The Basel problem ($1+\frac{1}{4}+\frac{1}{9}+\cdots=\frac{\pi^2}{6}$) and the Riemann–Lebesgue lemma

Martin Klazar*

December 8, 2015

In this write-up we give a (detailed and self-contained) proof of the famous formula of L. Euler,

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\ .$$

It follows from the following integral representation of the series tail.

Theorem. *For all integers $n>0$,*

$$\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}=\frac{1}{2\pi}\int_0^{\pi}x(x-2\pi)\frac{\sin((n+\frac{1}{2})x)}{2\sin(x/2)}\,dx\to 0,\ n\to\infty\ .$$

Proof. We derive the identity and then prove the convergence to 0. From $\exp(ix)=\cos x+i\sin x$, $x\in\mathbb{R}$, and properties of the exponential function we get

$$\begin{aligned}1+2\sum_{k=1}^{n}\cos(kx)=\sum_{k=-n}^{n}\exp(ikx)&=\exp(-inx)\frac{\exp(i(2n+1)x)-1}{\exp(ix)-1}\\&=\frac{\exp(i(n+\frac{1}{2})x)-\exp(-i(n+\frac{1}{2})x)}{\exp(ix/2)-\exp(-ix/2)}\\&=\frac{2i\sin((n+\frac{1}{2})x)}{2i\sin(x/2)}\end{aligned}$$

and so

$$\frac{1}{2}+\sum_{k=1}^{n}\cos(kx)=\frac{\sin((n+\frac{1}{2})x)}{2\sin(x/2)},\ x\in\mathbb{R}\ ,$$

where for $x=2m\pi$, $m\in\mathbb{Z}$, the fraction $\frac{0}{0}$ is set to its limit value. We compute the integrals $I_k=\int_0^{\pi}x\cos(kx)\,dx$ and $J_k=\int_0^{\pi}x^2\cos(kx)\,dx$, $k\in\mathbb{N}$. Integration

*klazar@kam.mff.cuni.cz

by parts gives

$$\begin{aligned}
I_k &= [x\sin(kx)/k]_0^\pi - \frac{1}{k}\int_0^\pi \sin(kx)\,dx = \left[\cos(kx)/k^2\right]_0^\pi \\
&= \frac{(-1)^k-1}{k^2} \quad \text{and} \\
J_k &= \left[x^2\sin(kx)/k\right]_0^\pi - \frac{2}{k}\int_0^\pi x\sin(kx)\,dx \\
&= \left[2x\cos(kx)/k^2\right]_0^\pi - \frac{2}{k^2}\int_0^\pi \cos(kx)\,dx \\
&= \frac{2\pi(-1)^k}{k^2}\ .
\end{aligned}$$

The integral of the theorem therefore equals

$$\begin{aligned}
\int_0^\pi \dots &= \frac{1}{2}\int_0^\pi (x^2-2\pi x)\,dx + \sum_{k=1}^n J_k - 2\pi\sum_{k=1}^n I_k \\
&= -\frac{\pi^3}{3} + 2\pi\sum_{k=1}^n \frac{1}{k^2}
\end{aligned}$$

and the identity is proven.

To prove that for $n\to\infty$ the integral approaches 0 we write it as

$$\int_0^\pi x(x-2\pi)\frac{\sin((n+\frac{1}{2})x)}{2\sin(x/2)}\,dx = \int_0^\pi f(x)\sin((n+1/2)x)\,dx$$

where the function $f(x) = x(x-2\pi)/2\sin(x/2)$ is continuous on $[0,\pi]$ (it has a finite limit at 0). Thus the convergence to 0 follows from

the Riemann–Lebesgue lemma (type result). If $f:\ [a,b]\to\mathbb{R}$ is a continuous function then

$$\lim_{n\to\infty}\int_a^b f(x)\sin((n+1/2)x)\,dx = 0\ .$$

We prove it. If $f\equiv c$ is constant then

$$\left|\int_a^b \dots\right| = |c|\cdot\left|\left[\frac{-\cos((n+1/2)x)}{n+1/2}\right]_a^b\right| \le \frac{2|c|}{n+1/2}\ .$$

In general f is even uniformly continuous because $[a,b]$ is compact, and for given $\varepsilon>0$ there is a $\delta>0$ such that $x,y\in[a,b], |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$. We divide $[a,b]$ by points $a=a_0<a_1<\cdots<a_k=b$ into subintervals $I_i=[a_i,a_{i+1}]$ with lengths $|I_i| = a_{i+1}-a_i<\delta$. Then for $x\in I_i$ we have $f(x)=f(a_i)+\Delta_i(x)$

with $|\Delta_i(x)| < \varepsilon$. So

$$\begin{aligned}
\left|\int_a^b \dots\right| &\leq \sum_{i=0}^{k-1}\left|\int_{I_i}(f(a_i)+\Delta_i(x))\sin((n+1/2)x)\,dx\right| \\
&\leq \sum_{i=0}^{k-1}\left|\int_{I_i} f(a_i)\sin((n+1/2)x)\,dx\right| + \\
&\quad + \sum_{i=0}^{k-1}\left|\int_{I_i}\Delta_i(x)\sin((n+1/2)x)\,dx\right| \\
&\leq \sum_{i=0}^{k-1}\frac{2|f(a_i)|}{n+1/2} + \sum_{i=0}^{k-1}\varepsilon|I_i| = \frac{2}{n+1/2}\sum_{i=0}^{k-1}|f(a_i)| + \varepsilon(b-a) \\
&< \varepsilon(b-a+1),\ n > n_0\ .
\end{aligned}$$

Thus the integral goes to 0 for $n \to \infty$ and Euler's formula is proven. □

The above proof is somewhat expanded and modified proof of Moreno [1] who gives more than 80 references to various other proofs of Euler's formula. But wait, what have we actually proven? What is π? How is this number defined? As the root of six times the sum of reciprocal squares? Then we would just claim the triviality $A = A$. Strictly speaking, without specifying the definition of π it is not clear whether anything nontrivial was achieved at all. But, surely, it was. Reflection upon the above proof shows that it in fact proves the identity

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{6}\left(\inf(\{x \in (0,+\infty) \mid x - x^3/3! + x^5/5! - x^7/7! + \cdots = 0\})\right)^2\ .$$

References

[1] S. G. Moreno, A one-sentence and truly elementary proof of the Basel problem, ArXiv:1502.07667v1, 7 pages (2015).