Lecture 10. Densities. Limiting distributions of arithmetic functions

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In the tenth lecture we cover Chapters III.1. *Densities* and III.2 *Limiting distributions of arithmetic functions* in G. Tenenbaum's book [3], up to page 445.

Chapter III.1. Densities

The following are Theorems 1.1, 1.2 and 1.3 in [3]. For $a \in \mathbb{N} = \{1, 2, \dots\}$ we define $a\mathbb{N} = \{a, 2a, 3a, \dots\}$ and $[a] = \{1, 2, \dots, a\}$.

Theorem 1 *There is no probability measure P on $\mathbb{N}$ such that for any $a \in \mathbb{N}$ one has that $P(a\mathbb{N}) = 1/a$.*

Proof. We define $A^c := \mathbb{N} \setminus A$. If $a, b \in \mathbb{N}$ and $(a, b) = 1$ then $a\mathbb{N} \cap b\mathbb{N} = ab\mathbb{N}$ and the events $A_a := a\mathbb{N}$ and $A_b = b\mathbb{N}$ are independent. By a lemma in probability theory, so are their complements A_a^c and A_b^c: $P(A_a^c \cap A_b^c) = (1 - \frac{1}{a})(1 - \frac{1}{b})$. Thus by induction we obtain for any pairwise coprime numbers $a_1, \dots, a_n$ that $P\big(\bigcap_{i=1}^n A_{a_i}^c\big) = \prod_{i=1}^n (1 - \frac{1}{a_i})$. But then for any $m, n \in \mathbb{N}$,

$$P(\{m\}) \le \prod_{m<p\le n} \big(1 - 1/p\big) \to 0 \ \text{ for } \ n \to \infty\,.$$

So $P(\{m\}) = 0$ for any $m \in \mathbb{N}$, which is a contradiction. $\square$

By lim we mean $\lim_{n\to\infty}$ and similarly for lim inf and lim sup. For $A \subset \mathbb{N}$ let $\mathbf{d}A := \lim \frac{1}{n}|A \cap [n]|$, $\underline{\mathbf{d}}A := \liminf \frac{1}{n}|A \cap [n]|$ and $\overline{\mathbf{d}}A := \limsup \frac{1}{n}|A \cap [n]|$. Let $\delta A := \lim \frac{1}{\log n} \sum_{a\in A, a\le n} \frac{1}{a}$, and similarly for $\underline{\delta}A$ and $\overline{\delta}A$. The former is the *natural density* (resp. lower and upper), and the latter is the *logarithmic density* $(\dots)$.

Theorem 2 *For any $A \subset \mathbb{N}$ one has that $\underline{\mathbf{d}}A \le \underline{\delta}A \le \overline{\delta}A \le \overline{\mathbf{d}}A$. Hence if $\mathbf{d}A$ exists then so does δA and both densities are equal.*

Theorem 3 *For any $A \subset \mathbb{N}$,*

$$\lim_{\sigma \to 1^+} (\sigma - 1) \sum_{n \in A} \frac{1}{n^\sigma} = \delta A$$

– if one side exists, so does the other and the equality holds.

The limit on the left side is the *analytic density* of A.

Chapter III.2. Limiting distributions of arithmetic functions

Finally, the following are Theorem 2.1 (Lebesgue decomposition theorem), Definition 2.2, Theorems 2.3 and 2.4 (Continuity theorem, Lévy, 1925), Lemma 2.5 and Theorem 2.6 in [3].

A *distribution function*, DF, is any function $F\colon \mathbb{R} \to [0,1]$ that is non-decreasing, right-continuous and that has the limits $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$. If F is a step function, we call it *atomic.* If $F(z) = \int_{-\infty}^{z} h(t)\,\mathrm{d}t$ where $h(t) \geq 0$ is Lebesgue-integrable and has $\|h\|_1 = 1$ then we call F *absolutely continuous.* Finally, F is *purely singular* if it is continuous and $\int_N 1 \cdot \mathrm{d}F(t) = 1$ for a null set $N \subset \mathbb{R}$.

