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Extremal Problems for Colored Trees and Davenport–Schinzel Sequences

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Abstract

In the theory of generalized Davenport–Schinzel sequences one estimates the maximum lengths of finite sequences containing no subsequence of a given pattern. Here we investigate a further generalization, in which the class of sequences is extended to the class of colored trees. We determine exactly the extremal functions associated with the properly 2-colored path of four vertices and with the monochromatic path of any length. We prove that the extremal function of any colored path grows almost linearly (this is a characteristic feature of DS sequences). Three problems are posed.

1 Introduction

We want to extend results on extremal problems concerning certain finite sequences to colored trees. The sequences are *Davenport–Schinzel sequences* (in short, *DS sequences*) introduced in [3]. We start by recalling them.

Let S^* denote the set of all finite sequences over a fixed infinite alphabet S. All our sequences will be from S^*. For $u \in S^*$ we denote by $|u|$ the length of u and by $\|u\|$ the number of symbols $x \in S$ appearing in u. We say that $u \in S^*$ is *alternating* if $u = xyxy\ldots$ where $x, y \in S$ are distinct. The sequence $aa\ldots a$ of i a's is denoted by a^i.

The set $\mathrm{DS}(s)$ of DS sequences with parameter s consists of all sequences $u = x_1x_2\ldots x_l$ such that (i) $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, l-1$ and (ii) no subsequence $x_{i_1}x_{i_2}\ldots x_{i_s}$ of length s is alternating. For example, 432134564 belongs to DS(4) but not to DS(3). As another example, $cb1bdd43b23 \not\in \mathrm{DS}(s)$ for any s, but omitting one of

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the d's we obtain a sequence from the family DS(5) (but not from DS(4), since it has the form $\dots b \dots 3b.3$).

The functions $N_s(n)$ measure maximum lengths of sequences in DS(s):

$$N_s(n) = \max\{|u| :\ u \in \mathrm{DS}(s)\ \&\ \|u\| \leq n\}.$$

Trivially, $N_1(n) = 0$, $N_2(n) = 1$, and $N_3(n) = n$. It is easy to show that $N_4(n) = 2n - 1$ (see [3] or Theorem 3.1 below). For $s > 4$ the situation is much more complicated. Davenport and Schinzel proved [3] $N_5(n) = O(n \log n)$ and, for any fixed s, $N_s(n) \leq n \exp(c(s)\sqrt{\log n}\ )$. Both results were improved by Szemerédi [12]: $N_s(n) = O(n \log^* n)$ ($\log^*(n)$ is the smallest number of iterations of the power function 2^x after which, starting with $x = 1$, a number $\geq n$ is reached). In a further improvement Hart and Sharir [4], inspired by some techniques used in [13], derived the upper and lower bounds $N_5(n) = \Theta(n\alpha(n))$ with $\alpha(n)$ being the inverse to the Ackermann function, so that $\alpha(n)$ goes to infinity but extremely slowly. Agarwal, Sharir and Shor [2] found that $N_6(n) = \Theta(n2^{\alpha(n)})$ and that, for $s > 6$, $N_s(n)$ is roughly $n2^{\alpha(n)^{s/2}}$: for the precise statement see [2, 11]. For $s > 6$ there is still some gap between the lower and upper bounds but the main problem posed by Davenport and Schinzel, to estimate $N_s(n)$ satisfactorily, was solved in [2].

DS sequences were rediscovered in the 1980s by researchers in computational geometry (interestingly, the original motivation of Davenport and Schinzel was also geometric and much the same as the modern one). Applications of DS sequences to algorithmic problems in geometry are discussed in the monograph by Sharir and Agarwal [11].

In [1], Adamec, Klazar and Valtr proposed a generalization of DS sequences. We say that $u = x_1x_2\dots x_l \in S^*$ and $v = y_1y_2\dots y_m \in S^*$ are *equivalent* if $l = m$ and there is a bijection $f :\ S \to S$ such that $x_i = f(y_i)$ for $i = 1, 2, \dots, l$. For a fixed sequence v with $|v| = s$ and $\|v\| = k$ we define the set DS(v) as consisting of all sequences $u = x_1x_2\dots x_l$ such that (i) $x_i, x_{i+1}, \dots, x_{i+k-1}$ are mutually distinct for each $i = 1, 2, \dots, l-k+1$ and (ii) no subsequence $x_{i_1}x_{i_2}\dots x_{i_s}$ of length s is equivalent to v. The general extremal function is defined [1] as

$$\mathrm{Ex}(v, n) = \max\{|u| :\ u \in \mathrm{DS}(v)\ \&\ \|u\| \leq n\}.$$

It is easy to see that $\mathrm{Ex}(v, n) < \infty$ for any v and n. For example, $\mathrm{Ex}(ababa, n) = N_5(n)$, $\mathrm{Ex}(a^4, n) = \mathrm{Ex}(aaaa, n) = 3n$, and $\mathrm{Ex}(abcda, n) = n$. Sequences $u \in S^*$ satisfying condition (i) in the definition of DS(v) are called *k-sparse* (in fact, *k-regular* in our other articles; we choose a different terminology here to avoid redefining a standard term in graph theory). In the case that u violates (ii) we say that u *contains* v. Sometimes it is useful to work with a more general function $\mathrm{Ex}(v, n, l)$, where $l \geq \|v\|$ is a new fixed parameter and the function is defined as the maximum length of an l-sparse sequence u not containing v with $\|u\| \leq n$. Thus $\mathrm{Ex}(v, n) = \mathrm{Ex}(v, n, \|v\|)$.

