Midsummer Combinatorial Workshop 2010

Martin Tancer (ed.)

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Preface

The 16th Prague Midsummer Combinatorial Workshop was held from July 26th to July 30th 2010 in our beautiful building Malostranské náměstí 25. This of course contributed to the comfort of the participants as all the activities (including the lunches) could be taken on the same site. Besides, as it was expressed by several participants, the renovated faculty building surely belongs to the most beautiful math and computer science departments in the world! The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with DIMATIA and ITI. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Peter J. Cameron among the participants. The list of speakers is included in this booklet.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students. For example six undergraduate students from the USA and six undergraduate students from Charles University, together with their mentors Aaron D. Jaggard, David Duncan from US side and Bernard Lidický from Prague side took part in the workshop, within the DIMATIA-DIMACS program International REU (supported jointly by NSF and Czech Ministry of Education ME 09074).

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest some of the atmosphere at the workshop from these proceedings, and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

This volume was edited by Martin Tancer. Most of the contributions were supplied by the authors in an electronic form. In a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

We gratefully acknowledge financial support of Czech research projects 1M0545 and MSM0021620838.

We hope to meet again in 2011 the same midsummer week!

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Figure 1: A photo of the locality where was the workshop held.

Cores, hulls and synchronization

Peter J. Cameron

The topic of *synchronization* arose originally in automata theory, turned into permutation group theory, and ended up connecting to graph homomorphisms. One thing that came from this is the notion of the "hull" of a graph, a concept which is in some way dual to the core.

This material is covered in much more detail in notes from an intensive course I gave recently. The notes are available from

`http://www.maths.qmul.ac.uk/~pjc/LTCC-2010-intensive3/`

1 Homomorphisms

I will be brief here since anything you want to know about graph homomorphisms can probably be found in the work of Jarik Nešetřil. All graphs here will be simple, and finite except in the final section.

A *homomorphism* between graphs is a map from the vertex set of one to that of the other which maps edges to edges. Two graphs are *homomorphism-equivalent* (or *hom-equivalent*, for brevity) if there are homomorphisms in both directions between them. Given a finite graph X, a *core* of X is a graph which has the minimal number of vertices among all graphs hom-equivalent to X. It is well-known that any finite graph has a core, unique up to homomorphism, which we denote by $\mathrm{Core}(X)$. If $X = \mathrm{Core}(X)$, we say that X is a core.

We remark that the core of X is the complete graph K_m if and only if $\omega(X) = \chi(X) = m$, where ω and χ are the clique number and chromatic number respectively.

It is known that the core of a vertex-transitive graph is vertex-transitive. The proof extends unchanged to other sorts of transitivity (edge-, nonedge-, etc.) with the proviso that the core may contain none of the objects in question (e.g. the core of a non-edge-transitive graph may be complete). One of the consequences of this work is:

Theorem 1.1. *If X is nonedge-transitive, then either X is a core, or the core of X is complete.*

For any finite graph X, we define the *hull* of X to be the graph $\mathrm{Hull}(X)$ which has the same vertex set as X, where vertices v and w are joined in $\mathrm{Hull}(X)$ if and only if there is no endomorphism f of X satisfying $v^f = w^f$.

Proposition 1.2. *(a) X is a spanning subgraph of* $\mathrm{Hull}(X)$.

(b) $\mathrm{End}(X) \leq \mathrm{End}(\mathrm{Hull}(X))$ *and* $\mathrm{Aut}(X) \leq \mathrm{Aut}(\mathrm{Hull}(X))$.

(c) $\mathrm{Core}(\mathrm{Hull}(X))$ *is a complete subgraph on the vertices of* $\mathrm{Core}(X)$.

For example, if X is the path of length 3, then $\mathrm{Hull}(X)$ is the cycle of length 4; the number of automorphisms increases from 2 to 8.

Proof of the Theorem Let X be nonedge-transitive. Then $\mathrm{Hull}(X)$ consists of X with possibly some orbits of nonedges changed to edges. There is only one such orbit, so $\mathrm{Hull}(X) = X$ or $\mathrm{Hull}(X)$ is complete.

In the first case, $\mathrm{Core}(X) = \mathrm{Core}(\mathrm{Hull}(X))$ is complete. In the second, $\mathrm{Core}(\mathrm{Hull}(X))$ has all the vertices of X, and hence so does $\mathrm{Core}(X)$; so X is a core.

2 Synchronization

A (finite deterministic) *automaton* is a set Ω of *states* with a set of *transitions*, each of which is a function on the set of states. Combinatorially we can regard it as an edge-coloured directed graph in which there is just one edge of each colour leaving each vertex. We can compose arbitrary sequences of transitions; so algebraically, an automaton is a transformation monoid on the set of states, with a distinguished set of generators.

A *reset word* is a word in the generators (the transitions) which takes the automaton to a fixed state from any starting state – that is, which evaluates to a constant function. The *Černý conjecture* from 1968, one of the oldest conjectures in automata theory, asserts that, if an n-state automaton has a reset word, then it has one of length at most $(n-1)^2$. (If true, this would be best possible.)

An approach to the conjecture devised by João Araújo and Ben Steinberg begins with the observation that the transitions which are permutations generate a permutation group which contains all elements of the corresponding monoid which are permutations. Now, if the addition of a non-permutation forces a reset word, there is some hope of using group-theoretic techniques to bound its length.

Accordingly, we say that a permutation group G on Ω is *synchronizing* if, whenever f is a transformation of Ω which is not a permutation, the monoid $\langle G, f\rangle$ contains a reset word (a transformation of rank 1).

Proposition 2.1. *(a) A synchronizing group is primitive.*

(b) A 2*-set transitive group is synchronizing.*

(c) Neither implication reverses.

So a new and interesting property has been added to the hierarchy of permutation group properties!

The next theorem gives the only practical test for this property. A graph is *trivial* if it is complete or null.

Theorem 2.2. *Let G be a permutation group on Ω. Then G is non-synchronizing if and only if there is a non-trivial G-invariant graph X on the vertex set Ω whose core is complete.*

Proof For the reverse implication, take f to be an endomorphism of X which is not an automorphism. Then $\langle G, f\rangle \leq \mathrm{End}(X)$ contains no constant function.

For the reverse, use the idea in the definition of a hull: if $\langle G, f\rangle$ contains no constant function, define the graph X by the rule that v is adjacent to w if and only if there is no element $h \in \langle G, f\rangle$ with $v^f = w^f$.

So the algorithm for testing the synchronization property of G is: list all the nontrivial G-invariant graphs; for each of them, test whether its clique number and chromatic number are equal. This is not a fast algorithm: there may be exponentially many graphs to check, and for each of them, we have a hard problem to solve. But in practice, the property can be checked for permutation groups with degrees in the thousands.

3 The infinite

What happens to these concepts in the infinite case?

To begin, despite the work of Bauslaugh, I believe that there is no satisfactory theory of infinite cores. Fortunately, for the other concepts, things are a bit better.

The usual definition of synchronization makes sense but is not interesting. If we adjoin a map f which is either surjective or injective, we will never generate a constant function, even if G is the symmetric group. Because synchronization involves collapsing points to the same place, we should presumably add a map which is not injective. Now in the finite case, a transformation monoid contains a map of rank 1 if and only if any two distinct points can be mapped to the same place by some element of the monoid. Accordingly we make the following definition:

The permutation group G on an infinite set Ω is *synchronizing* if and only if, whenever f is a non-injective transformation of Ω, it holds that for any two points v, w of Ω, there exists $h \in \langle G, f\rangle$ such that $v^h = w^h$.

Unfortunately we get nothing interesting in the countable case:

Proposition 3.1. *Let G be a permutation group of countable degree. Then G is synchronizing if and only if it is 2-set transitive.*

The essence of the proof is the forward implication. Suppose that G is not 2-set transitive. Then there is a non-trivial G-invariant graph X. By Ramsey's Theorem, replacing X by its complement if necessary, we can assume that X contains an infinite clique. Now it is a fairly simple exercise to find an endomorphism of X which collapses two non-adjacent vertices.

Some variations of the concept have been tried, but no really satisfactory definition is known. However, there is an interesting open problem here: is the proposition true for permutation groups of uncountable degree? (Of course the application of Ramsey's Theorem fails.)

For hulls, we could start with the usual definition: v and w are joined in $\mathrm{Hull}(X)$ if and only if there is no element $f \in \mathrm{End}(X)$ with $v^f = w^f$. Then

(a) every countable graph containing an infinite clique is a hull (arguing as above);

(b) if X is a hull and $\omega(X) < \infty$ then $\omega(X) = \chi(X)$.

4 Problems

The first of these problems was presented at the problem session. I am happy for the others to be included as well, if the editor thinks it worthwhile.[1]

[1] The editor happily agrees.

Problem 1 This is an attempt to define edge density for an infinite graph. Let G be an infinite graph. Let $\mathcal{G}_n$ be the set of graphs on the vertex set $\{1, 2, \ldots, n\}$ (that is, labelled graphs) which are embeddable in G. Let d_n be the proportion of graphs in $\mathcal{G}_n$ in which the vertices 1 and 2 are adjacent.

Does d_n tend to a limit as $n \to \infty$?

If so, then this limit is the "edge density" of G.