Theorem 4 *Each* DF F *has the unique decomposition*

$$F = \alpha_1 F_1 + \alpha_2 F_2 + \alpha_3 F_3$$

where the $\alpha_i \geq 0$ and sum up to 1*, every F_i is* DF*, F_1 is absolutely continuous, F_2 is purely singular and F_3 is atomic.*

For $f\colon \mathbb{N} \to \mathbb{R}$ and $N \in \mathbb{N}$ we define

$$F_N(z) := \tfrac{1}{N} |\{n \in \mathbb{N} \mid f(n) \leq z\}|\,.$$

It is an atomic DF.

Definition 5 *We say that $f\colon \mathbb{N} \to \mathbb{R}$ has a (limiting)* DF F *(or that it has a limit law with* DF F*) if the functions F_N converge weakly to the* DF F*, i.e. for any $z \in \mathbb{R}$ where F is continuous the limit $\lim_{N\to\infty} F_N(z) = F(z)$ holds.*

Theorem 6 *Let $f\colon \mathbb{N} \to \mathbb{R}$. Suppose that for every $\varepsilon > 0$ there is a function $a_\varepsilon\colon \mathbb{N} \to \mathbb{N}$ such that*

1. $\lim_{\varepsilon\to0^+} \limsup_{T\to+\infty} \overline{\mathbf{d}}\{n \mid a_\varepsilon(n) > T\} = 0$,
2. $\lim_{\varepsilon\to0^+} \overline{\mathbf{d}}\{n \mid |f(n) - f(a_\varepsilon(n))| > \varepsilon\} = 0$ *and*
3. *for every $a \in \mathbb{N}$ the density $\mathbf{d}\{n \mid a_\varepsilon(n) = a\}$ exists.*

Then f has a limit law.

By [3], this theorem is identical to Lemma A2 in [1].

For a DF F its *characteristic function* $\varphi\colon \mathbb{R} \to \mathbb{C}$, CHF, is

$$\varphi(\tau) = \int_{-\infty}^{+\infty} \mathrm{e}^{i\tau z}\, \mathrm{d}F(z)\,.$$

Theorem 7 *Let (F_n) be a sequence of distribution functions and (φ_n) be their characteristic functions. Then F_n converge weakly to a* DF *$F \iff \varphi_n \to \varphi$ (on $\mathbb{R}$) where φ is continuous at 0. Then φ is the characteristic function of F and $\varphi_n \rightrightarrows \varphi$ on any compact subset of $\mathbb{R}$.*

Paul Lévy (1886–1971) was a French probabilist ([2]).

Lemma 8 *If F is a* DF *and φ is its* CHF *then for any $z \in \mathbb{R}$ and $h > 0$,*

$$\tfrac{1}{h}\int_{z}^{z+h} F(t)\,\mathrm{d}t - \tfrac{1}{h}\int_{z-h}^{z} F(t)\,\mathrm{d}t = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \big(\tfrac{\sin(\tau/2)}{\tau/2}\big)^2 \mathrm{e}^{-i\tau z/h}\varphi(\tau/h)\,\mathrm{d}\tau\,.$$

Theorem 9 *$f\colon \mathbb{N} \to \mathbb{R}$ has a* DF *$F \iff$ the functions*

$$\varphi_N(\tau) := \tfrac{1}{N}\textstyle\sum_{n\le N} \mathrm{e}^{i\tau f(n)}$$

converge pointwisely on $\mathbb{R}$ to a function $\varphi(\tau)$ that is continuous at 0. Then φ is the CHF *of F.*

References

[1] R. R. Hall and G. Tenenbaum, *Divisors*, CUP, Cambridge 1988

[2] P. Lévy, *Calcul des probabilités*, Gauthier-Villars, Paris 1925

[3] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, AMS, Providence, RI 2015 (Third Edition. Translated by Patrick D. F. Fion)