Properties of $\mathrm{Ex}(v, n)$ were investigated in [1] and then in the author's thesis [8]. Further results were obtained in [5, 6, 7, 9]. More information and references can be found in the survey paper [10]. Recently Valtr [14, 15] found the following interesting application of $\mathrm{Ex}(v, n)$ in combinatorial geometry. Let a *geometric graph* be a graph whose vertices and edges are points and *straight segments* in the plane. The bound

$$\mathrm{Ex}(a_1a_2 \ldots a_{k-1}a_ka_{k-1} \ldots a_2a_1a_2 \ldots a_{k-1}a_k, n) = O(n)$$

proved in [7] implies that any geometric graph with n vertices and no k pairwise crossing edges has $O(n \log n)$ edges.

$\mathrm{Ex}(v, n)$ proved to be an interesting and useful generalization of $N_s(n)$. In this paper we investigate a further extension of it, in which the realm of sequences S^* is extended to the realm of colored trees. This way we try to connect DS sequences with graph theory; hopefully some applications in this field will be found.

The paper is organized as follows. In Section 2 we introduce the appropriate extension of $\mathrm{Ex}(v, n)$. In an important auxiliary result we determine exactly how many 'peripheral' vertices in certain colored trees there may be. In Section 3 the tree extremal functions of the path color patterns *abab* and $aa \ldots a$ are determined. In Section 4 it is shown that any path color pattern has an almost linearly growing extremal function. In Section 5 we pose some problems, to indicate a possible direction for further research.

2 Extremal functions $\mathrm{Ex_T}(\mathcal{P}, n)$ and $\mathrm{Ex_T}(\mathcal{P}, n, l)$

By a *tree* we mean always a finite undirected tree. By a *colored tree* we understand a pair $\mathcal{T} = (T, f)$ where T is a tree and $f: V(T) \to S$ is a vertex coloring. A coloring is *proper* if no edge is monochromatic. More generally, (T, f) is *k-sparse* if $f(u) = f(v), u \neq v$, implies that the path joining u and v has at least k edges. Symbols $|\mathcal{T}|$ and $\|\mathcal{T}\|$ denote the number of vertices in $\mathcal{T}$ and the number of colors appearing in $\mathcal{T}$.

If the tree T in $\mathcal{T}$ is a path P we speak of a *colored path*. Obviously, any sequence $u = x_1x_2 \ldots x_l \in S^*$ yields a colored path (P, f) where $P = (v_1, v_2, \ldots, v_l)$ and $f(v_i) = x_i$. We denote it by $\mathcal{P}_u$ but mostly will write simply u. Usually there is no danger of confusion, because the symbol for tree extremal function $\mathrm{Ex_T}$ indicates clearly the tree context.

Two colored trees $\mathcal{T}_1 = (T_1, f_1)$ and $\mathcal{T}_2 = (T_2, f_2)$ are *equivalent* if there is a graph isomorphism $G: V(T_1) \to V(T_2)$ and a bijection $F: S \to S$ such that $f_1(v) = F(f_2(G(v)))$ for each $v \in V(T_1)$. We say that $\mathcal{T}_1$ is *contained* in $\mathcal{T}_2$ if it is possible to subdivide some edges of $\mathcal{T}_1$ by colored vertices so that the colored tree obtained is equivalent to a subgraph of $\mathcal{T}_2$. The containment of colored trees generalizes the containment of sequences. More precisely, if v is contained in u then $\mathcal{P}_v$ is contained in $\mathcal{P}_u$, and if $\mathcal{P}_v$ is contained in $\mathcal{P}_u$ then v is contained in u or in the reversed u.

Forbidding the containment of a fixed colored path is not enough to give an interesting extremal problem, because any colored star avoids any path coloring with more than three vertices. A condition taking care of colored stars is necessary. We propose the *tripod condition*. This requires that $\mathcal{T}$ does not contain a specific colored tree (U, f) with U being the 4-vertex star and f being the proper 2-coloring. This colored tree is called a *tripod*. We will say that $\mathcal{T}$ meets the tripod condition, or that $\mathcal{T}$ is *tripod-free*.

We feel the tripod condition ensures that in the more general situation the basic features of DS sequences remain preserved.

Let $\mathcal{P}$ be a colored path with $\|\mathcal{P}\| = k$. The set $\mathrm{DS}(\mathcal{P})$ consists of all colored trees $\mathcal{T}$ such that (i) $\mathcal{T}$ is k-sparse and (ii) $\mathcal{T}$ contains neither $\mathcal{P}$ nor a tripod. The *tree extremal function* of $\mathcal{P}$ is

$$\mathrm{Ex}_{\mathrm{T}}(\mathcal{P}, n) = \max\{|\mathcal{T}| : \mathcal{T} \in \mathrm{DS}(\mathcal{P}) \ \&\ \|\mathcal{T}\| \leq n\}.$$

We will investigate also the more general extremal function

$$\mathrm{Ex}_{\mathrm{T}}(\mathcal{P}, n, l) = \max\{|\mathcal{T}| : \mathcal{T} \in \mathrm{DS}(\mathcal{P}) \ \&\ \|\mathcal{T}\| \leq n \ \&\ \mathcal{T} \text{ is } l\text{-sparse}\},$$

where $l \geq \|\mathcal{P}\|$. Thus $\mathrm{Ex}_{\mathrm{T}}(\mathcal{P}, n) = \mathrm{Ex}_{\mathrm{T}}(\mathcal{P}, n, \|\mathcal{P}\|)$.