Notes:

- The edge density (and the convergence question) only depends on the age of G (the class of finite graphs embeddble in G).
- One can replace an edge by an arbitrary graph H on the vertex set $\{1, 2, \ldots, k\}$; simply redefine d_n to be the proportion of graphs in $\mathcal{G}_n$ which induce H on $\{1, \ldots, k\}$.
- If these limits exist for all finite graphs, then one can define a probability measure on the class of countable graphs whose age is contained in that of G, by the following rule: take a countable set of vertices; decree that, for any vertices $v_1, \ldots, v_k$, the probability that the map $i \mapsto v_i$ for $i = 1, \ldots, k$ is an embedding of H in the random graph is $\lim_{n\to\infty} d_n$. Unlike most probability measures for such graphs, this one does not depend on the order in which vertices or edges are considered.

Problem 2 Consider a class of discrete minimization problems. Suppose that the cost of a solution is equal to the amount by which the objective function exceeds its true minimum value, and that the cost of computation is ϵ times the number of Turing machine steps reqired. What is the solution that minimizes the sum of these two costs? In particular, how does it depend on ϵ?

Problem 3

(a) Is there a description of the random graph R in which the vertices and edges are computable but we cannot proe in Peano arithmetic that the graph is R (because the witnesses are not computable functions of the data)?

(b) Consider the graph with vertex set $\mathbb{Z}$, in which x and y are joined if and only if, for $n = |x - y|$, the n-th odd prime is congruent to 3 (mod 4). Is this the random graph?

Problem 4 The theorem of Engeler, Ryll-Nardzewski and Svenonius asserts that a complete theory in a countable first-order language has the property that all its countable models are isomorphic if and only if the automorphism grou of a countable model of T is *oligomorphic* (that is, has only finitely many orbits on n-tuples for all n.

Is there an analogous result for homomorphisms? That is, is there a property of endomorphism monoids sch that this property of $\mathrm{End}(M)$ is equivalent to the assertion that all contable models of the theory of M are hom-equivalent? What about monomorphisms?

Problem 5 Take a network in which each edge has unit capacity. At each time step, perform the following operation:

(a) choose a random maximal flow f in the network;

(b) choose a random edge e;

(c) with probability $1 - f(e)$, the edge e "silts up" and is removed.

Eventually we are left with only those edges lying in minimal cuts, each carrying flow 1 in a maximal flow. What can be said about the distribution of the time for this to happen?

Complexity of contractions to paths on claw-free graphs

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Joint work with Daniël Paulusma[1].

As an open problem we ask, what is the computational complexity of the problem whether a claw-free graph can be contracted to a path of length four.

This result would be interesting due to the following facts:

It is well known that the decision whether a fixed graph H is a minor of a given graph G is solvable in polynomial time for general graphs [Robertson, Seymour]. Hence it makes sense to study graph manipulations derived from vertex and edge deletions, and edge contractions.

If edge contractions are forbidden, these problems become variations of the subgraph problem, which are trivially polynomial as long as H is fixed.

If only contractions and vertex deletions are permitted, i.e. we seek for an induced minor, it is known that there exist a particular graph H on about 50 vertices for which the problem is NP-complete [Fellows et al.]. On the other hand, it is easy when H is a path (not necessarily fixed).

It is known that for $H = P_4$ the contraction problem is NP-complete for general G [Brower et al.], while we can show that it becomes polynomially solvable when G is claw-free. This construction can be extended to any H formed from a clique and vertices of degree one. Our construction also extends to the induced minor problem.

On the other hand, if $H = P_7$ and G is a line graph, we can prove that the problem becomes NP-complete. Hence the twist in the computational copmplexity appears for some short path length. We would like to resolve the mising cases of $H = P_5$ and $H = P_6$ to determine it precisely when the twist happens.

[1]Durham University, UK

The flip graph of unique-sink orientations

Jan Foniok

A *hypercube* of dimension n is the graph Q_n with vertex set $V_n = \{0,1\}^n$ and an edge connecting any two vertices that differ in exactly one entry. Define $v \oplus C$ by

$$(v \oplus C)_i = \begin{cases} 1 - v_i & \text{if } i \in C \\ v_i & \text{if } i \notin C \end{cases}$$

for any $v \in V_n$ and $C \subseteq \{1, \ldots, n\}$. A *subcube* of Q_n is then any induced subgraph with vertex set $\{v \oplus D : D \subseteq C\}$ for some $v \in \{0,1\}^n$ and $C \subseteq \{1, \ldots, n\}$. Any vertex v of a subcube determines the same subcube (with the same C); the dimension of the subcube is $|C|$.

A *unique-sink orientation* (USO) is an orientation of Q_n such that every subcube has a unique vertex of out-degree zero; its *sink*. Unique-sink orientations of hypercubes are used for studying algorithms for various problems such as linear programming, linear complementarity problems, and the smallest enclosing ball of points or balls. The original problem is always reduced to the problem of finding the unique sink of a hypercube.

The object of our interest is the *flip graph of unique-sink orientations*: in a fixed dimension n, the vertices of the flip graph are all (labelled) unique-sink orientations of n-dimensional hypercubes; an edge connects two orientations differing in the direction of exactly one edge.

We hope that understanding the structure of the flip graph might bring new insights into the algorithmic questions. For instance, how much does the behaviour of an algorithm differ on two USOs that can be obtained from one another by flipping few edges (and thus there is a short path between them in the flip graph)?

In particular, we are interested in the following questions:

- Is the flip graph connected?
- Are there isolated vertices?
- What is the diameter of the flip graph?
- How far can a USO be form a *uniform orientation*, in which all edges are oriented “from 0 to 1”?

With K. Fukuda we supervised a bachelor student U. Müller, who was able to establish some facts.

- For dimensions $n = 2, 3$, the flip graph is connected.
- In dimension 2, the distance of any two vertices in the flip graph is equal to their Hamming distance (the number of differing edges), and thus the diameter is 4.
- In dimension 3, the above is not true. The diameter is at most 24.

Moreover, we have an interesting characterisation of "flippability".

Lemma 1.1. *Let Φ be a unique-sink orientation of the hypercube Q_n, $n \geq 2$, and let e be an edge. If the orientation obtained from Φ by reversing e is not a unique-sink orientation, then there exists a 2-dimensional subcube Ψ containing e such that the orientation obtained from Ψ by reversing e is not a unique-sink orientation.*

The rest of the questions remain open.

Partial Representation Extension

Pavel Klavík
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Joint work with Jan Kratochvíl and Tomáš Vyskočil.

Abstract

Recognition problems (Recog) are well studied in graph theory. For a fixed class $\mathcal{C}$, Recog($\mathcal{C}$) asks whether a given graph belongs to $\mathcal{C}$. For example, recognition of interval graphs (**INT**) can be done in a linear time. We introduce a new problem called *partial representation extension* (PRExt). For a given graph and a part of its representation fixed, it asks whether this representation can be extended to the whole graph. We show that interval graphs can be extended in time $\mathcal{O}(n^2)$.

In this talk, we consider only intersection representations of graphs. An *intersection representation* assigns sets to vertices in such a way that two vertices are adjacent if and only if the corresponding sets intersect. Classes of intersection graphs restrict these sets. For example, interval graphs (**INT**) are graphs with representations by (closed) intervals of the real line. A *partial representation* assigns sets to some vertices of a graph. We consider the following problem:

Problem:	Partial Representation Extension – PRExt($\mathcal{C}$).
Input:	A graph G with a partial representation R.
Output:	A representation of G extending R if exists, no otherwise.

Theorem 1.1. *The problem* PRExt(**INT**) *is polynomially solvable in time* $\mathcal{O}(n^2)$.

We sketch the proof. To solve PRExt(**INT**), we modify the PQ-tree Algorithm of Booth and Lueker [1]. This algorithm is based on the following characterization of interval graphs, due to Fulkerson and Gross [2]:

Lemma 1.2. *A graph is an interval graph if and only if there exists an ordering of the maximal cliques such that for every vertex the cliques containing this vertex appear consecutively in this ordering.*

Using PQ-trees, such an ordering is found if it exists. For an ordering, we place points representing maximal cliques on the real line. These points are called *clique-points*. Using clique-points, we construct the representation, see Figure 1.1.

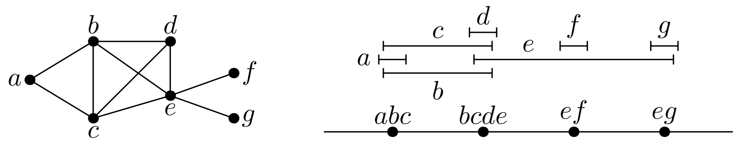

Figure 1.1: An interval graph and its representation with the corresponding ordering of maximal cliques. The clique-points are placed and each interval is placed between clique-points containing this interval.

The key observation is that a partial representation gives a partial ordering of clique-points. Intervals given by a partial representation split the real line to several parts, see Figure 1.2. For a clique-point a, we denote $I(a)$ the represented intervals contained in this clique. A clique-point a can be placed only to a part of the real line containing exactly represented intervals from $I(a)$. By $\curvearrowleft(a)$ (resp. $\curvearrowright(a)$) we denote the leftmost (resp. the rightmost) point of the real line where the clique-point a can be placed. We obtain a natural ordering $\blacktriangleleft$ of the maximal cliques:

$$\blacktriangleleft = \bigl\{(a,b) \mid \curvearrowright(a) \le \curvearrowleft(b)\bigr\}.$$

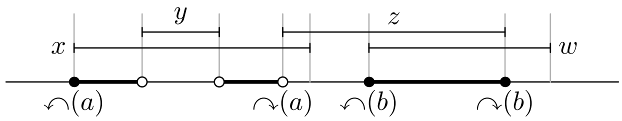

Figure 1.2: Clique-points a and b, having $I(a) = \{x\}$ and $I(b) = \{z, w\}$, can be placed to the bold parts of the real lines. We obtain $a \blacktriangleleft b$.

