Colouring Edges with many Colours in Cycles

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Abstract

The arboricity of a graph G is the minimum number of colours needed to colour the edges of G so that every cycle gets at least two colours. Given a positive integer p, we define the generalized p-arboricity $\mathrm{Arb}_p(G)$ of a graph G as the minimum number of colours needed to colour the edges of a multigraph G in such a way that every cycle C gets at least $\min(|C|, p+1)$ colours. In the particular case where G has girth at least $p+1$, $\mathrm{Arb}_p(G)$ is the minimum size of a partition of the edge set of G such that the union of any p parts induce a forest. If we require further that the edge colouring be proper, i.e., adjacent edges receive distinct colours, then the minimum number of colours needed is the generalized p-acyclic edge chromatic number of G. In this paper, we relate the generalized p-acyclic edge chromatic numbers and the generalized p-arboricities of a graph G to the density of

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the multigraphs having a shallow subdivision as a subgraph of G.

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1 Introduction

In this paper, we consider the following problem: given a graph G, how many colours do we need to colour the edges of G in such a way that every cycle gets "many" colours? Of course, the answer to this question depends on the precise meaning of "many". If we require that each cycle γ of length l of G gets l colours, i.e., every cycle is rainbow, then the minimum number of colours needed is equal to the maximum size of a block of G, as two edges of G belong to a common cycle if and only if they belong to the same block. If we require that every cycle gets at least 2 colours, i.e., every colour class induces a forest, then the minimum number of colours needed is the *arboricity* $\mathrm{Arb}(G)$ of G, and its determination is solved by the well-known Nash-Williams' theorem we recall now.

Denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G. Also denote by $|G| = |V(G)|$ (resp. $\|G\| = |E(G)|$) the *order* of G (resp. *size*). For $A \subseteq V(G)$ denote by $G[A]$ the subgraph of G induced by A. By Nash-Williams' theorem [7, 8], the arboricity of a graph G is given by the formula:

$$\mathrm{Arb}(G) = \max_{A \subseteq V(G), |A|>1} \left\lceil \frac{\|G[A]\|}{|A|-1} \right\rceil . \tag{1}$$

Here we consider a generalization of these two extreme cases. A general form of our problem is captured by the following:

> Given an unbounded non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ and an integer p, what is the minimum number $N_f(G, p)$ of colours needed to colour the edges of a graph G in such

a way that each cycle γ gets at least $\min(f(|\gamma|), p+1)$ colours?

Thus for $p = 1$ and $f(n) \geq 2$ we get $N_f(G,p) = \mathrm{Arb}(G)$. For an arbitrary graph G, it is usually difficult to determine $N_f(G,p)$. Our interest is to find upper bound for $N_f(G,p)$ in terms of other graph parameters, and upper bound for $N_f(G,p)$ for some nice classes of graphs and/or for some nice special functions f.

Many colouring parameters are bounded for proper minor closed classes of graphs. It is natural to ask for which functions f is $N_f(G,p)$ bounded for any proper minor closed class $\mathcal{C}$ of graphs. We shall prove (Lemma 1) that if $f(2^{p-1}) > p-1$ for some value of p then there is a (quite small) minor closed class of graphs $\mathcal{C}$, such that $N_f(G,p)$ is unbounded. On the other hand, we prove (Corollary 6) that if $f(x) \leq \lceil \log_2 x \rceil$ for all x then $N_f(G,p)$ is not only bounded on proper minor closed classes of graphs, but actually bounded on a class $\mathcal{C}$ if and only if $\mathcal{C}$ has *bounded expansion* (to be defined in Section 3).

Next we consider the special function $f(x) = x$. For this special function, the parameter $N_f(G,p+1)$ is denoted as $\mathrm{Arb}_p(G)$ and is called the *generalized p-arboricity* of G. So $\mathrm{Arb}_p(G)$ is the number of colours needed if we require that each cycle of G gets at least $p+1$ colours or is rainbow if its length is smaller than $p+1$. Note that if $p = 1$, then $\mathrm{Arb}_p(G)$ is the arboricity $\mathrm{Arb}(G)$ of G. We shall relate the generalized p-arboricities of a graph to the density of its shallow topological minors. Toward this end we define the following notions, which are analogous to those defined in [12] and [13]. The main difference is that here we consider multigraphs.

Let G be a multigraph and let r be a half integer. A multigraph H is a *shallow topological minor* of G at depth r if a $\leq 2r$-subdivision of H is a subgraph of G. We denote by $G\widetilde{\nabla} r$ the class of the multigraphs which are shallow topological minors of G at depth r. Hence we have

$$G \in G\widetilde{\nabla} 0 \subseteq G\widetilde{\nabla} \tfrac{1}{2} \subseteq \cdots \subseteq G\widetilde{\nabla} r \subseteq \ldots.$$

Notice that the class $G\widetilde{\nabla} 0$ is exactly the monotone closure of G, that

is the class of all the subgraphs of G.

We denote by $\widetilde{\nabla}_r(G)$ the maximum density of a graph in $G\widetilde{\triangledown} r$, that is:

$$\widetilde{\nabla}_r(G) = \max_{H \in G\widetilde{\triangledown} r} \frac{\|H\|}{|H|}. \tag{2}$$

In this paper, we will give lower and upper bounds for $\mathrm{Arb}_p(G)$ based on $\widetilde{\nabla}_{\frac{p-1}{2}}(G))$. For $p = 1$, notice that it is an easy consequence of Nash-Williams' Theorem that $\lceil \widetilde{\nabla}_0(G) \rceil \leq \mathrm{Arb}_1(G) \leq 2\lceil \widetilde{\nabla}_0(G) \rceil$. In this paper, we shall show (Theorem 15) that for any positive integer p, there is a polynomial P_p such that for any graph G,

$$\left(\widetilde{\nabla}_{\frac{p-1}{2}}(G) \right)^{1/p} \leq \mathrm{Arb}_p(G) \leq P_p(\widetilde{\nabla}_{\frac{p-1}{2}}(G)). \tag{3}$$

The paper is organized as follows: In Section 2, we consider the key case of graphs with bounded tree-depth. In particular, we establish that if $f(2^{p-1}) > p-1$ for some value of p then there is a minor closed class of graphs $\mathcal{C}$ (namely the class of graphs with tree-depth at most p) such that $N_f(G,p)$ is unbounded. In Section 3, we prove that if, for some unbounded non-decreasing function f and each fixed integer p, the value $N_f(G,p)$ is bounded for graphs in a class $\mathcal{C}$, then the class $\mathcal{C}$ has bounded expansion. We prove also that, conversely, if $\mathcal{C}$ has bounded expansion and $f_0(x) = \lceil \log_2 x \rceil$ then $\sup_{G\in\mathcal{C}} N_{f_0}(G,p)$ is bounded for each integer p. In Section 4, we establish (3), which is the main result of this paper. For the sake of improving the readability of this paper, the proofs of two difficult lemmas used in Section 4 are actually postponed to Section 5. In Section 6 we consider a dual version of the problem.