In the case of sequences the first and the last appearances of a symbol $x \in S$ in $u \in S^*$ are often important. Obviously, altogether there are at most $2\|u\|$ such appearances. We need to cope with the tree analogy of this phenomenon. A vertex v in a colored tree $\mathcal{T} = (T, f)$ is *peripheral* if there is no path P in T with the endvertices w and z such that v is an inner vertex of P and $f(w) = f(v) = f(z)$. Let $p(\mathcal{T})$ be the number of peripheral vertices in $\mathcal{T}$. The following lemma is perhaps of some interest as an extremal result in its own right. We use it later in Sections 3 and 4.

Lemma 2.1 *The maximum number of peripheral vertices $p(n,k) = \max p(\mathcal{T})$ taken over k-sparse tripod-free colored trees $\mathcal{T}$ with $\|\mathcal{T}\| = n$, is $p(n,k) = n$ for $n < k$ and, for $n \geq k$,*

$$p(n,k) = \left\lfloor \frac{2(n-1)k}{k-1} \right\rfloor. \tag{1}$$

Proof. The first bound is obvious, since then no color can be repeated. We prove the second formula by showing separately that the RHS of (1) is an upper and lower bound for the LHS.

(i) *The upper bound.* Suppose $\mathcal{T} = (T, f)$ is k-sparse, meets the tripod condition, and $\|\mathcal{T}\| = n$. A *tail* in $\mathcal{T}$ is a path $(v_0, v_1, \ldots, v_l)$ such that v_0 is a leaf, $v_1, \ldots, v_{l-1}$ have degree 2 and v_l has degree > 2. The *length* of the tail is l. Note that if T is a path then it has no tails.

In the first step we transform $\mathcal{T}$ into a colored tree $\mathcal{T}_1$ meeting the same hypothesis as $\mathcal{T}$ and with $p(\mathcal{T}_1) = p(\mathcal{T})$, but with all tails having length $\geq k$. Let $s(\mathcal{T})$ be the number of tails with length $< k$. If $s(\mathcal{T}) = 0$ there is nothing to be done. Otherwise let $P = (v_0, v_1, \ldots, v_l), l < k$, be one of the shortest tails. $T \backslash \{v_l\}$ has at least three components and the color $f(v_0)$ does not appear in at least one of them (since $f(v_0) \neq f(v_l)$ and the tripod condition holds). Let v be a leaf in one of those components. We cut off v_0 from P and attach it to v (see Figure 1(a)). The new colored tree is k-sparse (since v_0 is at least as far from v_l as it was before) and tripod-free, has the same number of peripheral vertices, and no tail length decreased except for P. P either disappeared or $(v_1, \ldots, v_l)$ is now the shortest tail. Repeating this until we eliminate P completely, the $\mathcal{T}'$ obtained has $s(\mathcal{T}') < s(\mathcal{T})$. Repeating the whole procedure we arrive at a $\mathcal{T}_1$ with $s(\mathcal{T}_1) = 0$. It may be the case that $\mathcal{T}_1$ has no tail at all because it is a colored path. Then $p(\mathcal{T}) = p(\mathcal{T}_1) \leq 2n$ which for $n \geq k$ is stronger than the bound claimed and we are done at once. Thus, we can assume the number of tails is the same as the number of leaves.

In the second step we process $\mathcal{T}_1$ to get rid of all vertices of degree > 3. A vertex v with degree $d > 3$ is replaced by a path of d new vertices (see Figure 1(b)), all with color $f(v)$ and each adjacent to one neighbour of v. Thus, each of the new vertices has degree at most 3. Repeating this procedure we obtain a $\mathcal{T}_2$ with all degrees at most 3. It is easy to check that $\mathcal{T}_2$ is tripod-free and $p(\mathcal{T}_2) \geq p(\mathcal{T}_1)$ (in each replacement $p(\cdot)$ could only increase). Also $s(\mathcal{T}_2) = 0$. The k-sparseness is lost but not completely, since each tail is colored k-sparsely. Obviously, $\|\mathcal{T}\| = \|\mathcal{T}_1\| = \|\mathcal{T}_2\| = n$.

Let $p = p(\mathcal{T}_2)$, let l be the number of leaves of $\mathcal{T}_2$, and let X be the set of vertices of $\mathcal{T}_2$ with degree 3. For a color $a \in S$ let $k(a)$ be the number of peripheral vertices in $\mathcal{T}_2$ colored a. We consider the sets of colors

$$A = \{a \in S : \ 1 \leq k(a) \leq 2\}, \ B = \{b \in S : \ k(b) \geq 3\}.$$

Clearly,

$$|A| + |B| = n \text{ and } |X| = l - 2. \tag{2}$$

Similarly, each $b \in B$ forces exactly $k(b) - 2$ vertices of X to be colored b (since the tripod condition holds and all degrees are at most 3), so that

$$\sum_{b \in B} k(b) \leq 2|B| + |X|. \tag{3}$$

Finally, since there are l tails and the first k vertices on each tail are peripheral,

$$l \leq p/k. \tag{4}$$

Combining the definitions of A and B with (2)–(4), we have

$$p \leq 2|A| + \sum_{b \in B} k(b) \leq 2|A| + 2|B| + |X| = 2n + l - 2 \leq 2n + \frac{p}{k} - 2.$$