The algorithm works in the following way:

(1) We find maximal cliques and construct a PQ-tree, independently of the partial representation.

(2) We compute $\curvearrowleft$ and $\curvearrowright$ for all the cliques and construct $\blacktriangleleft$.

(3) We search the PQ-tree and find an ordering of the cliques extending $\blacktriangleleft$.

(4) We place the clique-points greedily on the real line.

(5) Using the clique-points, we construct a representation.

Surprisingly, the ordering $\blacktriangleleft$ is sufficient to solve $\textsc{PRExt}(\mathbf{INT})$:

Lemma 1.3. *For an ordering of the cliques compatible with the PQ-tree extending $\blacktriangleleft$, the greedy algorithm in Step 4 never fails.*

We omit a proof of the above lemma. The algorithm can be easily implemented in $\mathcal{O}(n^2)$.

We conclude with two open problems. Two famous subclasses of interval graphs are studied. Proper interval graphs (**PROPER INT**) are interval graphs with a representation such that no interval is a proper subset of another interval. Unit interval graphs (**UNIT INT**) can be represented by intervals of a unit length. A well-known theorem of Roberts [3] shows that **PROPER INT** = **UNIT INT**.

Surprisingly, the Partial Representation Extension Problem distinguishes **PROPER INT** and **UNIT INT**. By another modification of PQ-trees, the problem $\textsc{PRExt}(\mathbf{PROPER\ INT})$ can be solved in time $\mathcal{O}(mn)$. The complexity of $\textsc{PRExt}(\mathbf{UNIT\ INT})$ remains open.

The other open problem is to solve $\textsc{PRExt}(\mathbf{INT})$ faster. We believe that a linear time algorithm can be constructed.

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Towards duality of multicommodity multiroute cuts and flows: Multilevel ball-growing

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Joint work with Christian Scheideler.

An *elementary h-route flow*, for an integer $h \geq 1$, is a set of h edge-disjoint paths between a source and a sink, each path carrying a unit of flow, and an *h-route flow* is a non-negative linear combination of elementary h-route flows. An *h-route cut* is a set of edges whose removal decreases the maximum h-route flow between a given source-sink pair (or between every source-sink pair in the multicommodity setting) to zero. The main result of this contribution is an approximate duality theorem for multicommodity h-route cuts and flows, for $h \leq 4$: The size of a minimum h-route cut is at least f/h and at most $O(\log^2 k \cdot f)$ where f is the size of the maximum h-route flow and k is the number of commodities. The main step towards the proof of this duality is the design and analysis of a polynomial-time approximation algorithm for the minimum h-route cut problem for $h \leq 4$ that has an approximation ratio of $O(\log^2 k)$. Previously, polylogarithmic approximation was known only for h-route cuts for $h \leq 2$. A key ingredient of our algorithm is a novel rounding technique that we call multilevel ball-growing. Though the proof of the duality relies on this algorithm, it is not a straightforward corollary of it as in the case of classical multicommodity flows and cuts. Similar results are shown also for the sparsest multiroute cut problem.

Linkless and flat embeddings in the 3-space

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Joint work with Ken-ichi Kawarabayashi[1] and Bojan Mohar [2] .

Abstract

We consider piecewise linear embeddings of graphs in the 3-space $\mathbb{R}^3$. Such an embedding is *linkless* if every pair of disjoint cycles forms a trivial link (in the sense of knot theory). Robertson, Seymour and Thomas [3] showed that a graph has a linkless embedding in $\mathbb{R}^3$ if and only if it does not contain as a minor any of seven graphs in Petersen's family (graphs obtained from K_6 by a series of $Y\Delta$ and ΔY operations). They also showed that a graph is linklessly embeddable in $\mathbb{R}^3$ if and only if it admits a *flat embedding* into $\mathbb{R}^3$, i.e. an embedding such that for every cycle C of G there exists a closed 2-disk $D \subseteq \mathbb{R}^3$ with $D \cap G = \partial D = C$. Clearly, every flat embedding is linkless, but the converse is not true. We consider the following algorithmic problem associated with embeddings in $\mathbb{R}^3$:

Flat Embedding: For a given graph G, either detect one of Petersen's family graphs as a minor in G, or return a flat (and hence linkless) embedding of G in $\mathbb{R}^3$.

The first outcome is a certificate that G has no linkless and no flat embeddings. Our main result is to give an $O(n^2)$ algorithm for this problem. While there is a known polynomial-time algorithm for constructing linkless embeddings [1], this is the first polynomial time algorithm for constructing flat embeddings in the 3-space. This settles a problem proposed by Lovász [2].

An extended abstract of this work has appeared at the 2010 Symposium on Computational Geometry (SOCG).

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Extending fractional precolorings

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Joint work with Daniel Kráľ, Matjaž Krnc, Borut Lužar, and Jan Volec.

Let G be a k-colorable graph. Consider an independent set $I \subseteq V(G)$ with minimum distance between its vertices at least d. A question asked by Thomassen was: "Does there exist d such that every precoloring of I with $k+1$ colors can be extended to a proper $(k+1)$-coloring of G?" Albertson[1] answered the question in affirmative with $d = 4$ which is optimal.

In [2], Albertson and West investigated extensions of circular colorings and gave optimal bounds for most choices of the parameters. Here, we focus on extensions of fractional colorings.

Definition 1.1. Let I be an interval of length d. A fractional d-coloring of graph G is a map $f : V(G) \to \mathcal{M}(I)$ such that

$$\forall x \in V(G)\ \mu(f(x)) \geq 1$$

and

$$\forall x, y \in E(G)\ f(x) \cap f(y) = \emptyset$$

where μ is the Lebesgue measure.

Our main result asserts the following.

Theorem 1.2 (Fractional coloring extention)**.** *Let G be a graph with fractional chromatic number χ, P an independent set in G and d the minimum distance between two vertices in P. If $d \geq 4$, then every fractional $(\chi+\varepsilon)$-precoloring of P can be extended to a fractional $(\chi+\varepsilon)$-coloring of G, where ε satisfies the following inequalities:*

$$\begin{array}{rccc}
d = 0 \texttt{ mod } 4: & \frac{1}{\chi+\varepsilon} & \geq & 1 - \frac{d}{4}\varepsilon \\
d = 1 \texttt{ mod } 4: & \frac{1}{\chi} & \geq & 1 - \frac{d-1}{4}\varepsilon \\
d = 2 \texttt{ mod } 4: & \frac{\chi-1}{\chi+\varepsilon} & \leq & \frac{d-2}{4}\varepsilon \\
d = 3 \texttt{ mod } 4: & \frac{(1-\varepsilon)(1-\chi)}{\chi} & \leq & \frac{d-3}{4}\varepsilon
\end{array}$$

For $\chi = 2$ and $\chi \geq 3$, the value of ε is the best possible.

Let us now briefly sketch the main idea of the proof. Our proof is based on a notion of universal graphs. For fractional colorings, the universal graphs are "fractional cliques" which are Kneser graphs.

Definition 1.3 ((p, q)-clique graph)**.** Let $p \geq q$ be positive integers and I an interval of length p/q divided into p subintervals of length $1/q$. A (p, q)-clique is a graph G where each vertex corresponds to a union of q of the subintervals and two vertices are adjacent if and only if all the subintervals corresponding to them are mutually different.

The notion of (p, q)-cliques is then modified to provide universal graphs for precolorings. Such a universal graph consists of a pool (p, q)-clique with several chains of (p, q)-cliques leading to precolored vertices attached. The length of the chains is proportional to d. The crucial property is that for every fractioanlly (p, q)-colorable graph with some vertices precolored can be homomorphically mapped to this graph (providing the number of chains is sufficient) in such a way that the precolored vertices are mapped to precolored vertices of the universal graph. The structure of the universal graphs is then exploited to provide both lower and upper bounds on ε.

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On computational complexity of homomorphism-homogeneity

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1 Introduction

A structure is *homogeneous* if every isomorphism between finite substructures of the structure extends to an automorphism of the structure. The theory of (countable) homogeneous structures gained its momentum in 1953 with the famous theorem of Fraïssé [9] which states that countable homogeneous structures can be recognized by the fact that their collections of finitely induced substructures have the amalgamation property. Nowdays it is a well-established theory with deep consequences in many areas of mathematics.

Homogeneous objects have been determined for many important classes of structures. For example, countably infinite homogeneous posets were characterized in [20]; countably infinite homogeneous graphs were described in [14], while the finite ones were determined in [10]; countably infinite homogeneous digraphs were described in [4] while finite and countably infinite homogeneous tournaments were described in [13]. As in this paper we are particularly interested in finite geometries, let us finally mention that homogeneous linear spaces were characterized in [6], and homogeneous semilinear spaces in [5].

In their recent paper [3] the authors discuss a variant of homogeneity with respect to various types of morphisms of structures, and in particular introduce the notion of homomorphism-homogeneous structures:

Definition 1.1 (Cameron, Nešetřil [3])**.** A structure is called *homomorphism-homogeneous* if every homomorphism between finite substructures of the structure extends to an endomorphism of the structure.

Not much is known about homomorphism-homogeneous structures. Homomorphism-homogeneous posets were characterized in [15] and the characterization of countable posets with respect to various types of morphisms can be found in [2]. Finite homomorphism-homogeneous tournaments (with loops) were characterized in [11], a some classes of finite point-line geometries in [16]. Moreover, a model-theoretic approach can be found in [3] as well as in [18].