2 Longest Cycles and Tree-Depth

Let us recall some definitions. The *height* of a rooted forest is the maximum number of vertices in a path from a root to a leaf. The *closure* of a forest F is the graph on $V(F)$ in which xy is an edge

if and only if x is an ancestor of y or y is an ancestor of x. The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest F such that G is a subgraph of its closure.

In this section we establish how the concept of tree-depth introduced in [11] is related to the length of the longest cycle of a graph. It will follow that if $f(2^{p-1}) > p-1$ for some value of p then there is a minor closed class of graphs $\mathcal{C}$ on which $N_f(G,p)$ is unbounded.

Lemma 1. *Let p be an integer such that the function f satisfies $f(2^{p-1}) > p-1$. Let $\mathcal{C}$ be the class of graphs with tree-depth at most p. Then $N_f(G,p)$ is unbounded on $\mathcal{C}$.*

Proof. Let N be an arbitrarily large integer. Let G be the closure of a rooted complete q-ary tree Y of height p, where $q = N^{\binom{p}{2}} + 1$. Let r be the root of Y. Given a leaf v of Y, there are $N^{\binom{p}{2}}$ ways to colour the edges of the subgraph of G induced by v and its ancestors with N colours, and let $\phi(v) \in \{1, \ldots, N^{\binom{p}{2}}\}$ be the encoding of this colouration corresponding to the leaf vertex v. For non leaf vertices v we define $\phi(v)$ by induction on the descending height as the majority value of $\phi(x)$ among the children of v. The root r has at least $\lceil q/N^{\binom{p}{2}} \rceil = 2$ sons v with $\phi(v) = \phi(r)$. Inductively, the root r is the root of a complete binary subtree Y' of Y, all vertices of which have the same ϕ-value as r. The closure of Y' contains a cycle γ of length 2^{p-1}, and this cycle gets at most $p-1$ colours (see Fig 2). As $\min(f(|\gamma|), p) = p$ we conclude that $N_f(G,p) > N$ and thus $N_f(G,p)$ is unbounded. □

Remark that the proof of Lemma 1 is a variant of an old trick of R. Goldblatt. Lemma 1 shows that we cannot expect $N_f(G,p)$ to be bounded on proper minor closed classes of graphs if $f(x) > \lceil \log_2 x \rceil$. In Section 3, we shall show that if $f(x) \leq \lceil \log_2 x \rceil$ then $N_f(G,p)$ are not only bounded on proper minor closed classes of graphs, but actually bounded on a class $\mathcal{C}$ if and only if $\mathcal{C}$ has bounded expansion. This provides yet another characterization of this robust notion.

We now prove that the connection with tree-depth shown in Lemma 1 is actually deeper in the sense that a 2-connected graph

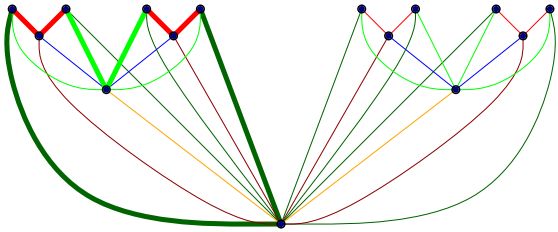

Figure 1: The closure of Y' contains a cycle γ of length 2^{p-1}, and this cycle gets exactly $p-1$ colours

has no long cycles if and only if it has a small tree-depth. (Note that 2-connectivity has to be assumed.)

Lemma 2. *Let G be 2-connected graph and let L be the maximum length of a cycle in G. Then*

$$1 + \lceil \log_2 L \rceil \leq \mathrm{td}(G) \leq \binom{L-1}{2} + 2.$$

Proof. The first inequality is a consequence of the monotonicity of tree-depth and the exact values of tree-depth for cycles: $\mathrm{td}(C_n) = 1 + \lceil \log_2 n \rceil$.

The remaining of the proof will concern the second inequality.

Consider a Depth-First Search tree Y of G and let r be the root of Y. Let $h = \mathrm{height}(Y)$. For a vertex x let $\mathrm{level}(x)$ be the height of x in Y. The rooted tree Y naturally defines a partial order $\preceq$ by $x \preceq y$ if x belongs to the tree-path from r to y. A basic property of DFS-trees is that two adjacent vertices are always comparable with respect to $\preceq$ (DFS-trees have no cross edges), so that $G \subseteq \mathrm{Clos}(Y)$ thus $\mathrm{td}(G) \leq h$.

For a vertex x, let $\mathrm{low}(x)$ be the smallest vertex y (with respect to $\preceq$) which is adjacent by an edge not in Y to a vertex $z \succeq x$. Notice that such a vertex exists as G is 2-connected and that for $x \neq r$ it holds $\mathrm{low}(x) \prec x$. Moreover, if $\mathrm{low}(x) \neq r$, the fact that

$\text{low}(x)$ is not a cut-vertex of G implies that there exists y such that $\text{low}(y) \prec \text{low}(x) \prec y \prec x$.

