Comments on a result of Trotter and Winkler in combinatorial probability

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We present an asymptotic upper bound and then an exact formula, both in elementary combinatorial probability. Trotter and Winkler have shown in [4], among other things, that in each sequence $A_1, A_2, \ldots, A_n$ of events in a probability space $\mathcal{P} = (\Omega, \mathcal{A}, \mathrm{Pr})$ there are two events A_i and A_j, $i < j$, such that $\mathrm{Pr}(A_i\overline{A}_j) < \frac{1}{4} + o(1)$; here $\frac{1}{4}$ is clearly best possible and the $o(1)$ error is with respect to $n \to \infty$.

A quick proof (different from the one in [4]) goes like this. Let σ_k be, as usual, the sum of probabilities

$$\sigma_k = \sum \mathrm{Pr}(A_{i_1} A_{i_2} \ldots A_{i_k})$$

taken over all k-subsets of $[n] = \{1, 2, \ldots, n\}$. It is well known that $\sigma_2 \geq \binom{\sigma_1}{2}$, and in general $\sigma_k \geq \binom{\sigma_1}{k}$ (this bound is not an optimum one, more about this later). Therefore if the A_i are equiprobable with $\mathrm{Pr}(A_i) = p$, we must have two, $i \neq j$, such that $\mathrm{Pr}(A_iA_j) \geq \binom{np}{2}/\binom{n}{2}$. Thus $\mathrm{Pr}(A_i\overline{A}_j) = \mathrm{Pr}(A_j\overline{A}_i) = p - \mathrm{Pr}(A_iA_j) \leq p - p^2 + \frac{p(1-p)}{n-1}$ and $\mathrm{Pr}(A_i\overline{A}_j) = \mathrm{Pr}(A_j\overline{A}_i) \leq \frac{1}{4} + \frac{1}{4(n-1)}$. In the general situation we apply this to some $\lfloor\sqrt{n}\rfloor$ events whose probabilities differ by at most $1/\sqrt{n}$ and obtain the T–W theorem, with $O(n^{-1/2})$ in place of $o(1)$. We sketch the proof of the following strengthening.

Theorem 1 *Among each n events $A_1, A_2, \ldots, A_n$ there are two, $i < j$, such that* $\mathrm{Pr}(A_i\overline{A}_j) < \frac{1}{4} + O(n^{-2/3})$.

Proof of Theorem 1 (Sketch). Using the argument with σ_2, we prove first a lemma saying that if $A_1, \ldots, A_m$ are events satisfying $|\mathrm{Pr}(A_i) - p| < \Delta$ for some $\Delta > 0$ and $0 \leq p \leq 1$, then $\mathrm{Pr}(A_i\overline{A}_j) < p - p^2 + \frac{1}{4(m-1)} + 6\Delta$ for some $i < j$. Then we define the function f as a constant $cn^{1/3}$ in $[\frac{1}{2} - n^{-1/3}, \frac{1}{2} + n^{-1/3}]$ and as $f(x) = cn^{-1/3}(x - 1/2)^{-2}$ in the rest of $[0, 1]$, where $c = 1/(4 - 4n^{-1/3})$. Clearly $\int_0^1 f(x)dx = 1$ and f is continuous. The pigeon hole principle tells us then that for any n events $A_1, \ldots, A_n$ there is an interval $I \subset [0, 1]$, $|I| = 2n^{-2/3}$, and an $x \in I$ such that $\mathrm{Pr}(A_i) \in I$ for at least $2f(x)n^{1/3}$ of the events. Finally, using the lemma with $m = 2f(x)n^{1/3}$ and $\Delta = 2n^{-2/3}$, we obtain the error term $O(n^{-2/3})$. □

If $i < j$ in the T–W theorem is replaced by $i \neq j$ then the best upper bound on $\min \mathrm{Pr}(A_i\overline{A}_j)$ can be determined exactly for each n. This follows easily from the instance $k = 2$ of our second result.

By $\lfloor\alpha\rfloor$ and $\{\alpha\}$ we denote the integral and the fractional part of α and by $(x)_k$ the product $x(x-1)\cdots(x-k+1)$. Let $[n]^k$ be all k-subsets of $[n]$.