2 Graphs

Finite homomorphism-homogeneous graphs without loops were characterized in [3]:

Theorem 2.1. *A finite graph G with no loops is homomorphism-homogeneous if and only if $G \cong k \cdot K_n$ for some $k, n \geq 1$.*

This result was slightly improved in [17]:

Theorem 2.2. *Let $D = (V, \rho)$ be a finite irreflexive binary relational system. Then D is homomorphism-homogeneous if and only if*

- *$D \cong k \cdot K_n$ for some $k, n \geq 1$, or*
- *$D \cong k \cdot C_3$ for some $k \geq 1$,*

where C_3 denotes the oriented 3-cycle.

The situation concerning graphs with loops is much more involved.

Theorem 2.3 ([19])**.** *Deciding whether a finite graph with loops is homomorphism-homogeneous is a coNP-complete problem.*

Using the same strategy as in [19] the following two related results were shown in [12]:

Theorem 2.4. (a) *Deciding whether a finite metric space is homomorphism-homogeneous is a coNP-complete problem.*

(b) *Deciding homomorphism-homogeneity of a finite point-line geometry in which no two thick lines (= lines with ≥ 3 points) intersect is a coNP-complete problem.*

(Interestingly, both the countable Urysohn's space U and its completion U^* are homomorphism-homogeneous [7].)

From [19] it follows that if a graph G with loops is *not* homomorphism-homogeneous, then there exists an obstacle in the following sense. We say that a subgraph H of G *has a cone in* G if there exists a vertex v in G which is adjacent to every vertex of H. We say that H is an *obstacle* in G if there exists an embedding $e : H \to G$ and an injective homomorphism $j : H \to G$ such that $e(H)$ has a cone in G and $j(H)$ does not.

Let $\mathcal{K}$ be a class of finite graphs, let $\mathcal{O}$ be the class of all the obstacles that appear in $\mathcal{K}$ and assume that both $\mathcal{K}$ and $\mathcal{O}$ are maximal with respect to each other in the following sense:

- if G is in $\mathcal{K}$, then all the obstacles of G, if any, belong to $\mathcal{O}$;
- if G is a graph such that all the obstacles of G, if any, belong to $\mathcal{O}$, then $G \in \mathcal{K}$.

Then it is easy to see the following:

Fact 2.5. *If there are only finitely many isomorphism types in $\mathcal{O}$ then deciding whether a graph from $\mathcal{K}$ is homomorphism-homogeneous is in P.*

Problem 2.6. *If there are infinitely many isomorphism types in $\mathcal{O}$ then deciding whether a graph from $\mathcal{K}$ is homomorphism-homogeneous is coNP-complete.*

3 Algebras

A notion related to homomorphism-homogeneity has been closely investigated for algebras. An algebra A is *quasi-injective* if every homomorphism $f : S \to A$ from a subalgebra S of A into A extends to an endomorphism of A. It is easy to see that if A is a finite algebra, then A is quasi-injective iff A is homomorphism-homogeneous. Finite quasi-injective (= homomorphism-homogeneous) groups have been characterized in 1979 by Bertholf and Walls [1].

Homomorphism-homogeneous lattices have been described in [8]:

Theorem 3.1. *A lattice L is homomorphism-homogeneous if and only if it is either a chain, or every interval of L is a boolean lattice.*

Corollary 3.2 ([8])**.** *A finite lattice L is hom-hom if and only if it is either a chain, or a direct power of* $0 < 1$.

In contast to that, it has been shown in [15] that every lattice $(L, \leq)$ understood as a relational structure is homomorphism-homogeneous.

The characterization of homomorphism-homogeneous semilattices is still an open problem.

Lemma 3.3 ([8])**.** *Let S be a hom-hom semilattice. Then S is either a tree, or it is locally bounded.*

Consequently, if S is a finite hom-hom semilattice, then it is either a tree, or the $\wedge$-semilattice reduct of a lattice.

Proposition 3.4 ([8])**.** *Every tree is a hom-hom semilattice.*

Theorem 3.5 ([8])**.** *Let $(L, \wedge, \vee)$ be a distributive lattice. Then $(L, \wedge)$ is a hom-hom semilattice.*

However, there are nondistributive lattices whose reducts are nevertheless homomorphism-homogeneous.

Lemma 3.6 ([8])**.** *$(N_5, \wedge)$ and $(M_3, \wedge)$ are homomorphism-homogeneous.*

Problem 3.7. *Characterize homomorphism-homogeneous semilattices.*

Alternatively, using the strategies from [19], show that deciding homomorphism-homogeneity of finite semilattices is coNP-complete.

Note that it was shown in [7] that the universal homogeneous countable semilattice is homomorphism-homogeneous.

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Also sprach Gromov: On finding heavily covered points

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First we formulate an extremal graph-theoretic problem. Let $G = (V, E)$ be a graph on n vertices. An *odd triple* of G is a triple $\{u, v, w\}$ of vertices such that the subgraph of G induced by $\{u, v, w\}$ has an odd number of edges (that is, it is a triangle or a single edge plus an isolated vertex). We say that G is *minimal* if, for every partition of the vertex set V into subsets A and $V \setminus A$, the number of edges of G going between A and its complement is at most $\frac{1}{2}|A| \cdot |V \setminus A|$ (i.e., at most half of all possible edges). For the readers familiar with the notion of Seidel switching, we remark that the minimality condition says that G should have the minimum number of edges among the graphs of its Seidel switching class.

> **Problem:** What is the smallest possible number of odd triples for a minimal graph G with n vertices and at least $\alpha\binom{n}{2}$ edges?

Here $\alpha > 0$ is a parameter, and the answer should depend on α. For a reason not explained here, the range of interest for α is $(0, \frac{2}{9})$.

The best *upper bound* we know is obtained for the following graph. We divide the vertex set into three subsets V_1, V_2, V_3, where $|V_1| \leq |V_2| = |V_3|$, and G is the complete bipartite graph with color classes V_1 and V_2. The sizes of V_1, V_2 is set so that the number of edges is $\alpha\binom{n}{2} \pm O(n)$. It is easily checked that this graph is minimal (the "critical" partition is with $A = V_1$, where the density is exactly $\frac{1}{2}$).

Optimistically, we conjecture the following.

1. The upper bound above is the truth, and the described example is the only extremal graph (at least for those α where the number of edges is exactly $\alpha\binom{n}{2}$).

[1] Research done during a visit at the ETH Zurich, whose support is gratefully acknowledged.

2. The triangles can be ignored: that is, if we want to minimize only the number of triples with a single edge, instead of the number of all odd triples.

The best lower bound we can currently prove is that a minimal graph with $\alpha\binom{n}{2}$ edges has at least $\max\left\{\alpha\binom{n}{3}, \psi(\alpha)\binom{n}{3}\right\}$ odd triples, where $\psi(\alpha) = \frac{3}{4}\alpha\left(1+\sqrt{1-4\alpha}\right)$.

Motivation. The problem above is motivated by the following basic geometric problem. Let X be a set of n points in $\mathbf{R}^d$ in general position, and let us consider the $\binom{n}{d+1}$ d-dimensional simplices spanned by the points of X. By results of Boros and Füredi (for $d = 2$) and of Bárány (for $d \geq 3$), there always exists a "heavily covered" point c (typically not belonging to X) that lies in at least $c_d\binom{n}{d+1}$ of these simplices, where $c_d > 0$ depends only on d. The best possible value of c_d is known only for $d = 2$, where $c_2 = \frac{2}{9}$.

Gromov developed, in a recent major work [Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry, *Geom. Funct. Anal.*, **20(2)**:416-526, 2010] a new, topological proof of the result above, which yields the current best lower bounds for c_d. The extremal problem formulated above is relevant for a further slight improvement of the lower bounds for the c_d, and generally for a better understanding of the possibilities of Gromov's method.

Counting substructures

Dhruv Mubayi

Turán's theorem determines the maximum number of edges in a graph with n vertices and no clique of a fixed size, and extremal graph theory has grown through extensions and generalizations of this result. One such direction is to count the number of copies of a specified clique in a graph with more edges than in the Turán bound. We take this approach further by extending classical results of Rademacher, Erdös, Simonovits, and Lovász-Simonovits to the class of color critical graphs. The techniques are new and quite general, and they yield similar results for hypergraphs. Here is a sample theorem:

Füredi-Simonovits and independently Keevash-Sudakov settled an old conjecture of Sós by proving that the maximum number of triples in an n vertex triple system (for n sufficiently large) that contains no copy of the Fano plane is $p(n) = \binom{\lceil n/2 \rceil}{2}\lfloor n/2 \rfloor + \binom{\lfloor n/2 \rfloor}{2}\lceil n/2 \rceil$.

We prove that there is an absolute constant c such that if n is sufficiently large and $1 \leq q \leq cn^2$, then every n vertex triple system with $p(n)+q$ edges contains at least

$$6q\left(\binom{\lfloor n/2 \rfloor}{4} + (\lceil n/2 \rceil - 3)\binom{\lfloor n/2 \rfloor}{3}\right)$$

copies of the Fano plane. This is sharp for $q \leq n/2 - 2$.

One modern ingredient of our approach is the use of the removal lemma, which is a consequence of the hypergraph regularity lemma. In many cases, our results so far use ad hoc methods for each hypergraph F, and one open problem is to prove general results that apply to large classes of hypergraphs. Another open problem is to count induced copies of graphs or hypergraphs, which is a more challenging problem. A specific case is to consider the enumerative questions for the configurations studied recently by Razborov and Pikhurko, which are closely related to the famous Turán conjecture for hypergraphs.