Let $P = (v_1 = r, \ldots, v_h)$ be a longest root-to-leaf path of Y. We inductively define indexes $a_1, b_1, \ldots, a_k, b_k \in \{1, \ldots, h\}$ as follows: let $a_1 = h$ and let $v_{b_1} = \text{low}(v_{a_1})$. For $i \geq 1$, if $b_i \neq 1$ then let $v_{b_{i+1}}$ be the minimum low of a vertex z such that $v_{b_i} \prec z \prec v_{a_i}$, let $v_{a_{i+1}}$ be such that $v_{b_i} \prec v_{a_{i+1}} \prec v_{a_i}$ and $\text{low}(a_{i+1}) = b_{i+1}$, and let e_i be a non-tree edge linking some $w \succeq v_{a_i}$ and v_{b_i}. This process stops at some value k such that $b_k = 1$. Notice that $a_{i+2} \preceq b_i$ for $1 \leq i \leq k-2$.

Let γ_i be the fundamental cycle of e_i and let γ be the symmetric difference of all the γ_i's. Then γ is a cycle and each edge of $P \cup \{e_1, \ldots, e_k\}$ belongs to 2 of the cycles $\gamma_1, \ldots, \gamma_k, \gamma$. Hence $2(h+k-1) \leq |\gamma_1| + \cdots + |\gamma_k| + |\gamma| \leq (k+1)L$, i.e. $h \leq \frac{k+1}{2}(L-2) + 2$ (see Fig. 2).

Moreover, γ contains at least two tree edges (either $k = 1$ and $\gamma = \gamma_1$ or $k > 1$ and then $a_2 < a_1$ and $b_k < b_{k-1}$) thus $L \geq |\gamma| \geq k+2$ hence $k \leq L-2$. Altogether, we obtain $h \leq \binom{L-1}{2} + 2$. □

It is not clear whether the quadratic bound of Lemma 2 is tight. We propose the following problem (compare with [5] where the chromatic number of a graph is bounded using the length of the largest odd cycle).

Problem 1. For a 2-connected graph G, denote by $L(G)$ the length of the longest cycle in G.

Does there exist a constant C such that for every 2-connected graph the following inequality holds:

$$\text{td}(G) \leq C\, L(G)?$$

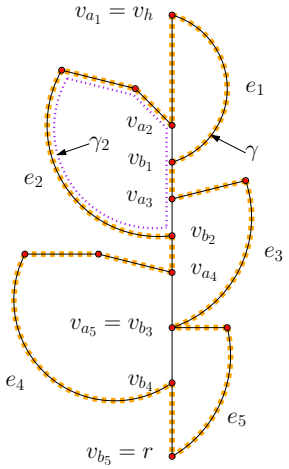

Figure 2: Illustration of the proof of Lemma 2. Cotree edge e_i links v_{a_i} (or one of its symmetric difference of the γ_i's

3 Classes of graphs with bounded expansion

Classes with bounded expansion have been introduced in [9, 10, 12] and are based on the boundedness of graph invariants similar to $\widetilde{\nabla}_r(G)$.

We denote [12, 13] by $G \triangledown r$ (resp. $G \widetilde{\triangledown} r$) the class of the simple graphs which are shallow minors (resp. simple shallow topological minors) of G at depth r, and we denote by $\nabla_r(G)$ (resp. $\widetilde{\nabla}_r(G)$) the maximum density of a graph in $G \triangledown r$ (resp. in $G \widetilde{\triangledown} r$), that is:

$$\nabla_r(G) = \max_{H \in G \triangledown r} \frac{\|H\|}{|H|} \qquad \widetilde{\nabla}_r(G) = \max_{H \in G \widetilde{\triangledown} r} \frac{\|H\|}{|H|}. \tag{4}$$

Notice that the main difference between the definition of $\widetilde{\nabla}_r(G)$ and the one of $\widetilde{\nabla}_r(G)$ stands in the way parallel edges are handled.

A class $\mathcal{C}$ has *bounded expansion* if $\sup_{G\in\mathcal{C}} \nabla_r(G)$ is bounded for each value of r. It is obvious that $\nabla_r(G) \geq \widetilde{\nabla}_r(G)$. However, it has been proved by Dvořák [1, 2] that for each integer r, $\nabla_r(G)$ is bounded by a polynomial function of $\widetilde{\nabla}_r(G)$. Hence a class $\mathcal{C}$ has bounded expansion if and only if $\sup_{G\in\mathcal{C}} \widetilde{\nabla}_r(G)$ is bounded .

Theorem 3. *Let $f : \mathbb{N} \to \mathbb{N}$ be an unbounded non-decreasing function with $f(x) \leq x$ and let $g : \mathbb{N} \to \mathbb{N}$ be defined by*

$$g(p) = \max\{i, f(i) \leq p\}.$$

Then for every graph G and every integer r we have:

$$\nabla_r(G) \leq N_f(G, 2r+1)^{2r+1} g(2r+1)^2.$$

Proof. Let $N = N_f(G, 2r+1)$ and let $c : E(G) \to [N]$ be a colouring of the edges of G such that each cycle γ gets at least $\min(f(|\gamma|), 2r+2)$ colours. For a subset $I \in \binom{[N]}{2r+1}$ of $2r+1$ colours, let G_I be the subgraph of G whose edges are coloured by colours in I. Then the maximum length of a cycle of G_I is $g(2r+1)$. According to Lemma 2, blocks of G_I have tree-depth at most $\binom{g(2r+1)-1}{2} + 2$.

Let $K \in G \triangledown r$ be such that $\|K\|/|K| = \nabla_r(G)$, let $x_1, \ldots, x_k$ be the roots of trees $T_1, \ldots, T_k$ of height at most r (corresponding to vertices $h_1, \ldots, h_k$ of K) and $H \subseteq G[V(T_1) \cup \cdots \cup V(T_k)]$ be such that $K \cong H/(E(T_1) \cup \cdots \cup E(T_k))$.