Solving this inequality for p we obtain

$$p(\mathcal{T}) = p(\mathcal{T}_1) \leq p(\mathcal{T}_2) = p \leq \frac{2(n-1)k}{k-1}.$$

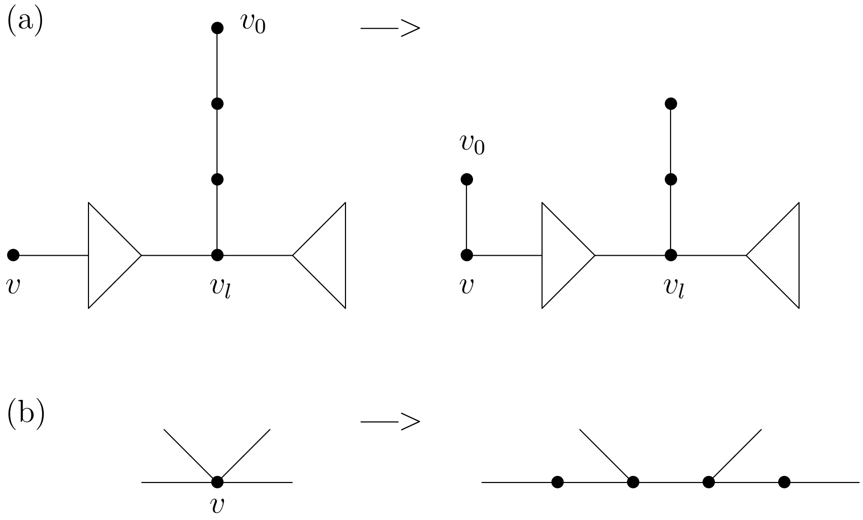

Figure 1: The proof of Lemma 2.1.

(ii) *The lower bound.* By $S(p,q)$ we denote the tree (sometimes called a *long star*) with $pq+1$ vertices which arises by taking p paths having q vertices each and joining their endpoints to one common central vertex. Write $n-1 = m(k-1)+r$ where $0 \le r < k-1$ and $m > 0$. Then

$$\left\lfloor \frac{2(n-1)k}{k-1} \right\rfloor = 2mk + 2r + \left\lfloor \frac{2r}{k-1} \right\rfloor . \tag{5}$$

Case 1: $r < (k-1)/2$. Then the RHS of (5) becomes $2mk+2r$. We color $S(2m,k)$ as follows. Let m of its rays be called *type 1* and the other m rays *type 2*. One color, say c_0, is used for the central vertex and for all the leaves. The remaining $2m(k-1)$ vertices are colored with $m(k-1)$ colors, each used for one vertex in a type-1 ray and one vertex in a type-2 ray, these vertices being distance exactly k apart; clearly such a coloring exists. Then we attach $2r$ new vertices as leaves and color two of them with each of r new colors, the only restriction to their attachment being that the resulting colored tree $\mathcal{T}_3$ is k-sparse. Clearly this is possible, and $\mathcal{T}_3$ is tripod-free. Moreover, $\mathcal{T}_3$ has $1+2mk+2r$ vertices, and each vertex except v_0 is peripheral. Thus $p(\mathcal{T}_3) = 2mk+2r$.

Case 2: $r \ge (k-1)/2$. The same construction works, but it is not optimal since now we should have $2mk+2r+1$ peripheral vertices. This time we start with $S(2m+1,k)$. Let $m-1$ of its rays be called *type 1*, other $m-1$ rays *type 2*, and the remaining three rays *type 3*. Let $\Delta = k-1-r \le r$, so that $k-1 = \Delta + r$. As before, one color, say c_0, is used for the central vertex v_0 and for all the leaves. The remaining vertices of type-1 rays and type-2 rays are colored in pairs as in Case 1. Type-3 rays are colored with new colors $a_1,\ldots,a_r,b_1,\ldots,b_r,c_1,\ldots,c_\Delta$ in the following pattern, where v_0 is to the left of each ray and the leaves are to the right:

$$\begin{array}{l} a_1, a_2, \ldots, a_r, b_\Delta, b_{\Delta-1}, \ldots, b_1 \\ b_1, b_2, \ldots, b_r, c_\Delta, c_{\Delta-1}, \ldots, c_1 \\ c_1, c_2, \ldots, c_\Delta, a_r, a_{r-1}, \ldots, a_1 \end{array}$$

It is not difficult to check that this coloring is k-sparse. Finally we attach $r - \Delta$ new vertices as leaves and give them colors $b_{\Delta+1}, \ldots, b_r$, the only restriction to their attachment being that the resulting colored tree $\mathcal{T}_4$ is k-sparse. Clearly this is possible, and $\mathcal{T}_4$ is tripod-free. Moreover, $\mathcal{T}_4$ has $1 + (2m+1)k + r - \Delta = 1 + 2mk + 2r + 1$ vertices, and each vertex except v_0 is peripheral. Thus $p(\mathcal{T}_4) = 2mk + 2r + 1$. □

The immediate corollary of (1) is that for $k = 2$ (proper colorings) and $k = 3$ we have $p(n, 2) = 4n - 4$ ($n > 1$) and $p(n, 3) = 3n - 3$ ($n > 2$). In any colored tree $\mathcal{T}$ the number $p(\mathcal{T})$ majorizes the number of leaves and that in turn majorizes the number of vertices of degree > 2. So, by the above lemma, any tripod-free properly colored tree $\mathcal{T}$ has $O(\|\mathcal{T}\|)$ leaves and $O(\|\mathcal{T}\|)$ path segments connecting vertices with degree $\neq 2$.