Mapping planar graphs into projective cubes

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Problem What is the smallest subgraph of the *projective (folded) cube* of dimension $2r$ to which every planar graph of odd-girth $2k+1$ admits a homomorphism?

Projective cube of dimension r, denoted $PC(r)$, is obtained from $r+1$-dimensional hypercube by identifying antipodal vertices or equivalently from r-dimensional hypercube by adding an edge between every pair of antipodal vertices. It can also be viewed as a Cayley graph on $\mathbb{Z}_2^r$, or on $\mathbb{Z}_2^{r+1}$. These graphs are known by many names, the term folded cube is used most frequent.

The problem posed above captures several interesting theorems and challenging conjectures. Even formation of a sensible conjecture may give insight to these conjectures.

The $2r$-dimensional projective cube has odd-girth $2r+1$. Thus there is no such subgraph for $r > k$. For $r = k$ it is a conjecture of the author that $PC(2r)$ itself is the answer. The existence of homomorphism in this case relates to the edge colouring of planar graphs and is related to a conjecture of P. Seymour, see [2]. That no proper subgraph of $PC(2r)$ works in this case follows from [3]. In fact we have conjectured that $PC(2r)$ is the smallest graph of odd-girth $2r+1$ to which every planar graph of odd-girth $2r+1$ admits a homomorphism. Using the four colour theorem, this is easily true for $r = 1$ and is prved for $r = 2$ [4].

For $r = k-1$ we have conjectured in [3] that the Kneser graph $K(2k-1, k-1)$ (also known as odd graph) is the answer. This is related to the fractional chromatic number of planar graphs and some partial results are proved in [3].

The smallest subgraph of the projective cube $PC(2r)$ which is not bipartite is C_{2r+1}. It is a classical result that when k is much larger than r then C_{2r+1} is the answer to our question. It is an strengthening of a conjecture of Jeager by Zhang in [5], that this holds for $k = 2r$. X. Zhu proved in [6] that for $k \geq 4r-2$, C_{2r+1} is the answer.

When $r = 1$ and $k \geq 2$ the Grotzsch's theorem states that C_3, the triangle, is the answer. For $r = 2$ and $k \geq 5$, M. DeVos and A. Deckelbaum claim C_5 is the answer.

Finally for $r = 3$ and $k = 5$ we conjecture that the Coxeter graph is the answer.

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On a generalization of the greedy flip property for greedy pseudotriangulations

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The complex of atoms of a prime network. By a *network* we mean a factorization $X \to Y \to Z$ of a finite (not necessarily connected) covering $X \to Z$ of $Z \approx \mathbb{S}^1$ such that the non trivial classes of the equivalence relation $\mathcal{R}$ on X induced by $X \to Y$ are finitely many and of size 2. By construction $Y \approx X/\mathcal{R}$ is the underlying space of a finite 1-dimensional cell complex and the images Y_i under $X \to Y$ of the connected components X_i of X are edge-disjoint supports of cycles of edges in Y that cover Y, i.e., $\bigcup Y_i = Y$. The Y_i are called the *cycles* of the network. By a *decomposition* of a network $X \to Y \to Z$ we mean the set of cycles of a network of type $X' \to Y \to Z$. In particular the set of cycles of a network is a decomposition of that network, that we shall called its canonical decomposition. The *status* of a vertex of a network in a decomposition of that network is termed *crossing* or *touching* depending on whether the decomposition coincides or not with the canonical decomposition of the network in the vicinity of that vertex. Clearly a decomposition depends only on the status of the vertices of the network; conversely any assignment to the vertices of a network of a crossing or touching status defines a unique decomposition of that network. A network $X \to Y \to Z$ is termed *prime* if (1) it has at least one prime decomposition, i.e., a decomposition whose cycles are homeomorphic to Z via $Y \to Z$ and whose intersecting cycles cross exactly once; (2) for any prime decomposition $\mathcal{A}$ and any touching vertex u of $\mathcal{A}$ the decomposition $\mathcal{A}'$ obtained from $\mathcal{A}$ by interchanging the touching status of u with the crossing status of the crossing vertex v of the two cycles of $\mathcal{A}$ touching at u is prime (we will say that $\mathcal{A}'$ is obtained by flipping the touching vertex u in $\mathcal{A}$). By definition the *atoms* of a prime network are the sets of contact points of its prime decompositions, and the *complex of atoms* of a prime network is the set of subsets of its atoms ordered by inclusion and augmented with a maximum element. By construction the complex of atoms of a prime network is a pure simplicial complex and satisfies the "diamond property". In the next paragraph we show that it is strongly flag-connected [McM94].

Theorem 1.1. *The complex of atoms of a prime network is an abstract simplicial m-polytope where m is the number of touching vertices of its* *atoms.* □

Example 1.2. Let P be a finite set of points in general position of a real two-dimensional affine geometry, let P^* be the set of lines incident to P, let $\mathcal{C}^*$ be the set of supporting lines of the convex hull $\mathcal{C}$ of P, let Y be the topological closure of $P^* \setminus \mathcal{C}^*$, let Γ be the decomposition into cycles of Y induced by the P_i^*, $P_i \in P$, let $X_i \to \Gamma_i$, $X_i = \mathbb{S}^1$, be a parametrization of Γ_i which is one-to-one except at its self-intersection points, let $X \to Y$ be the disjoint union of the $X_i \to \Gamma_i$, and let $Y \to Z(\approx \mathbb{S}^1)$ be the map that assigns to a line its direction. Then $X \to Y \to Z$ is a prime network and its complex of atoms is isomorphic to the complex of pointed pseudotriangulations of P [PP10]. The complex of pointed pseudotriangulations is known to be polytopal in the case where the underlying affine geometry is the standard affine geometry [RSS03].

Example 1.3. Let P be a finite set of points in convex position of a real two-dimensional affine geometry, let P^* be the set of lines incident to (at least one point of) P, let $\mathcal{C}_k^*$ be the set of lines of P^* with the property that one of their open sides contains at most k points of P, let Y be the topological closure of $P^* \setminus \mathcal{C}_k^*$, let Γ be the decomposition into cycles of Y induced by the P_i^*, $P_i \in P$, let $X_i \to \Gamma_i$, $X_i = \mathbb{S}^1$, be a parametrization of Γ_i which is one-to-one except at its self-intersection points, let $X \to Y$ be the disjoint union of the $X_i \to \Gamma_i$, and let $Y \to Z(\approx \mathbb{S}^1)$ be the map that assigns to a line its direction. Then $X \to Y \to Z$ is a prime network and its complex of atoms is isomorphic to the complex of k-triangulations of P [PP10]). The complex of k-triangulations is a vertex-decomposable triangulated sphere [SS10, Stu10], and the complex of 2-triangulations of the octogon is polytopal [BP09].

Example 1.4. Let $\mathcal{M}$ be a noncompact Möbius strip, i.e., a topological space homeomorphic to the real two-dimensional projective plane with one point deleted. An *arrangement of pseudoline with contact points* in $\mathcal{M}$ is a finite family of pseudolines in $\mathcal{M}$ such any two pseudolines have finitely many intersection points of which exactly one is transversal. Let $Y = \bigcup \mathcal{A}$ be the support of a simple arrangement of pseudolines with contact points $\mathcal{A}$, let Γ be the decomposition into cycles of Y defined by the condition that the intersection points of the cycles are transversal in $\mathcal{M}$, let $X_i \to \Gamma_i$, $X_i = \mathbb{S}^1$, be a parametrization of Γ_i which is one-to-one except at the self-intersection points of Γ_i, let $X \to Y$ be the disjoint union of the $X_i \to \Gamma_i$, and let $\mathcal{M} \to Z$ be a retraction of $\mathcal{M}$ onto one of its core circle Z compatible with the arrangement $\mathcal{A}$ in the sense that the restriction of $\mathcal{M} \to Z$ to any pseudoline of $\mathcal{A}$ is one-to-one. Then $X \to Y \to Z$ is a prime network [PP10]. The complex of atoms of (the prime network defined by) the support of an arrangement of two pseudolines with m contact points is the lattice of the m-simplex. The complex of atoms of the support of an arrangement of three pseudolines with contact points is the opposite of the lattice of a simple polytope with three more facets than its dimension; futhermore any simple polytope with three more facets than its dimension can be realized like this (F. Santos 09, personnal communication).

Example 1.5. Let $\mathbb{B}$ be a finite branched covering space of an affine real two-dimensional geometry $\mathbb{A}$. Let $\mathcal{D}$ be a finite family of pairwise disjoint convex bodies of $\mathbb{B}$ in general position with the property that any branch point belongs to the interior of a body of the family and any body contains at most one branched point, and let $\mathbb{F}$ be the complement in $\mathbb{B}$ of the interiors of the bodies. The space $\mathbb{F}$ is endowed with the point-line incidence structure inherited from the point-line incidence structure of $\mathbb{A}$, i.e., the lines of $\mathbb{F}$ are the traces on $\mathbb{F}$ of the subsets of $\mathbb{B}$ homeomorphic to the lines of $\mathbb{A}$ via the covering map $\mathbb{B} \to \mathbb{A}$. Let $\mathcal{D}_i^*$ be the set of lines of $\mathbb{F}$ that are tangent to the convex body $\mathcal{D}_i$, let $\mathcal{C}^*$ be the set of lines of $\mathbb{F}$ that are tangent to the convex hull $\mathcal{C}$ of the $\mathcal{D}_i^*$, let Y be the topological closure of $\bigcup \mathcal{D}^*$ minus C^*, let Γ be the decomposition into cycles of Y induced by the $\mathcal{D}_i^*$, let $X_i \to \Gamma_i$, $X_i = \mathbb{S}^1$, be a parametrization of Γ_i which is one-to-one except at its self-intersection points, let $X \to Y$ be the disjoint union of the $X_i \to \Gamma_i$, and let $Y \to Z(\approx \mathbb{S}^1)$ be the map that assigns to a line its direction. Then $X \to Y \to Z$ is a prime network (that we shall called a *visibility network*) and its complex of atoms is isomorphic to the complex of pseudotriangulations of $\mathcal{D}$.