If h_i and h_j are adjacent in K then there exists in H a path $P_{i,j}$ of length at most $2r+1$ linking x_i and x_j. Denote by $A_{i,j}$ a subset of $2r+1$ colours such that all the edges of $P_{i,j}$ have their colour in $A_{i,j}$. For $I \in \binom{[N]}{2r+1}$, let H_I be subgraph of H containing the edges of the paths $P_{i,j}$ for which $A_{i,j} = I$. Let K_I be the corresponding subgraph of K. As the blocks of H_I are included in blocks of G_I, they have tree-depth at most $\binom{g(2r+1)-1}{2} + 2$. As tree-depth is minor monotone, this bound also applies to the blocks of K_I. Observe that a graph of tree depth k has density at most $k-1$, because if all edges are oriented from higher level end vertex to lower level end vertex, then each vertex has out-degree at most $k-1$. It follows that

$\|K_I\|/|K| \leq \|K_I\|/|K_I| \leq \binom{g(2r+1)-1}{2} + 1$. By summing up over the possible choices of I we obtain

$$\|K\|/|K| \leq \binom{N}{2r+1}\left(\binom{g(2r+1)-1}{2}+1\right) \leq N_f(G, 2r+1)^{2r+1} g(2r+1)^2.$$

□

Corollary 4. *Let $\mathcal{C}$ be a class of graph and let $f : \mathbb{N} \to \mathbb{N}$ be an unbounded non-decreasing function with $f(x) \leq x$.*

If $\sup_{G\in\mathcal{C}} N_f(G, r) < \infty$ for every $r \in \mathbb{N}$ then $\mathcal{C}$ has bounded expansion.

Theorem 5. *Let $f_0(x) = \lceil \log_2 x \rceil$ and let r be an integer. There exists a polynomial P_r such that for every graph G it holds*

$$N_{f_0}(G, r) \leq P_r(\widetilde{\nabla}_{2^r}(G)).$$

Proof. According to [15] there exists for each $p \in \mathbb{N}$ a polynomial Q_p such that every graph G has a vertex colouring $c : V(G) \to [Q_p(\widetilde{\nabla}_{2^{p-1}}(G))]$ such that the subgraph induced by any $i \leq p$ colours has tree-depth at most i. Let $p = r + 1$ and let $P_r(X) = \binom{Q_{r+1}(X)}{2}$. The graph G admits a vertex-colouring $c : V(G) \to [Q_{r+1}(\widetilde{\nabla}_{2^r}(G))]$ such that the subgraph induced by any $i \leq r + 1$ colours has tree-depth at most i. Colour each edge $\{x, y\}$ of G by the set $\{c(x), c(y)\}$ (hence using $P_r(\widetilde{\nabla}_{2^r}(G))$ colours). Let γ be a cycle of G with $i \leq r$ colours. By construction, the vertices of γ use at most $i + 1 \leq r + 1$ colours hence $\mathrm{td}(\gamma) \leq i + 1$, thus $|\gamma| \leq 2^i$. It follows that every cycle γ of G gets at least $\min(r + 1, f_0(|\gamma|)$ colours. □

Corollary 6. *Let $\mathcal{C}$ be a class of graphs. Then the following are equivalent:*

1. *There exists non-decreasing unbounded $f : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall p \in \mathbb{N}, \qquad \sup_{G\in\mathcal{C}} N_f(G, p) < \infty$$

2. *Let* $f_0(x) = \lceil \log_2 x \rceil$. *Then*

$$\forall p \in \mathbb{N}, \qquad \sup_{G \in \mathcal{C}} N_{f_0}(G, p) < \infty$$

3. *the class* $\mathcal{C}$ *has bounded expansion.*

Notice that the second condition is tight, according to Lemma 1.

4 Bounds on $\mathrm{Arb}_p(G)$

The lower bound $\left(\widetilde{\nabla}_{\frac{p-1}{2}}(G)\right)^{1/p}$ for $\mathrm{Arb}_p(G)$ is easy.

Lemma 7. *Let* G *be a graph and let* p *be a positive integer. Then* $(\mathrm{Arb}_p(G))^p$ *is greater than or equal to the maximum arboricity of a multigraph* H *such that* G *contains a* $\leq (p-1)$*-subdivision of* H*, that is,*

$$(\mathrm{Arb}_p(G))^p \geq \max\{\mathrm{Arb}(H), H \in G\widetilde{\nabla}\left(\tfrac{p-1}{2}\right)\} \geq \widetilde{\nabla}_{\frac{p-1}{2}}(G). \qquad (5)$$

Proof. Let c be an edge colouring of G with a set J of $\mathrm{Arb}_p(G)$ colours such that every cycle C of G gets at least $\min(|C|, p+1)$ colours. Assume that G includes a $\leq (p-1)$-subdivision S of a multigraph H. Colour each edge e of H by the set X of colours used by the edges of the path of length at most p in G corresponding to e in S. The total number of colours used by edges of H is at most the number of $\leq p$-subsets of the J, which is less than $(\mathrm{Arb}_p(G))^p$. If $(\mathrm{Arb}_p(G))^p < \mathrm{Arb}(H)$, then H has a monochromatic cycle, each edge being coloured by a set X of at most p colours. Then the corresponding cycle C of G uses at most $|X| < p+1$ colours and has length at least $2|X|$, contradicting the colouring assumption. Thus $(\mathrm{Arb}_p(G))^p \geq \mathrm{Arb}(H)$. □

The upper bound is more involved. First, we introduce the admittedly rather technical definition of fraternal completion of oriented multigraphs.

A digraph $\vec{G}$ is *fraternally oriented* if $(x,z) \in E(\vec{G})$ and $(y,z) \in E(\vec{G})$ implies $(x,y) \in E(\vec{G})$ or $(y,x) \in E(\vec{G})$. This concept was introduced by Skrien [14] and a characterization of fraternally oriented digraphs having no symmetrical arcs has been obtained by Gavril and Urrutia [4], who also proved that triangulated graphs and circular arc graphs are all fraternally orientable graphs.