3 Colored paths $abab$ and a^i

We analyze extremal functions of the simplest colored paths, namely $abab$, the 4-vertex path colored properly with two colors, and a^i, the monochromatic path of i vertices. We start with $\mathrm{Ex_T}(abab, n, k)$. Recall that this function is defined as the maximum number of vertices of a k-sparse tripod-free colored tree $\mathcal{T}$ that does not contain $abab$ and satisfies $\|\mathcal{T}\| \leq n$. It is not difficult to prove [9] that for sequences we have $\mathrm{Ex}(abab, n, k) = 2n - k + 1$ (for $n \geq k - 1$). In contrast, the tree extremal function is independent of k and equals $2n - 1$.

The following result was published already in [10], for completeness we sketch the proof.

Theorem 3.1 *For any integer $n \geq 1$ we have* $\mathrm{Ex_T}(abab, n) = \mathrm{Ex_T}(abab, n, 2) = 2n - 1$.

Proof. (Sketch). First we recall the proof that $\mathrm{Ex}(abab, n) = 2n - 1$. The length $2n-1$ is attained, for instance, by the sequences $1\ 2\ \ldots\ n-1\ n\ n-1\ \ldots\ 2\ 1$. That $2n - 1$ is also the upper bound follows by induction on n: just delete the symbol appearing only once.

The order $2n-1$ is attained already by paths. Hence, to prove $\mathrm{Ex_T}(abab, n) = 2n-1$ it suffices to show that any proper tripod-free $\mathcal{T} = (T, f)$ that does not contain $abab$ has at most $2\|\mathcal{T}\| - 1$ vertices. We proceed by induction on the sum $\|\mathcal{T}\| + \#\{v \in V(T) : \deg(v) > 2\}$. If $\mathcal{T}$ is not a colored path (otherwise we are done by the previous paragraph) there is a vertex v such that $T \backslash \{v\}$ has ≥ 3 components of which at most one is not a path. It is easy to see that we can assume that no color appears more than once on the union of the path components and v (otherwise $\|\mathcal{T}\|$ can be decreased by deleting some vertices from the paths). It can be shown that the path components can be assembled into one path so that the pattern *abab* is not created: see [10]. The number of vertices of degree > 2 decreased and we are done by induction. $\square$

We remark that there are plenty of extremal configurations: any tree with $2n-1$ vertices can be colored properly with n colors so that the coloring meets the tripod condition and does not contain *abab*.

Theorem 3.2 *For any* $n \geq (k-1)^2+1$ *we have* $\mathrm{Ex_T}(abab, n, k) = 2n-1$.

Proof. The inequality $\mathrm{Ex_T}(abab, n, k) \leq 2n-1$ follows from the previous theorem. We show that for n large enough the number of vertices $2n-1$ is attained. Let $n \geq (k-1)^2+1$ and write $n-1 = (k-1)l + m, 0 \leq m < k-1$. Thus $l \geq k-1$. The basic structure is the tree $S(2l, k-1)$ (defined as in the second part of the proof of Lemma 2.1). Let its rays be $R_1, R_2, \ldots, R_{2l}$. By $v(i,j)$ we denote the vertex on R_i at distance j from the centre.

Give distinct colors to the central vertex and all vertices on the rays $R_1, \ldots, R_l$. For $i = 1, \ldots, l$ and $j = 1, \ldots, k-1$, the color used on $v(i,j)$ is used again on $v(l+i+j, k-j)$ if $i+j \leq l$ and on $v(i+j, k-j)$ otherwise. This ensures that two different rays can have at most one color in common, so there is no *abab*. Also, the coloring is k-sparse and tripod-free. We finish the construction by appending in an appropriate way $2m$ leaves and coloring two of them with each of m new colors. Altogether we have used $2(n-1)+1 = 2n-1$ vertices. $\square$

We proceed to the colored path a^i. For sequences the problem is trivial, $\mathrm{Ex}(a^i, n, k) = (i-1)n$ (for $n \geq k \geq 1$). For trees the situation is more interesting. Note that $\mathrm{Ex_T}(a^i, n) = \mathrm{Ex_T}(a^i, n, 1) = \infty$ if $i > 3$, so that we investigate $\mathrm{Ex_T}(a^i, n, k)$ only for $k > 1$.

Recall that $p(n,k)$ denotes the maximum number of peripheral vertices, which was determined in Lemma 2.1. For a colored tree $\mathcal{T} = (T, f)$ the *depth* of a vertex v is the *largest* i such that there is a path P in T which has v as an inner vertex and which contains at least $i-1$ vertices with color $f(v)$ on each side of v (hence at least $2i-1$ such vertices in total). Clearly the vertices with depth 1 are peripheral vertices, and so if $d_i = d_i(\mathcal{T})$ denotes the number of vertices with depth i, then $p(\mathcal{T}) = d_1 \geq d_2 \geq \cdots$

Theorem 3.3 *For any* $n \geq k \geq 2$

$$\mathrm{Ex_T}(a^i, n, k) = \begin{cases} \frac{i-2}{2} \cdot p(n,k) + n & \text{if } i \geq 2 \text{ is even} \\ \frac{i-3}{2} \cdot p(n,k) + 2n & \text{if } i \geq 3 \text{ is odd.} \end{cases}$$

Proof. (i) *The upper bound.* Suppose first that i is even. In any k-sparse $\mathcal{T} \in \mathrm{DS}(\mathcal{P}_{a^i})$ there can be at most $\|\mathcal{T}\|$ vertices with depth $\geq i/2$. Otherwise a color, say a, would be repeated on two such vertices v_1 and v_2. By the above definition, each of v_1 and v_2 is covered by a path with $i/2-1$ vertices of color a on both sides of the vertex. These paths together with the path joining v_1 and v_2 produce a^i. Thus, $|\mathcal{T}| \leq (i/2-1)p(\mathcal{T}) + \|\mathcal{T}\|$.