Problem 1.6. *Investigate the polytopality (shellability, etc) of the complexes of atoms of prime networks.*

Greedy atoms and the greedy flip property. Let $\mathcal{N}$ be a prime network defined by the factorization $X \to X/\mathcal{R} \to Z$, let $X \to X/\mathcal{S} \to X/\mathcal{R}$ be a factorization of $X \to X/\mathcal{R}$ with the property that $\mathcal{S}$ contains the free vertices of $\mathcal{N}$ (a vertex of a prime network is termed *free* if it is contact point of an atom of the network), and let $\mathcal{N}_{\mathcal{S}}$ be the induced network $X \to X/\mathcal{S} \to Z$. It should be clear that $\mathcal{N}_{\mathcal{S}}$ is also prime and that the flip graph $\mathcal{P}(\mathcal{N}_{\mathcal{S}})$ on the atoms of $\mathcal{N}_{\mathcal{S}}$ and the flip graph $\mathcal{P}(\mathcal{N})$ on the atoms of $\mathcal{N}$ are isomorphic via $\rho : X/\mathcal{S} \to X/\mathcal{R}$.

We set $\mathcal{M} = X/\mathcal{S}$. Let $\pi : \widetilde{Z} \approx \mathbb{R} \to Z$ be a universal covering of Z, let $\tau : \widetilde{Z} \to \widetilde{Z}$ be a generator of its automorphism group (which is infinite cyclic) and let $\widetilde{\mathcal{M}}$ be the induced covering of $\mathcal{M}$ defined as the set of pair $(x, y) \in \mathcal{M} \times \widetilde{Z}$ such that $p \circ \rho(x) = \pi(y)$, and let $\iota : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ defined by $\iota(x, y) = (x, \tau(y))$. By construction $\widetilde{\mathcal{M}}$, oriented according to ι, is acyclic and we denote by $\preccurlyeq$ the induced partial order on its set of vertices and edges. Given a filter I of the poset $(\widetilde{\mathcal{M}}_0, \preccurlyeq)$ bounded from below (i.e., with no infinite descending chain) we introduce

1. the maximal antichain $\hat{I}$ of the poset $(\widetilde{\mathcal{M}}_1, \preccurlyeq)$ associated with I, i.e., the set of 1-cells of $\widetilde{\mathcal{M}}$ whose sinks belong to I but not their sources;
2. the section $s_I : \mathcal{M}_0 \to \widetilde{\mathcal{M}}_0$ of the canonical projection $\widetilde{\mathcal{M}}_0 \to \mathcal{M}_0$ with range the set of vertices of I not in $\iota(I)$, i.e., $s_I(u)$, $u \in \mathcal{M}_0$, is the unique element of I not in $\iota(I)$ whose image under the canonical projection $\widetilde{\mathcal{M}}_0 \to \mathcal{M}_0$ is u;
3. the directed version $\mathcal{P}(I)$ of $\mathcal{P}(\mathcal{N}_\mathcal{S})$ obtained by orienting its edges according to the following rule: the edge $\{\mathcal{A}, \mathcal{A}'\}$ is oriented from $\mathcal{A}$ to $\mathcal{A}'$ if $s_I(u) \preccurlyeq s_I(v)$ where u is the contact point of $\mathcal{A}$ not in $\mathcal{A}'$ and v is the contact point of $\mathcal{A}'$ not in $\mathcal{A}$;
4. the family of curves $\lambda(e; I)$, $e \in \widetilde{\mathcal{M}}_1$, $\mathrm{sink}(e) \in I$, $\mathrm{sour}(e) \notin \iota(I)$, defined inductively as follows: If e belongs to the antichain associated with $\iota(I)$ we set $\lambda(e; I) = e$; otherwise we introduce the two 1-cells f and f' with initial vertex the terminal vertex v of e with the convention that f and e are supported by the same cycle of $\mathcal{N}_\mathcal{S}$, and we set $\lambda(e; I) = ev\lambda(f'; I)$ or $\lambda(e; I) = ev\lambda(f; I)$ depending on whether the curves $\lambda(f; I)$ and $\lambda(f'; I)$ share a vertex or not;
5. the image $G(I)$ under the canonical projection $\widetilde{\mathcal{M}} \to \mathcal{M}$ of the family $\lambda(e; I)$, $e \in \hat{I}$.

The following theorem generalizes to the setting of prime networks the greedy flip property for greedy pseudotriangulations of [PV96, AP03, HP07].

Theorem 1.7. *Let $\mathcal{N}$ be a prime network, let $\mathcal{S}$ as above, let I be a filter of $\widetilde{\mathcal{M}}_0$, and let u be a minimal element of I. Then (1) $\mathcal{P}(I)$ is acyclic; (2) $G(I)$ is an atom of $\mathcal{N}_\mathcal{S}$ and is the unique source of $\mathcal{P}(I)$; (3) $\mathcal{P}(\mathcal{N}_\mathcal{S})$ is connected; (4) u is a contact point of $G(I)$ if and only if u is free; in* *particular if $\mathcal{S}$ contains only free vertices then u is a contact point of $G(I)$; (5) if u is free then $G(I \setminus \{u\})$ is obtained from $G(I)$ by flipping u; otherwise $G(I \setminus \{u\})$ and $G(I)$ coincide.* $\square$

In particular we derive from the greedy flip property that the operator Φ defined on the set of free vertices of $\widetilde{\mathcal{M}}$ as the one that assigns to u the sole element of $G(I \setminus u) \setminus G(I)$, u minimal in I, is well-defined (independant of the choice of I), one-to-one and onto.

Problem 1.8. *Study the properties of the Φ-operator at the light of the properties of the Φ-operator for visibility networks reported in [AP03].*

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The surviving rate of planar graphs

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The following *firefighter problem* on a finite graph $G = (V, E)$ was introduced by Hartnell at the conference in 1995 [3]. Suppose that a fire breaks out at a given vertex $v \in V$. In each subsequent time unit, a firefighter protects one vertex which is not yet on fire, and then fire spreads to all unprotected neighbours of the vertices on fire. (Once a vertex is on fire or gets protected it stays in such state forever.) Since the graph is finite, at some point each vertex is either on fire or is protected by the firefighter, and the process is finished. (Alternatively, one can stop the process when no neighbour of the vertices on fire is unprotected. The fire will no longer spread.) The objective of the firefighter is to save as many vertices as possible. Today, 15 years later, our knowledge about this problem is much greater and a number of papers have been published. We would like to refer the reader to the survey of Finbow and MacGillivray for more information [6].

We would like to focus on the following property. Let $\mathrm{sn}(G, v)$ denote the number of vertices in G the firefighter can save when a fire breaks out at vertex $v \in V$, assuming the best strategy is used. The surviving rate $\rho(G)$ of G, introduced in [5], is defined as the expected percentage of vertices that can be saved when a fire breaks out at a random vertex of G (uniform distribution is used), that is, $\rho(G) = \frac{1}{n^2} \sum_{v \in V} \mathrm{sn}(G, v)$. It is not difficult to see that for cliques $\rho(K_n) = \frac{1}{n}$, since no matter where a fire breaks out only one vertex can be saved. For paths we get that

$$\rho(P_n) = \frac{1}{n^2} \sum_{v \in V} \mathrm{sn}(G, v) = \frac{1}{n^2} \left(2(n-1) + (n-2)(n-2)\right) = 1 - \frac{2}{n} + \frac{2}{n^2}$$

(one can save all but one vertex when a fire breaks out at one of the leaves; otherwise two vertices are burned). It is not surprising that a path can be easily protected, and in fact, all trees have this property. Cai, Cheng, Verbin, and Zhou [1] proved that the greedy strategy of Hartnell and Li [4] for trees saves at least $1 - \Theta(\log n / n)$ percentage of vertices on average for an n-vertex tree. Moreover, they managed to prove that for every outerplanar graph G, $\rho(G) \geq 1 - \Theta(\log n / n)$. Both results are asymptotically tight and improved earlier results of Cai and Wang [2]. Let us note that there is no hope for similar result for planar graphs, since, for example, $\rho(K_{2,n}) = 2/(n+2) = o(1)$.

Let us stay focused on sparse graphs. It is clear that sparse graphs are easier to control so their survival rates should be relatively large. Finbow, Wang, and Wang [7] showed that any graph G with average degree strictly smaller than 8/3 has the surviving rate bounded away from zero. Formally, it has been shown that any graph G with $n \geq 2$ vertices and $m \leq (\frac{4}{3} - \varepsilon)n$ edges satisfies $\rho(G) \geq \frac{6\varepsilon}{5} > 0$, where $0 < \varepsilon < \frac{5}{24}$ is a fixed number. This result was recently improved by the author of this extended abstract to show that any graph G with average degree strictly smaller than 30/11 has the surviving rate bounded away from zero [8].