In the context of multigraphs, this notion may be extended as follows:

Definition 1. Let $\vec{G}$ be a directed multigraph and let a be a positive integer. A *fraternal completion* of $\vec{G}$ of *depth* a is a triple $\mathfrak{f} = ((E_1, \ldots, E_a), w, \kappa)$, where

- $E_1 = E(\vec{G})$ is the arc set of $\vec{G}$; for each $2 \leq i \leq a$, E_i is the arc set of a multigraph having $V(\vec{G})$ as its vertex set; for every $1 \leq i < j \leq a$, $E_i \cap E_j = \emptyset$ (although **different** arcs of E_i and E_j may have the same head and tail);
- for $e \in \bigcup_{1 \leq i \leq a} E_i$, the *weight* $w(e)$ of e is the integer $i \in [a]$ such that $e \in E_i$;
- $\kappa : \bigcup_{1 < i \leq a} E_i \rightarrow \left(\bigcup_{1 \leq i \leq a} E_i\right)^2$ is such that for every $e \in \bigcup_{1 < i \leq a} E_i$ with $\kappa(e) = (f,g)$ we have

$$\begin{aligned} \mathrm{tail}(f) &\neq \mathrm{tail}(g) \\ w(e) &= w(f) + w(g) \\ \mathrm{tail}(e) &= \mathrm{tail}(f) \\ \mathrm{head}(e) &= \mathrm{tail}(g) \\ \mathrm{head}(f) &= \mathrm{head}(g); \end{aligned}$$

- conversely, for every $i, j \in \mathbb{N}, f \in E_i$ and $g \in E_j$ such that $i + j \leq a, \mathrm{tail}(f) \neq \mathrm{tail}(g)$ and $\mathrm{head}(f) = \mathrm{head}(g)$ there exists a unique $e \in E_{i+j}$ such that $\kappa(e) \in \{(f,g),(g,f)\}$.

We also define the *arc set* $E_\mathfrak{f}$ of the fraternal completion $\mathfrak{f}$ by $E_\mathfrak{f} = \bigcup_{1 \leq i \leq a} E_i$ (notice that $E_\mathfrak{f}$ includes no loop) and define $a = \mathrm{depth}(\mathfrak{f})$

as the depth of $\mathfrak{f}$. A fraternal completion $\mathfrak{f}' = ((E'_1, \ldots, E'_{a'}), w', \kappa')$ of $\vec{G}$ *extends* another fraternal completion $\mathfrak{f} = ((E_1, \ldots, E_a), w, \kappa)$ of $\vec{G}$ (or is an *extension* of $\mathfrak{f}$) if

- $a' = \text{depth}(\mathfrak{f}') > a = \text{depth}(\mathfrak{f})$,
- for every $1 \leq i \leq \text{depth}(\mathfrak{f})$ we have $E'_i = E_i$,
- the restrictions of κ and κ' to $E_{\mathfrak{f}}$ coincide.

We now state an easy lemma of fraternal completions:

Lemma 8. *For every oriented multigraph $\vec{G}$ and every positive integer a,*

- *$\vec{G}$ has a unique fraternal completion of depth 1 defined by $E_1 = E(\vec{G})$,*
- *every fraternal completion $\mathfrak{f}$ of depth a has an extension of depth $a+1$.*

Proof. The first item is direct from the definition.

For the second item, let $\mathfrak{f} = ((E_1, \ldots, E_a), w, \kappa)$ be a fraternal completion of $\vec{G}$ of depth a. Consider an arbitrary numbering ν of $E_{\mathfrak{f}}$. Define

$$E_{a+1} = \{e_{f,g} : (f,g) \in E_{\mathfrak{f}}^2, \quad w(f) + w(g) = a+1, \nu(f) < \nu(g), \\ \text{tail}(f) \neq \text{tail}(g), \text{ and } \text{head}(f) = \text{head}(g)\},$$

where $e_{f,g}$ is an arc with $\text{tail}(e_{f,g}) = \text{tail}(f)$ and $\text{head}(e_{f,g}) = \text{tail}(g)$; define the mapping $\kappa' : \bigcup_{1 \leq i \leq a+1} E_i \to E_{\mathfrak{f}}$ by $\kappa'(e) = \kappa(e)$ if $w(e) \leq a$ and $\kappa'(e_{f,g}) = (f,g)$ for $e_{f,g} \in E_{a+1}$; also define $w' : E_{\mathfrak{f}} \cup E_{a+1}$ by $w'(e) = i$ if $e \in E_i$. Then $\mathfrak{f}' = ((E_1, \ldots, E_{a+1}, w', \kappa')$ is obviously an extension of $\mathfrak{f}$ of depth $a+1$. □

Suppose $\mathfrak{f} = ((E_1, \ldots, E_a), w, \kappa)$ is a fraternal completion of $\vec{G}$ of depth a. We associate to each arc $e \in E_{\mathfrak{f}}$ a walk $W(e)$ of $\vec{G}$ defined as follows:

- If $w(e) = 1$ then $W(e)$ is the walk e,
- Otherwise, if $\kappa(e) = (f, g)$ then $W(e)$ is the walk $W(f)$ followed by the reverse $\overline{W}(g)$ of the walk $W(g)$, what we denote by $W(e) = W(f) \cdot \overline{W}(g)$.

This way, each arc e of $E_\mathfrak{f}$ is associated a walk $W(e)$ in $\vec{G}$ of length $w(e)$ which has the same endpoints as e. An arc $e \in E_\mathfrak{f}$ is called *simple* if $W(e)$ is a path.

Let $\vec{G}$ be a directed multigraph, let $\mathfrak{f} = ((E_1, \dots, E_a), w, \kappa)$ be a fraternal completion of $\vec{G}$ of depth a. Let $\prec$ be the partial order on $E_\mathfrak{f}$ defined by transitivity from the conditions

$$\kappa(e) = (f, g) \quad \Longrightarrow \quad f \prec e \text{ and } g \prec e.$$

Notice that if $e \in E_\mathfrak{f}$ and $f \in E_1$ we have $e \succeq f$ if and only if f belongs to the walk $W(e)$.

For $i = 1, 2, \dots, a$, let $\vec{H}_i$ (resp. $\vec{H}_{\leq i}$) be the multigraphs with vertex set $V(\vec{G})$ and arc set E_i (resp. $\bigcup_{j \leq i} E_j$). In particular, $\vec{G} = \vec{H}_1$. For arcs e_1, e_2 of $\vec{G}$ and a fraternal completion $\mathfrak{f} = ((E_1, \dots, E_a), w, \kappa)$ of $\vec{G}$ of depth a of $\vec{G}$, say that a pair $(e_1, e_2) \in E_1^2$ is a *conflict* if there exists arcs $f_1 \succeq e_1$ and $f_2 \succeq e_2$ and a directed path of length at most a of $\vec{H}_{\leq a}$ starting with f_1 and ending at one of the endpoints of f_2 (notice that we allow $f_1 = f_2$).