For i odd the number of vertices with depth $\geq (i-1)/2$ is at most $2\|\mathcal{T}\|$. Otherwise three such vertices would have the same color. We can assume they lie on a path because of the tripod condition. The rest of the argument is the same as for i even. Thus $|\mathcal{T}| \leq \frac{i-3}{2}p(\mathcal{T}) + 2\|\mathcal{T}\|$.

(ii) *The lower bound.* The extremal configurations are modifications of those from the second part of the proof of Lemma 2.1. We use the notation we have introduced there.

Let first $r < (k-1)/2$. For simplicity we explain the construction when $r = 0$. So $\mathcal{T}_3$ is a coloring of $S(2m, k)$. Suppose i is even. Each of the m type-1 rays is replicated $i/2$ times so as to form a 'long ray' of length $ki/2$, and each of the m rays of type 2 is replicated similarly $i/2-1$ times. The leaves of type-1 long rays are deleted. The colored tree obtained is k-sparse, belongs to $\mathrm{DS}(\mathcal{P}_{a^i})$, and has $(i/2-1)p(n,k)+n$ vertices. For i odd all rays of $\mathcal{T}_3$ are replicated $(i-1)/2$ times and then all leaves but one are deleted. The colored tree obtained is k-sparse, belongs to $\mathrm{DS}(\mathcal{P}_{a^i})$, and has $((i-3)/2)p(n,k)+2n$ vertices.

The other case $r \geq (k-1)/2$ is similar. To make it simpler, we again assume $r - \Delta = 0$, so that $\mathcal{T}_4$ is a coloring of $S(2m+1, k)$. Let i be even. Type-1 and type-3 rays are replicated $i/2$ times and type-2 rays $i/2-1$ times. Leaves of type-1 and type-3 long rays are deleted. Finally the last $r = \Delta$ vertices of each of the three type-3 long rays are deleted. The colored tree obtained is tripod-free, k-sparse, does not contain a^i, and has $(i/2-1)p(n,k)+n$ vertices. For i odd the construction is exactly the same as for $r < (k-1)/2$. We leave to the interested reader the easy task of attaching the extra leaves needed when $r > 0$ or $r - \Delta > 0$. □

4 A general colored path

In this section we prove Theorem 4.1, which states that the extremal function of any colored path grows almost linearly. We should remark that the proof is a reduction to sequences, namely to the general upper bound for $\mathrm{Ex}(v, n)$.

Theorem 4.1 *Let $\mathcal{P}$ be a colored path with $|\mathcal{P}| = s$ and $\|\mathcal{P}\| = k \geq 2$. There is a constant $c > 0$ depending only on s such that*

$$\mathrm{Ex}_{\mathrm{T}}(\mathcal{P}, n) \leq n2^{\alpha(n)^c}.$$

In the proof we will make use of the following result, which avoids the need to work with the definition of $\alpha(n)$.

Theorem 4.2 ([5]) *Let $v \in S^*$ have length 5 or more (if not, $\mathrm{Ex}(v,n) = O(n)$). Then*

$$\mathrm{Ex}(v, n) \leq n2^{c\alpha(n)^{|v|-4}},$$

where $c > 0$ depends only on $|v|$.

Let $\mathcal{T} = (T, f)$ be an element of $\mathrm{DS}(\mathcal{P})$ with $|\mathcal{T}| = m$ and $\|\mathcal{T}\| = n$. Recall that $\mathcal{T}$ is k-sparse. Distinguishing a vertex $r \in V(T)$ we change T into a rooted tree T_r with the root r. We have a poset $(V(T_r), \prec)$, where $v_1 \prec v_2$ iff the path joining r and v_2 contains v_1. The merging point of the r-v_1 and r-v_2 paths is denoted by $v_1 \wedge v_2$. We say that v is a *child* of w if $v \succ w$ and there is no z such that $v \succ z \succ w$. We fix a linear order on each set of children of a vertex. Those linear orders determine a unique linear extension $(V(T_r), <)$ of $(V(T_r), \prec)$ in which $v_1 < v_2$ iff $v_1 \prec v_2$ or (if they are incomparable by $\prec$) the child of $v = v_1 \wedge v_2$ lying on the v-v_1 path is smaller in the linear order than the child of v lying on the v-v_2 path. This linear extension will be called the *postorder*.

We turn $\mathcal{T} = (T_r, f)$ into a sequence $u_1 \in S^*$ of length $m = |\mathcal{T}|$ by listing the vertices in the postorder. This means, $u_1 = (f(v_1), f(v_2), \ldots, f(v_m))$, where $v_1 < v_2 < \cdots < v_m$. A problem is that u_1 may not be k-sparse. This is fixed by Lemma 4.3.

