC are mutually equivalent.*

Proof. We prove first that the AFC implies the FHC. If $v = a_1 a_2 \ldots a_l \in \mathbf{N}^*$, $1 \leq i \leq j < k \leq l$, and $a_j > a_{j+1}$, we say that a_i and a_k are in v separated by a fall. Let p be a permutation and G a bipartite graph on $[n], [n]'$ not containing

G_p. Consider the sequence $v = L_1 L_2 \dots L_n \in [n]^*$ of the previous proof. Recall that each L_i is increasing. It may happen that $v \supset_< p$ because one L_i can contribute to the subsequence isomorphic to p by more than one element. To prevent this, we take a larger permutation $p' \supset_< p$, $|p'| = k'$, such that each two consecutive elements of the subsequence of p' that is isomorphic to p are in p' separated by a fall. Now $v \supset_< p'$ is impossible. As we know, by deleting $< k'n$ elements we can obtain a k'-sparse subsequence w. By the AFC for p', we have a linear bound on $|w|$. So $|E(G)| = |v| < k'n + |w| = O(n)$.

We prove that the FHC implies the AFC. If we are content in the AFC with *any* bound, the (unconditional) proof is easy. Suppose $u \in [k]^*$, $v \in [n]^*$ is k-sparse, and $v \not\supset_< u$. We derive a bound on $|v|$ in terms of k, $|u|$, and n. Notice that u is any sequence, not just a permutation. Split v into intervals of length k and the remainder of length $< k$. By the pigeonhole principle ($v \not\supset_< u$), there are at most $(|u|-1)\binom{n}{k}$ intervals. Therefore

$$|v| < k((|u|-1)\binom{n}{k} + 1) \ .$$

Now suppose that p is a k-permutation and $v \in [n]^*$ is a k-sparse sequence not containing p. Using the FHC, we prove a linear bound on $|v|$. There is a constant $c > 0$ such that $|E(G)| < cn$ holds for each bipartite graph on $[n], [n]'$ not containing G_p. If $k > c$, we set $l = k$ and $w = v$. Else we fix a positive integer $l, l > c$, and take w as the longest l-sparse subsequence of v. We show first that $|w|$ is proportional to $|v|$. The subsequence w splits v into nonempty intervals disjoint to w. Let I be one of them. Since no term of I can be used to extend w, each of them equals to one of the $\le l-1$ terms of w preceding I or following it. So there are only $\le 2l-2$ distinct numbers in I and, since I is k-sparse and $I \not\supset_< p$, $|I| < k((k-1)\binom{2l-2}{k} + 1) = d(k,l)$ (by the above trivial bound). Thus

$$|v| < (d(k,l)+1)(|w|+1) \ .$$

Now split w into m intervals w_i of length l and the remainder r of length $< l$: $w = w_1 w_2 \dots w_m r$. Consider the bipartite graph G on $[n], [m]'$, defined by

$$\{i, j'\} \in E(G) \iff i \text{ appears in } w_j \ .$$

Clearly, G does not contain G_p and each j' has degree l. If $m \ge n$, after adding some isolated vertices to $[n]$ G can be regarded as a graph on $[m], [m]'$. But G has $lm > cm$ edges, a contradiction. Hence $m < n$ and we have the bound

$$|v| < (d(k,l)+1)(|w|+1) < (d(k,l)+1)(ln+1) = O(n) \ .$$

□

Conclusion and remarks. In (1) one can set, for $n > n(u)$, $c = 1000k^3$ and $d = |u|-4$, see [5]. Thus in Theorem 2 one can set, for $n > n(k)$, $c = 1000k^3$ and $d = 2k^2$. The bound (1) cannot in general be improved to $O(n)$. For example, it is known that if $v_1, v_2 \in [n]^*$ are 2-sparse and have the maximum length with respect to $v_1 \not\supset_= 12121, v_2 \not\supset_= 121212$, then $n\alpha(n) \ll |v_1| \ll n\alpha(n)$ and $n2^{\alpha(n)} \ll |v_2| \ll n2^{\alpha(n)}$. For more information see the book [8] of Sharir and Agarwal. So it is possible that the FHC is false. If the FHC is true, to prove it seems to require tools far stronger than are those presented in [3].

The paper [1] teaches us that we understand the SWC better by viewing permutations as elements of $\mathbf{N}^*$. We have seen that we can view them also as bipartite graphs. The situation can be generalized further to the class of ordered hypergraphs, see [6] and [7].

In the end of [4] Gessel mentions the problem to decide if for each permutation p the sequence $\{S_n(p)\}_{n\geq 1}$ is P-recursive. The above results are not relevant to the problem and it appears to be very difficult. On the other hand, a strong optimism in this respect was expressed by Zeilberger [9].

References

1. N. Alon and E. Friedgut, On the number of permutations avoiding a given pattern, *J. Comb. Theory, Ser. A* **89** (2000), 133–140.
2. M. Bóna, The solution of a conjecture of Stanley and Wilf for all layered patterns, *J. Comb. Theory, Ser. A* **85** (1999), 96–104.
3. Z. Füredi and P. Hajnal, Davenport–Schinzel theory of matrices, *Discrete Math.* **103** (1992), 233–251.
4. I.M. Gessel, Symmetric functions and P-recursiveness, *J. Comb. Theory, Ser. A* **53** (1990), 257–285.
5. M. Klazar, A general upper bound in extremal theory of sequences, *Commentat. Math. Univ. Carol.* **33** (1992), 737–746.
6. M. Klazar, Counting pattern-free set partitions I: A generalization of Stirling numbers of the second kind, *Eur. J. Comb.*, to appear.
7. M. Klazar, Counting pattern-free set partitions II: Noncrossing and other hypergraphs, submitted.
8. M. Sharir and P.K. Agarwal, *Davenport–Schinzel sequences and their geometric applications,* Cambridge University Press, Cambridge, UK 1995.
9. D. Zeilberger, Enumeration schemes, and more importantly, their automatic generation, *Ann. Comb.* **2** (1998), 185–195.