Theorem 1.1 ([8]). *Suppose that graph G has $n \geq 2$ vertices and $m \leq (\frac{15}{11} - \varepsilon)n$ edges, for some $0 < \varepsilon < \frac{1}{2}$. Then, $\rho(G) \geq \frac{\varepsilon}{60}$.*

(Let us note that the goal was to show that the surviving rate is bounded away from zero, not to show the best lower bound for $\rho(G)$. The constant $\frac{1}{60}$ can be easily improved with more careful calculations.)

On the other hand there are some dense graphs with large survival rates (take, for example, a large collection of cliques). However, in [8] a construction of a sparse random graph on n vertices with the survival rate tending to zero as n goes to infinity is proposed. Hence the result is tight and the constant $\frac{15}{11}$ cannot be improved.

It would be nice to find the threshold for other families of graphs, including planar graphs.

Question 1.2. *Determine the largest real number M such that every planar graph G with $n \geq 2$ vertices and $m \leq (M - \varepsilon)n$ edges has $\rho(G) \geq c \cdot \varepsilon$ for some $c > 0$.*

It follows from Theorem 1.1 and the fact that $\rho(K_{2,n}) = o(1)$ that $\frac{15}{11} \leq M \leq 2$.

The second question was asked in [7].

Question 1.3. *Determine the least integer g^* such that there is a constant $0 < c < 1$ such that every planar graph G with girth at least g^* has $\rho(G) \geq c$.*

Note that a connected planar graph with n vertices and girth g can have at most $\frac{g}{g-2}(n-2)$ edges (see, for example, [7]). Thus, from Theorem 1.1 it follows that $g^* \leq 8$. Using the fact that $\rho(K_{2,n}) = o(1)$ one more time, we conclude that $5 \leq g^* \leq 8$.

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Partitioning 3-colored complete graphs into three monochromatic cycles

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Abstract

We show in this paper that in every 3-coloring of the edges of K^n all but $o(n)$ of its vertices can be partitioned into three monochromatic cycles. From this, using our earlier results, actually it follows that we can partition *all* the vertices into at most 10 monochromatic cycles, improving the best known bounds. If the colors of the three monochromatic cycles must be different then one can cover $(\frac{3}{4}-o(1))n$ vertices and this is close to best possible. These are joint results of András Gyárfás, Gábor Sárközy, Endre Szemerédi and myself.

1 Introduction

It was conjectured in [3] that in every r-coloring of a complete graph, the vertex set can be covered by r vertex disjoint monochromatic cycles (where vertices, edges and the empty set are accepted as cycles).

Conjecture 1.1. *(Erdős, Gyárfás, Pyber, [3]) In every r-coloring of the edges of K_n its vertex set can be partitioned into r monochromatic cycles.*

For general r, the $O(r^2 \log r)$ bound of Erdős, Gyárfás, and Pyber [3] has been improved to $O(r \log r)$ by Gyárfás, Ruszinkó, Sárközy and Szemerédi [4]. The case $r = 2$ was conjectured earlier by Lehel and was settled for large n using the Regularity Lemma by Łuczak, Rödl and Szemerédi [6]. Recently Allen [1] gave a proof without the Regularity Lemma, Bessy and S. Thomassé [2] found an elementary argument that works for every n.

The main result of this paper confirms Conjecture 1.1 in case $r = 3$ in asymptotic sense.

Theorem 1.2. *In every 3-coloring of the edges of K_n all but $o(n)$ of its vertices can be partitioned into three monochromatic cycles.*

Combining Theorem 1.2 with some of our earlier results from [4] we can actually prove that we can partition *all* the vertices into at most 10 monochromatic cycles, improving the best known bounds for $r = 3$.

Theorem 1.3. *In every* 3*-coloring of the edges of* K_n *the vertices can be partitioned into at most 10 monochromatic cycles.*

Note that in the same way for a general r if one could prove the corresponding asymptotic result as in Theorem 1.2 (even with a weaker linear bound on the number of cycles needed; unfortunately we are not there yet), then we would have a linear bound overall. This makes the asymptotic result interesting.

In the proof of Theorem 1.2 our main tools are the Regularity Lemma [7] and the following lemma. A *connected matching* in a graph G is a matching M such that all edges of M are in the same component of G.

Lemma 1.4. *If* n *is even then in every* 3*-coloring of the edges of* K_n *the vertex set can be partitioned into three monochromatic connected matchings.*

Theorem 1.2 fails if we insist that the monochromatic cycles must have different colors. The following result is tight.

Theorem 1.5. *In every* 3*-coloring of the edges of* K_n*, at least* $(\frac{3}{4} - o(1))n$ *vertices can be covered by vertex disjoint monochromatic cycles having distinct colors.*

Theorem 1.5 relies on the following variant of Lemma 1.4.

Lemma 1.6. *In every* 3*-coloring of the edges of* K_n *vertex disjoint monochromatic connected matchings of distinct colors cover at least* $\frac{3n}{4} - 1$ *vertices.*

One can find the proofs of the stated results in [5]. Basically, we prove Lemmata 1.4 and 1.6 and then we use standard methods based on the Regularity Lemma. However working in the reduced graph we usually face the difficulty of extending Ramsey-type results from the case when the host graph is K_n to the case when the host graph is an arbitrary almost complete one. Therefore, it would be technically very useful to prove something in this direction. I don't know exactly what would be the right statement, therefore I pose a problem in the following vague form.

Problem 1.7. *Assume that a graph* G *has a 'good reason' to have a linear Ramsey number, i.e., in every* r*-coloring of the edges of* K_n *there is* *a monochromatic copy of G of order $f(r)n$ where f is a function depending only on r. Is it true then that in every r-edge-coloring of a graph on n vertices having almost all edges there is a monochromatic copy of G of asymptotically similar order?*

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Degree and substructure in infinite graphs

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Abstract

We are interested in the relation between the average/minimum degree and the appearance of substructures in infinite graphs.

In finite graphs, the impact of the average degree on the appearance of certain substructures is well studied. The specific substructure that will play the lead role here is the complete graph K^k, for $k \in \mathbb{N}$, which may appear as a subgraph, as a minor, or as a topological minor. So a good example of a result in the direction we aim at is Turán's classical theorem, which states that an average degree of more than $\frac{k-2}{k-1}n$, where n is the number of vertices of the host graph G, ensures the existence of a finite subgraph of G that is isomorphic to K^k.

While the function from Turán's theorem depends on n, for forcing 'weaker' substructures the degree bound may depend only on k: An average degree of at least c_1k^2 ensures a topological minor isomorphic to K^k, and an average degree of at least $c_2k\sqrt{\log k}$ ensures a minor isomorphic to K^k. (The c_i are some constants from $\mathbb{R}_+$.)

Further, related substructures such as the complete k-partite graph K^k_s with partition classes of size s, or k-connected subgraphs, can be forced with stronger/weaker assumptions. In the following table, where we assume G to be a graph on n vertices, the reader finds an overview of some well-known results we would like to extend to infinite graphs. For the sake of brevity, in the first of these results, the quantifiers are missing: for every ε, k and s there is an n_0 so that for all $n \geq n_0$ the implication below is valid.

	Erdős-Stone	Turán	Bollobás & Thomason	Kostochka	Mader
$d(G)$	$> (\frac{k-2}{k-1}+\varepsilon)n$	$> \frac{k-2}{k-1}n$	$\geq c_1k^2$	$\geq c_2k\sqrt{\log k}$	$\geq 4k$
$\Rightarrow$	$K^k_s \subseteq G$	$K^k \subseteq G$	$K^k \preceq_{top} G$	$K^k \preceq G$	$H \subseteq G$, H is $(k+1)$-connected.

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So how do these results extend to infinite graphs? To answer this question, we must ask first of all how the average degree translates to an infinite graph, as we now deal with an infinite number of vertices. Of course, the average degree is closely related to the density $|E(G)|/\binom{|V(G)|}{2}$, and this notion is reflected in the *upper density* of an infinite graph.

The upper density $ud(G)$ of a graph G is defined as the supremum of the subgraph densities, taken over all sequences of finite subgraphs of G whose order tends to infinity. That is,

$$ud(G) := \sup_{(H_i)_{i\in\mathbb{N}}} \limsup_{i\to\infty} \left(|E(H_i)|/\binom{V(H_i)}{2} \right),$$

where the sequences $(H_i)_{i\in\mathbb{N}}$ range over all sequences of finite subgraphs $H_i \subseteq G$ with $\lim_{i\to\infty} |V(H_i)| = \infty$ (see for instance Bollobás [1]).

Now, it is not difficult to calculate that if (for $k > 1$) the upper density of a graph G is greater than $\frac{k-2}{k-1}$, say $ud(G) \geq (1+\delta)\frac{k-2}{k-1}$, then G has a finite subgraph H of average degree at least $(1+\frac{\delta}{2})\frac{k-2}{k-1}|V(H)|$, and thus, by Turán's theorem, contains a K^k-subgraph. Actually, as the order of the subgraph H may be assumed to exceed any given integer, we may apply to H the Erdős-Stone theorem for any s, and obtain a K^k_s-subgraph. So in this sense, both the Turán and the Erdős-Stone theorem do extend to infinite graphs.

From the existence of arbitrarily large complete k-partite subgraphs once the threshold upper density $\frac{k-2}{k-1}$ is surpassed, it follows that the upper density of any infinite graph takes one of the following (countably many) values: $1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$, that is, one of the Turán densities. So it seems that the graphs for which it would be interesting to extend the latter three results discussed above, all have upper density 0. In other words, the upper density is not fine enough a measure for a generalisation of e.g. Kostochka's theorem to infinite graphs.