Theorem 2 *For all triples n, k, p, $0 \le p \le 1$, we have*

$$\min \max_{X\in[n]^k} \Pr(\bigwedge_{i\in X} A_i) = \frac{(\lfloor pn\rfloor)_{k-1}(\lfloor pn\rfloor - k + 1 + k\{pn\})}{(n)_k} =: P(n,k,p),$$

where the minimum is taken over all $\mathcal{P}$ and n equiprobable events $A_1, \ldots, A_n$, $\Pr(A_i) = p$.

The proof uses the following bound.

Theorem 3 *We have the inequality*

$$\sigma_k \ge \binom{\lfloor\sigma_1\rfloor}{k-1}\sigma_1 - (k-1)\binom{\lfloor\sigma_1\rfloor+1}{k}.$$

Proof of Theorem 3 (Sketch). Set $m = \lfloor\sigma_1\rfloor$ in $\sigma_k \ge \binom{m}{k-1}\sigma_1 - (k-1)\binom{m+1}{k}$. The latter inequality reduces by the Rényi's 0-1 principle to an easily verifiable inequality for binomial coefficients. □

Proof of Theorem 2 (Sketch). That $\min\max \ge P(n,k,p)$ follows immediately from Theorem 3 as in the above proof of the T–W theorem. To prove $\min\max \le P(n,k,p)$ we define a $\mathcal{P}$ and events $A_1, \ldots, A_n$ such that $\Pr(A_i) = p$ for all i and $\Pr(A_{i_1}A_{i_2}\ldots A_{i_k}) = P(n,k,p)$ for all k-subsets of $[n]$. We set $m = \lfloor pn\rfloor$, $\Omega = [n]^m \cup [n]^{m+1}$, $\mathcal{A} = \exp(\Omega)$, $\Pr(A) = \sum_{\omega\in A} w(\omega)/\sum_{\omega\in\Omega} w(\omega)$, where the weight is $w(\omega) = 1$ on $[n]^m$ and $w(\omega) = \frac{m+1}{n-m}\cdot\frac{\{pn\}}{1-\{pn\}}$ (which is zero for integral pn) on $[n]^{m+1}$. Finally, $A_i = \{\omega \in \Omega : \ i \in \omega\}$. Straightforward calculations show that A_i and $A_{i_1}A_{i_2}\ldots A_{i_k}$ have the stated probabilities . □

One can derive from the formula in Theorem 2 that $P(n,k,p) = P(n+1,k,p)$ iff (i) $p(n+1)$ is an integer or (ii) $p \ge n/(n+1)$ or (iii) $p \le (k-1)/(n+1)$ or (iv) $k = 1$. The construction of $\mathcal{P}$ also shows that the inequality in Theorem 3 is best possible. For example, for $k = 2$ it improves $\sigma_2 \ge \binom{\sigma_1}{2} = (\sigma_1^2 - \{\sigma_1\} - \lfloor\sigma_1\rfloor)/2$ to $\sigma_2 \ge (\sigma_1^2 - \{\sigma_1\}^2 - \lfloor\sigma_1\rfloor)/2$.

As an interesting problem we mention the question what is the right order of magnitude of the error in the T–W theorem. The above example gives the $\gg 1/n$ lower bound but it is suited for the symmetric ($i \ne j$) case and can be probably improved in the asymetric ($i < j$) situation.

Another problem, in the spirit of [4]. If G is a graph on $[n]$, set $P(G,p) = \min\max\Pr(A_iA_j)$, where the max is taken over all edges $\{i,j\}$ of G and the min as above. It can be seen that the maximum value of p such that $P(G,p) = 0$ is $1/\chi^*(G)$, where $\chi^*(G)$ is the fractional chromatic number of G. What else can be said about the function $P(G,p)$?

A problem closely related to the case $k = 2$ of Theorem 2 was investigated already by Erdős, Neveu and Rényi in [1].

Final remark. I was informed by prof. J. Galambos that Theorems 2 and 3 are very close to some results in [2] and [3]. So it might be that these are already known results (in which case their authors have my apologies). When writing this extended abstract I had neither [2] nor [3] to my disposal and was not able to clarify this matter.

References

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