For k, l positive integers we define the sequence of length kl

$$z(k,l) = a_1 a_2 \ldots a_k a_1 a_2 \ldots a_k \ldots a_1 a_2 \ldots a_k,$$

where $a_1, a_2, \ldots, a_k$ are k distinct symbols in S. Notice that if $\mathcal{T}$ contains $\mathcal{P}_{z(k,s)}$, it contains $\mathcal{P}$.

Lemma 4.3 *There is a k-sparse subsequence u_2 of u_1 such that*

$$|u_2| \geq |u_1| - cn,$$

where $c > 0$ is an absolute constant.

Proof. Suppose the leaves of T_r are $l_1 < l_2 < \cdots < l_p$. They determine a partition of $V(T_r)$ into the paths $P_1, P_2, \ldots, P_p$:

$$P_0 = \emptyset\ ,\ \ P_i = \{v \in V(T_r) :\ v \preceq l_i\} \setminus \bigcup_{j=0}^{i-1} V(P_j).$$

Clearly u_1 is the juxtaposition of the sequences $f(P_1), f(P_2), \ldots, f(P_p)$, where $f(P_i)$ is the sequence of colors on P_i taken in the postorder (i.e., in $\prec$). By Lemma 2.1, $p \leq p(\mathcal{T}) \leq 4n - 4$. The k-sparseness of u_1 may be violated only on the transitions $f(P_i)f(P_{i+1})$. Let $K = \mathrm{Ex}(z(k,s), 3k-4) + 1$. We can assume $|f(P_i)| \geq K + k - 1$ for all P_i, short $f(P_i)$ are deleted. We set $f(Q_i), i = 2, \ldots, p$, to be the initial segment of $f(P_i)$ of length K. If $f(Q_i)$ contains $z(k,s)$ then, by our remark, $\mathcal{T}$ contains $\mathcal{P}$ and we have a contradiction. Thus, $f(Q_i) \in \mathrm{DS}(z(k,s))$ and therefore $\|f(Q_i)\| \geq 3k - 3$. For each $i = 2, 3, \ldots, p$ there are $k - 1$ symbols, denote them S_i, appearing in $f(Q_i)$, which are distinct from the $k - 1$ symbols preceding $f(Q_i)$ and from the $k - 1$ symbols following after $f(Q_i)$. We delete all terms of $f(Q_i)$ except for some $k - 1$ terms which are appearances of the symbols in S_i. The resulting sequence u_2 is k-sparse. Only at most $p(K + k - 2) + (p - 1)(K - k + 1) = O(n)$ elements have been deleted. $\square$

Let

$$\hat{s} = (5^{s+4} - 1)(s + 2) + 1. \tag{6}$$

We prove the implication

$$\mathcal{P} \text{ is not contained in } \mathcal{T} \Rightarrow z(k, \hat{s}) \text{ is not contained in } u_2. \tag{7}$$

This proves Theorem 4.1, because Theorem 4.2 provides a strong upper bound on $\mathrm{Ex}(z(k, \hat{s}), n) \geq |u_2|$ and, by Lemma 4.3, $m = |\mathcal{T}|$ differs only little from $|u_2|$.

We suppose that u_2 contains $z(k, \hat{s})$ and deduce that $\mathcal{T}$ contains $\mathcal{P}$. Let $w_i, i = 1, 2, \ldots, \hat{s}k$, be $\hat{s}k$ vertices in T_r such that $w_1 < w_2 < \cdots < w_{\hat{s}k}$ and the subsequence $(f(w_i) :\ i = 1, 2, \ldots, \hat{s}k)$ of u_2 is equivalent to $z(k, \hat{s})$. We recall the following result from folklore (the easy companion of Dilworth's theorem).

Lemma 4.4 *In any poset $(X, <)$ the minimum number of parts in a partition of X into antichains is the same as the maximum size of a chain. In particular, if $|X| \geq (a - 1)(b - 1) + 1$ then there is in X either an antichain of size a or a chain of size b.*

Let

$$W = \{w_1, w_{k+1}, w_{2k+1}, \ldots, w_{(\hat{s}-1)k+1}\}.$$

These $\hat{s}$ vertices have the same color, say a_1. We distinguish two cases: (a) there are $s + 3$ vertices in W which are mutually comparable by $\prec$ or (b) there are 5^{s+4} vertices in W which are mutually incomparable by $\prec$. By the previous lemma and by (6), one of (a) or (b) must occur. We show that in both cases $\mathcal{P}$ is contained in $\mathcal{T}$; this will finish the proof of (7) and thus the whole proof of Theorem 4.1.

First a simple lemma that will be applied in both cases.

Lemma 4.5 *Suppose x, y, and w are three distinct vertices of T_r such that $x < y < w$. Then $x \wedge w \preceq y \wedge w$.*

Proof. This is immediate from the definition of the postorder. $\square$

Case (a). We have $s + 3$ vertices $z_1 \prec z_2 \prec \cdots \prec z_{s+3}$ in W with color a_1. For each of the remaining $k - 1$ colors $a_j, j = 2, \ldots, k$, we have the vertices z_i^j

$$z_1 < z_1^2 < \cdots < z_1^k < z_2 < z_2^2 < \cdots < z_2^k < z_3 < \cdots < z_{s+2}^k < z_{s+3}$$

with color $f(z_i^j) = a_j$. Let R be the r-z_{s+3} path and let $r_i^j = z_i^j \wedge z_{s+3}, i = 1, \ldots, s+2, j = 2, \ldots, k$. By Lemma 4.5, the vertices $z_i, r_i^2, \ldots, r_i^k, z_{i+1}$ lie on R in this order ($\prec$). For $i = 2, \ldots, s + 1$ we have $f(r_i^j) = a_j$, because either $z_i^j = r_i^j$ or they are distinct but then the tripod condition forces r_i^j to have the color a_j. In particular, all $r_i^j, j > 1$, and $z_i, i = 2, \ldots, s + 1$, are distinct. Thus, $\mathcal{T}$ contains $\mathcal{P}_{z(k,s)}$ and hence $\mathcal{P}$.