Lemma 9. *Assume $\vec{G}$ is an orientation of G, $\mathfrak{f}$ is a fraternal completion of $\vec{G}$ of depth a. Assume that $c : E(\vec{G}) \to [N]$ is a colouration of the edges of $\vec{G}$ such that for every conflict (e_1, e_2) we have $c(e_1) \neq c(e_2)$. Then every cycle γ gets at least $\min(|\gamma|, a+1)$ colours.*

Proof. Assume for contradiction that there exist in $\vec{G}$ a cycle $\gamma = (v_1, \dots, v_{|\gamma|})$ which gets less than $\min(|\gamma|, a + 1)$ colours. We say a sequence $(e_1, e_2, \dots, e_{q+1})$ of simple arcs in $E_\mathfrak{f}$ is *admissible* if the $W(e_i)$ are pairwise arc-disjoint and form consecutive subpaths of γ and $e_1, \dots, e_q$ form a directed path of $\vec{H}_{\leq a}$.

Choose an admissible sequence $(e_1, e_2, \ldots, e_{q+1})$ in such a way that $\sum_i w(e_i)$ is maximal and then that q is minimum. Without loss of generality, we may assume that for $1 \leq i \leq q$ we have $e_i = (v_{a_i}, v_{a_{i+1}})$ with $1 = a_1 < a_2 < \cdots < a_{q+1} \leq |\gamma|$.

First we show that $\sum_{i=1}^{q+1} w(e_i) \geq \min(a+1, |\gamma|)$. Assume for contradiction that $\sum_{i=1}^{q+1} w(e_i) < \min(a+1, |\gamma|)$. Then e_{q+1} has $v_{a_{q+1}}$ and $v_{a_{q+2}}$ as endpoints with $a_{q+1} < a_{q+2} < |\gamma|$ (actually we have $a_{q+2} = \sum_i w(e_i) - 1$). Let g be the arc of $\vec{G}$ linking $v_{a_{q+2}}$ to the next vertex of γ. According to the maximality of $\sum_i w(e_i)$, the sequence $(e_1, \ldots, e_{q+1}, g)$ is not admissible hence $e_1, \ldots, e_{q+1}$ is not a directed path, that is: $e_{q+1} = (v_{a_{q+2}}, v_{a_{q+1}})$. As $w(e_q) + w(e_{q+1}) \leq \sum_i w(e_i) \leq a$ there exists an arc f in $E_{\mathfrak{f}}$ such that $\kappa(f) = (e_q, e_{q+1})$ (see Fig 3). The arc f is clearly simple ($W(f) = W(e_q)\overline{W}(e_{q+1})$ or $W(f) = W(e_{q+1})\overline{W}(e_q)$). If $\kappa(f) = (e_q, e_{q+1})$ then $(e_1, \ldots, e_{q-1}, f, g)$ is admissible, and $\sum_{i=1}^{q-1} w(e_i) + w(f) + w(g) = \sum_{i=1}^{q+1} w(e_i) + w(g) > \sum_{i=1}^{q+1} w(e_i)$, what contradicts the maximality of $\sum_i w(e_i)$. Otherwise, $\kappa(f) = (e_{q+1}, e_q)$. Then, $(e_1, \ldots, e_{q-1}, f)$ is an admissible sequence such that $\sum_{i=1}^{q-1} w(e_i) + w(f) = \sum_{i=1}^{q+1} w(e_i)$, what contradicts the minimality of q (for given $\sum_i w(e_i)$).

Thus $\left|\bigcup_i W(e_i)\right| = \sum_{i=1}^{q+1} w(e_i) \geq \min(a+1, |\gamma|)$. By assumption, the cycle γ which gets less than $\min(|\gamma|, a+1)$ colours hence there exist $f_1, f_2 \in \bigcup_i W(e_i)$ such that $c(f_1) = c(f_2)$. Let b_1, b_2 be such that $f_1 \in W(e_{b_1})$ and $f_2 \in W(e_{b_2})$. Without loss of generality we assume $b_1 \leq b_2$. As $f_1 \preceq e_{b_1}, f_2 \preceq e_{b_2}$ and as there exists by construction a (maybe empty) directed path of length at most a of $\vec{H}_{\leq a}$ starting with e_{b_1} and ending at one of the endpoints of e_{b_2} we deduce that (f_1, f_2) is a conflict, contradicting the hypothesis that $c(f_1) = c(f_2)$. □

To prove that $\mathrm{Arb}_p(G) \leq P_p(\widetilde{\nabla}_{\frac{p-1}{2}}(G))$ for some polynomial P_p, it suffices to find a fraternal completion $\mathfrak{f}$ of an orientation $\vec{G}$ of G of depth p so that each edge e of G is in conflicts with at most $P_p(\widetilde{\nabla}_{\frac{p-1}{2}}(G))$ other edges. We shall see that this problem can be

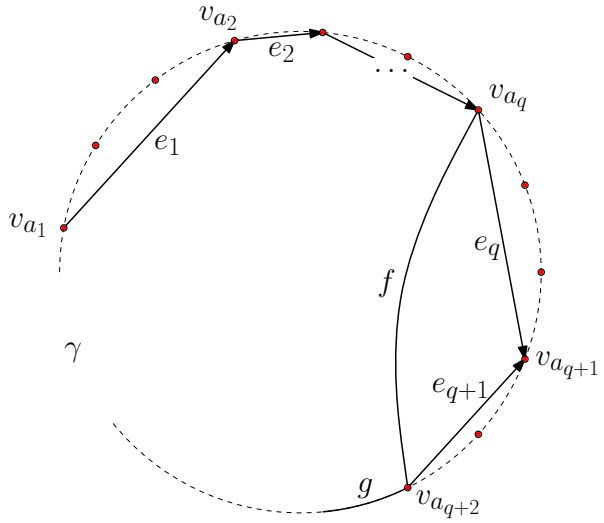

Figure 3: Illustration for the proof of Lemma 9.