One possible way out of this dilemma is replacing the average degree with something that quite obviously does exist in infinite graphs, the minimum degree. For rayless graphs, this is an excellent option, as we have the following result, which is not difficult to prove. Write $\delta^V(G)$ for the minimum degree taken over all vertices of the graph G.

Proposition 1.1. [4] *Let $k \in \mathbb{N}$ and let G be a rayless graph with $\delta^V(G) \geq k$. Then G has a finite subgraph of average degree at least k.*

This means that the latter three results from the table above extend literally to rayless graphs, if we replace the average degree with the minimum degree.

In general, however, we are not that lucky. Just consider an infinite tree, whose vertices may attain any minimal degree condition, while the tree does not contain any interesting substructure. The example suggests that we need some additional condition that prevents 'the density from escaping to infinity', in other words, that makes the vertices send their edges 'back' instead of 'further out'. Following recent developments (see [3]), the most natural way to impose such an additional condition is to impose it on the ends[2] of the graph.

In [2, 5], see also [3], the *vertex-degree*[3] $d_v(\omega)$ of an end ω is defined as the supremum of the cardinalities of the sets of vertex-disjoint rays from ω. This intuitive notion allows us to extend Mader's theorem from above to infinite graphs. For this, let us write $\delta^{V,\Omega_v}(G)$ for the minimum of the degrees or vertex-degrees, taken over all vertices and ends of the graph G.

Theorem 1.2. [5] *Let G be a graph. If $\delta^{V,\Omega_v}(G) \geq 2k(k+3)$ then G has a $(k+1)$-connected subgraph.*

We remark that the $(k+1)$-connected subgraph can neither be guaranteed to be finite nor to be infinite. The bound $2k(k+3)$ may possibly be lowered, but not to less than $\frac{k}{5}\log\frac{k}{5}$. See [5].

The vertex-degree, however, does not serve for forcing large complete (topological) minors. One can see this by considering the following example. Take, for $r \in \mathbb{N}$, $r > 2$, the infinite r-regular tree, and add a spanning cycle in each level. The resulting graph G_r has one end of infinite vertex-degree, while all vertices have degree at least r. Now, although r may be arbitrarily large, G_r is planar, and thus has no complete minor of order greater than 4.

So, a different road has to be taken for forcing minors and topological minors in graphs with rays. In [4], the *relative degree* of an end was introduced for locally finite graphs. The idea is to calculate the ratios of the

[2]The *ends* of a graph are the equivalence classes of the rays (the one-way infinite paths) of the graph under the following equivalence relation. Two rays are equivalent if no finite set of vertices separates them. For more on the end space of an infinite graph, see [3].

[3]There, also the *edge-degree* of ω is defined quite analogously, as the supremum of the cardinalities of the sets of edge-disjoint rays from ω. The edge-degree allows for an extension of the edge-version of Mader's theorem. In this edge-version, only linear bounds on the minimum (edge)-degree are needed. For details, see [5].

cardinality of the edge-boundary[4] $\partial_e H_i$ versus the cardinality of the vertex-boundary[5] $\partial_v H_i$ of certain subgraphs H_i of G, and then define the relative degree to be the limit of these ratios as the H_i in some sense converge to ω. This intuitive idea can be formalised as follows.

We call a subgraph H of a graph G an *ω-region* if $\partial_v H$ is finite and H contains a ray of the end $\omega \in \Omega(G)$. We write $\Omega^G(H)$ for the sets of all ends of G that have a ray in H.

Now, for a locally finite graph G, write $(H_i)_{i\in\mathbb{N}} \to \omega$ if $(H_i)_{i\in\mathbb{N}}$ is an infinite sequence of distinct ω-regions of G such that $H_{i+1} \subseteq H_i - \partial_v H_i$ and $\partial_v H_{i+1}$ is an inclusion-minimal $\partial_v H_i$–$\Omega^G(H_{i+1})$ separator, for each $i \in \mathbb{N}$. Note that by the local finiteness of G such sequences do exist. Define

$$d_{e/v}(\omega) := \inf_{(H_i)_{i\in\mathbb{N}}\to\omega} \liminf_{i\to\infty} \frac{|\partial_e H_i|}{|\partial_v H_i|}.$$

This definition leads to the desired results for locally finite graphs. Let $\delta^{V,\Omega_{e/v}}(G)$ denote the minimum (relative) degree, taken over all vertices and ends of the graph G. The constants $c_1, c_2 \in \mathbb{R}_+$ are as in the corresponding theorems for finite graphs.

Theorem 1.3. [4] *Let $k \in \mathbb{N}$ and let G be a locally finite graph.*

(a) If $\delta^{V,\Omega_{e/v}}(G) \geq c_1 k^2$, then K^k is a topological minor of G.

(b) If $\delta^{V,\Omega_{e/v}}(G) \geq c_2 k\sqrt{\log k}$, then K^k is a minor of G.

For arbitrary infinite graphs it is necessary to adapt the definition of the relative degree. This is so as now there may be vertices dominating[6] ends. In that case, the sequences $(H_i)_{i\in\mathbb{N}}$ cannot satisfy the condition that $H_{i+1} \subseteq H_i - \partial_v H_i$. We thus ask:

Question 1.4. *Does Theorem 1.3 extend to arbitrary infinite graphs? How does the (relative) end degree have to be defined in this case?*

A partial answer to Question 1.4 will be provided in [6]. A second not less interesting question is:

Question 1.5. *Are there extensions of the results mentioned in the beginning, if we let k be an infinite cardinal?*

[4]The *edge-boundary* of a subgraph H of a graph G is the set $\partial_e H := E(H, G - H)$.

[5]The *vertex-boundary* of a subgraph H of a graph G is the set $\partial_v H := N_G(G - H)$.

[6]A vertex is said to *dominate* an end ω if for some ray $R \in \omega$ there are infinitely many v–$V(R)$ paths, disjoint except in v.

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Fractional colorings in cubic graphs with large girth

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Joint work with Daniel Kráľ, and František Kardoš.

We show that every (sub)cubic n-vertex graph with sufficiently large girth has a fractional chromatic number at most 2.2978, which improves the result of Hatami and Zhu [1]. As a corollary, we obtain that it also contains an independent set of size $0.4352n$. This improves the previous lower bound on the independence number of cubic graphs with large girth given by Hoppen [2] and translates also to random cubic graphs.

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Matroid representation over the reals

Geoff Whittle

Historically most research in structural matroid theory has focused on matroids representable over finite fields. No doubt there are a variety of reasons for this. Nonetheless Whitney probably had the real numbers in mind as the canonical field over which to represent matroids. Unfortunately almost all of the general theorems that hold for finite fields and the general conjectures that we believe hold for finite fields fail spectacularly for infinite fields.

Consider some examples. Recall that an *excluded minor* for a minor closed class of matroid $\mathcal{M}$ is a matroid not in $\mathcal{M}$ with the property that all of its proper minors belong to $\mathcal{M}$. Perhaps the most famous open problem in matroid theory is Rota's Conjecture which states that if $\mathbb{F}$ is a finite field, then the class of $\mathbb{F}$-representable matroids has a finite number of excluded minors. Rota's Conjecture is known to be true for $GF(2)$, $GF(3)$ and $GF(4)$ and it is widely believed to hold in general. Consider the non-finite case. Assume that $\mathbb{F}$ is an infinite field. Then, it is known that there are an infinite number of excluded minors of $\mathbb{F}$-representability. Indeed, it is even known that every $\mathbb{F}$-representable matroid is a minor of an excluded minor for $\mathbb{F}$-representability.

Similarly, it is believed that many of the results of the graph-minors project of Robertson and Seymour extend to matroids representable over finite fields. For example we believe that, if $\mathbb{F}$ is a finite field, then the class of $\mathbb{F}$-representable matroids is well-quasi-ordered and that membership of any minor-closed subclass of $\mathbb{F}$-representable matroids can be recognised in polynomial time. Again, for infinite fields, these conjectures fail spectacularly. Indeed, if $\mathbb{F}$ is infinite, there are an uncountable number of minor-closed classes of $\mathbb{F}$-representable matroids. But there are only a countable number of algorithms, so recognising membership of a minor-closed subclass of $\mathbb{F}$-representable matroids is, in general, not even decidable.

Another problem is related to logic. Inspired by a comment of Whitney, it is sensible to ask if, for a given field $\mathbb{F}$, one can add a finite number of axioms *in a logic that is natural for matroids* that characterises $\mathbb{F}$-representability. This question was addressed by Vámos, but he used first order logic which is not natural for matroids and cannot even be used to distinguish binary matroids. If Rota's Conjecture holds then it is routine to show that representability over finite fields can be characterised in a logic that is natural for matroids, but evidence is accumulating that this is not the case for infinite fields.

Just how bad can things be? Here is a very grim conjecture. Let $\mathcal{M}$ be a minor-closed class of matroids and let $\mathcal{M}^+$ denote the matroids that are either in $\mathcal{M}$ or are excluded minors for $\mathcal{M}$. Call $\mathcal{M}$ a *fractal class*, if for all $\epsilon > 0$, there is an N such that, for $n \geq N$, the proportion of n-element members of $\mathcal{M}^+$ that belong to $\mathcal{M}$ is less than ϵ. Together with Dillon Mayhew and Mike Newman, I conjecture that the matroids representable over any fixed infinite field form a fractal class.

The contrast between the behaviour of matroid representation over finite and infinite fields is particularly striking when one considers that matroids representable over the real numbers are particularly natural from the point of view of human intuition—they are the configurations of classical projective geometry. Is there a sensible general structure theory for these matroids? I believe that this is a question of great interest for future research.