Case (b). We have 5^{s+4} incomparable vertices in W, we denote their set by Z. Let T_1 be the rooted subtree of T_r that is spanned by the smallest connected subset of vertices containing Z.

First we show there is no vertex in T_1 with more than five children. Otherwise there would be six vertices $z_1 < z_2 < \cdots < z_6$ in Z with color a_1 and a vertex $v \in V(T_1)$ such that $v \prec z_i$ and all six v-z_i paths merge at v. But there are another 5 vertices y_i, $z_1 < y_1 < z_2 < y_2 < z_3 < \cdots < z_6$, all with a different color a_2. Consider y_1, y_3, and y_5. Clearly, $v \prec y_1, y_3, y_5$ and all three v-y_i paths merge at v. There is a tripod with the central vertex v, which is a contradiction.

The leaves of T_1 are the vertices of Z. We call the vertices of T_1 with more than one child *branching* vertices. The tripod condition forces that each branching vertex that majorizes in $<$ a branching vertex must have color a_1. Since we have $|Z| = 5^{s+4}$ leaves in T_1 and each branching vertex has at most five children, there must be $s+3$ branching vertices v_i such that

- all have color a_1,
- $v_1 \prec v_2 \prec \cdots \prec v_{s+3}$,
- for each there is a vertex z_i in Z such that $z_i \wedge v_{s+3} = v_i$, and
- $z_1 < z_2 < \cdots < z_{s+2} < v_{s+3}$.

Again, for each $j = 2, \ldots, k$ there are vertices z_i^j

$$z_1 < z_1^2 < \cdots < z_1^k < z_2 < z_2^2 < \cdots < z_2^k < z_3 < \cdots < z_{s+2}^k < z_{s+3}$$

with color $f(z_i^j) = a_j$. Let $r_i^j = z_i^j \wedge v_{s+3}, i = 1, \ldots, s+2, j = 2, \ldots, k$. By Lemma 4.5, the vertices $v_i, r_i^2, \ldots, r_i^k, v_{i+1}$ lie on the r-v_{s+3} path in this order. For $i = 2, \ldots, s+1$, r_i^j must have color a^j. As in the case (a), the containment of $\mathcal{P}$ is forced and the proof of Theorem 4.1 is complete.

5 Some problems

The following result was proved in [1]. For any positive integer i,

$$\mathrm{Ex}(a^i b^i a^i b^i, n) = O(n).$$

Together with the lower bound $\mathrm{Ex}(ababa, n) = \Omega(n\alpha(n))$ due to Hart and Sharir [4], this completely characterizes the sequences u with $\|u\| \leq 2$ and $\mathrm{Ex}(u, n) = O(n)$: if $\|u\| \leq 2$ then $\mathrm{Ex}(u, n) = O(n)$ iff u does not contain *ababa*. We want to extend this to trees.

Problem 5.1 *Let i be a fixed positive integer. Prove*

$$\mathrm{Ex}_{\mathrm{T}}(a^i b^i a^i b^i, n) = O(n).$$

In [8, 10] (also in the first version of this paper) the problem was posed to prove $\mathrm{Ex}_\mathrm{T}(abba, n) = O(n)$. Recently P. Valtr [16] proved an even stronger result:

$$\mathrm{Ex}_\mathrm{T}(a^i b^i a^i, n) \leq 24in.$$

Thus,

$$5n - 8 \leq \mathrm{Ex}_\mathrm{T}(abba, n) \leq 48n,$$

where the lower bound follows from a simple construction due to Ch. Vogt.

Problem 5.2 *Determine* $\mathrm{Ex}_\mathrm{T}(abba, n)$ *and* $\mathrm{Ex}_\mathrm{T}(abba, n, k)$ *exactly.*

It is not difficult to show that $\mathrm{Ex}(abba, n) = 3n - 2$. In general, $\mathrm{Ex}(abba, n, k) = 2n - 1 + \lfloor (n-1)/(k-1) \rfloor$ ($n \geq k$, see [9]).

The last problem considers the first superlinear extremal function.

Problem 5.3 *Is it true that*

$$\mathrm{Ex}_\mathrm{T}(ababa, n) = \Theta(n\alpha(n))?$$

Of course,

$$\mathrm{Ex}_\mathrm{T}(ababa, n) \geq \mathrm{Ex}(ababa, n) = N_5(n) = \Omega(n\alpha(n)).$$

As to the upper bound, from the proof of Theorem 4.1 we know that $\mathrm{Ex}_\mathrm{T}(ababa, n) = O(N_t(n))$ for a big t, say $t = (5^9 - 1)7 + 1$. In fact, already $t = (5^7 - 1)5 + 1 = 390621$ works. With more work one could probably get a much better result; we did not try to optimize the value of $\hat{s}$ in (6).

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