reduced to finding a fraternal completion with bounded in-degrees. Toward this end, let

$$C(\mathfrak{f}) = \sum \left\{ \prod_j \Delta^-(\vec{H}_{i_j}) : \sum_j i_j < \operatorname{depth}(\mathfrak{f}) \right\}.$$

Lemma 10. *For every arc e of $\vec{G}$ we have*

$$\left|\{f \in E_{\mathfrak{f}} : f \succeq e\}\right| \leq C(\mathfrak{f}).$$

Proof. For each $f \succeq e$, there is a sequence of arcs $g_1, g_2, \ldots, g_t$ in $\mathfrak{f}$ such that $g_1 = e$, $g_t = f$ and g_i covers g_{i-1} in $\prec$. So it suffices to show that for any $g \succeq e$ with $w(f) = i$, for any $j > i$, there are at most $\Delta^-(\vec{H}_{j-i})$ arcs g' with $w(g') = j$ such that g' covers g in $\prec$.

This is so because the number of such arcs g' is at most the number of arcs f' such that $\kappa(g') \in \{(g, f'), (f', g)\}$ and hence at most the number of arcs f' with $\text{head}(f') = \text{head}(g)$ in $\vec{H}_{j-i}$, which is at most $\Delta^-(\vec{H}_{j-i})$. □

Lemma 11. *For every arc e_2 of $\vec{G}$ there exist at most $3pC(\mathfrak{f})(\max(2, \Delta^-(\vec{H}_p))^a$ arcs e_1 of $\vec{G}$ such that (e_1, e_2) is a conflict.*

Proof. According to Lemma 10 there exists at most $C(\mathfrak{f})$ arcs $f_2 \in \bigcup_{i=1}^{p} E_i$ such that $f_2 \succeq e_2$. Given an arc f_2 of $\vec{H}_{\leq p}$ there exist at most $(\Delta^-(\vec{H}_p) - 1)(1 + \cdots + \Delta^-(\vec{H}_p)^{p-1}) = \Delta^-(\vec{H}_p)^p - 1$ arcs f_1 such that there exists in $\vec{H}_{\leq p}$ a directed path of length at most p starting with f_1 and ending at the head of f_2. Similarly we have at most $1 + \cdots + \Delta^-(\vec{H}_p)^p$ arcs f_1 such that there exists in $\vec{H}_p$ a directed path of length at most p starting with f_1 and ending at the tail of f_2. Hence for each f_2 we have at most $3\max(2, \Delta^-(\vec{H}_p)^p$ possibilities for f_1. As there are $|W(f_1)| \leq a$ arcs e_1 such that $f_1 \succeq e_1$, we conclude. □

Lemma 12. *Let $Q_1(X), \ldots, Q_p(X)$ be polynomials, and let P_p be the polynomial defined by:*

$$P_p(X) = 6p(2 + Q_p(X))^p \left(\sum_{\sum i_j < p} \prod_j Q_{i_j}(X) \right) + 1. \tag{6}$$

Let G be a multigraph with a fraternal completion $\mathfrak{f}$ of depth p such that

$$\forall 1 \leq i \leq p, \qquad \Delta^-(\vec{H}_i) \leq Q_i(\widetilde{\nabla}_{(p-1)/2}(G)). \tag{7}$$

Then

$$\text{Arb}_p(G) \leq P_p(\widetilde{\nabla}_{\frac{p-2}{2}}(G)). \tag{8}$$

(Of course, the polynomial P_p depends on polynomials $Q_1, \ldots, Q_p$. This dependence will be clear from the context.)

Proof. According to Lemma 11 and Lemma 9, G has a colouration of its edges by at most $P_p(\widetilde{\nabla}_{(p-1)/2}(G))$ colours such that every cycle γ gets at least $\min(p+1, |\gamma|)$ colours, that is

$$\mathrm{Arb}_p(G) \leq P_p(\widetilde{\nabla}_{\frac{p-1}{2}}(G)).$$

□

By Lemma 12, to prove that for some polynomial P_p, $\mathrm{Arb}_p(G) \leq P_p(\widetilde{\nabla}_{\frac{p-1}{2}}(G))$, it suffices to prove that one can find a fraternal completion $\mathfrak{f}$ of an orientation $\vec{G}$ of G of depth p so that $\Delta^-(\vec{H}_i)$ is bounded by some polynomial function Q_i of $\widetilde{\nabla}_{\frac{p-1}{2}}(G)$. The construction of such a fraternal completion $\mathfrak{f}$ is easy: Let H_i be the underlying graph of $\vec{H}_i$. By definition, $H_1 = G$ and for $i = 1, 2, \ldots, p-1$, H_{i+1} is uniquely determined by $\vec{H}_j$ for $j = 1, 2, \ldots, i$. For $i = 1, 2, \ldots, p$, we orient the edges of H_i to obtain $\vec{H}_i$ so that $\Delta^-(\vec{H}_i) = \lceil \widetilde{\nabla}_0(H_i) \rceil$. This defines a fraternal completion $\mathfrak{f}$ of an orientation $\vec{G}$ of G of depth p. In the following, we shall show that for this fraternal completion $\mathfrak{f}$, $\Delta^-(\vec{H}_i)$ is bounded by some polynomial function of $\widetilde{\nabla}_{\frac{p-1}{2}}(G)$.

For $i = 1, 2, \ldots, p$, let T_i be the $(i-1)$-subdivision of the underlying graph H_i of $\vec{H}_i$. Hence

$$H_i \in T_i \widetilde{\nabla}(\tfrac{i-1}{2}).$$

In particular, $H_1 = T_1 = G$.

For integer m, let $G \bullet m$ be the multigraph with vertex set $V(G) \times [m]$ where $\{(x,i),(y,j)\}$ is an edge of $G \bullet m$ of multiplicity k if and only if $\{x,y\}$ is an edge of G of multiplicity k. In the following section, we shall prove the following two lemmas.

Lemma 13. *Let G be a multigraph, let m be a positive integer and let r be a positive half-integer. Then*

$$\widetilde{\nabla}_r(G \bullet m) \leq (2r(m-1)+1)\,\widetilde{\nabla}_r(G) + m^2\,\widetilde{\nabla}_0(G) + m - 1. \qquad (9)$$

Lemma 14. *For every integer $p \geq 2$, there is a polynomial N_p such that T_p is a subgraph of $G \bullet N_p(\widetilde{\nabla}_{\frac{p-2}{2}}(G))$.*

From these lemmas will then follow the main result of this paper:

Theorem 15. *For each integer p there exists a polynomial P_p such that for every multigraph G it holds*

$$\left(\widetilde{\nabla}_{\frac{p-1}{2}}(G)\right)^{1/p} \leq \mathrm{Arb}_p(G) \leq P_p(\widetilde{\nabla}_{\frac{p-1}{2}}(G)).$$

In particular, Arb_p *and* $\widetilde{\nabla}_{\frac{p-1}{2}}$ *are two polynomially equivalent multigraph invariants.*

Proof. The inequality $\left(\widetilde{\nabla}_{\frac{p-1}{2}}(G)\right)^{1/p} \leq \mathrm{Arb}_p(G)$ follows from Lemma 7. By the argument above, Lemma 13 and Lemma 14 imply that $\Delta^-(\vec{H}_i)$ is bounded by some polynomial function of $\widetilde{\nabla}_{\frac{p-1}{2}}(G)$, and hence, by Lemma 12, there exists a polynomial P_p such that $\mathrm{Arb}_p(G) \leq P_p(\widetilde{\nabla}_{\frac{p-1}{2}}(G))$. □

5 Proofs of Lemmas 13 and 14

Proof of Lemma 13. The vertices of $K \bullet m$ are the pairs (v, i), v a vertex of G and $1 \leq i \leq m$. The every vertex v of G, we say that (v, i) and (v, j) are *twins* in $G \bullet m$ and we denote by π the projection of $G \bullet m$ into G which maps (v, i) to v.

Let S be a subgraph of $G \bullet m$ which is ($\leq 2r$)-subdivision of a multigraph H such that

$$\widetilde{\nabla}_r(G \bullet m) = \frac{\|H\|}{|H|}.$$

Choose S with the minimal number of vertices.

A path of S corresponding to an edge of H is called a *branch*. The vertices of S corresponding to vertices of H are called *principal vertices*. The other vertices of S are *subdivision vertices*.

Let S_0 be the graph obtained from S by deleting all the branches which are not subdivided and let H_0 be the corresponding subgraph of H (S_0 is a $(\leq 2r)$-subdivision of H_0 and every branch of S_0 is a path of length at least 2). Then we have $\|H\| \leq \|H_0\| + m^2\|G[\pi(V(H))]\|$ hence

$$\frac{\|H\|}{|H|} \leq \frac{\|H_0\|}{|H|} + m^2 \frac{\|G[\pi(V(H))]\|}{|\pi(V(H))|}.$$

Thus

$$\widetilde{\nabla}_r(G \bullet m) \leq \frac{\|H_0\|}{|H_0|} + m^2\, \widetilde{\nabla}_0(G).$$

First notice that no branch of $S(H)$ contains two twin vertices, except if the branch is a single edge path linking two twin vertices (otherwise we can shorten the branch without changing $\|H\|$ and $|H|$, see Fig 4).

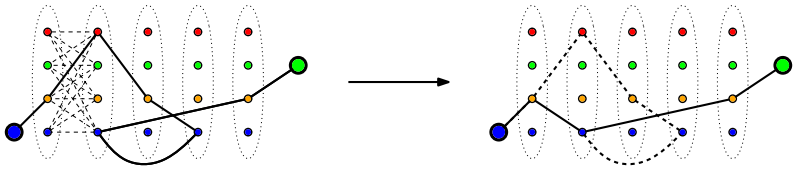

Figure 4: If a branch contains twin vertices, we shorten it.

We define the multigraph H_1 and its $(\leq 2r)$-subdivision S_1 by the following procedure: Start with $H_1 = H_0$ and $S_1 = S_0$. Then, for each subdivision vertex $v \in S_1$ having a twin which is a principal vertex of S_1, delete the branch of S_1 containing v and the corresponding edge of H_1. In this way, we delete at most $(m-1)|H_0|$ edges of H_1. Thus $\frac{\|H_1\|}{|H_1|} \geq \frac{\|H_0\|}{|H_0|} - (m-1)$ and S_1 is such that no subdivision vertex is a twin of a principal vertex.

Given H_1 we construct the conflict graph C of H_1 as follows: the vertex set of C is the edge set of H_1 and the edges of C are the

pairs of edges $\{e_1, e_2\}$ such that one of the subdivision vertices of the branch corresponding to e_1 is a twin of one of the subdivision vertex of the branch corresponding to e_2. Note that graph C has maximum degree at most $2r(m-1)$ hence it is $(2r(m-1)+1)$-colourable. Let H_2 be the subgraph of H_1 induced by a monochromatic set of edges of H_1 of size at least $\frac{\|H_1\|}{2r(m-1)+1}$. So $\frac{\|H_1\|}{|H_1|} \leq (2r(m-1)+1)\frac{\|H_2\|}{|H_2|}$.

Let S_2 be the corresponding subgraph of S_1. If v is a principal vertex of S_2, then two edges incident to v cannot have their other endpoints equal or twins (because of the colouration). Let $S_3 = \pi(S_2)$ be the projection of S_2 on G. Because of the above colouration, S_3 is a $(\leq 2r)$ subdivision of H_2. Hence

$$\begin{aligned}\widetilde{\nabla}_r(G) &\geq \frac{\|H_2\|}{|H_2|} \geq \frac{\|H_1\|}{(2r(m-1)+1)|H_1|} \\ &\geq \frac{\|H_0\|}{(2r(m-1)+1)|H_0|} - \frac{m-1}{2r(m-1)+1}\end{aligned}$$

and the result follows. □

Proof of Lemma 14

Let $\mathfrak{f} = ((E_1, \ldots, E_p), w, \kappa)$ be a fraternal completion of $\vec{G}$ of depth p constructed in such a way that for $i = 1, 2, \ldots, p$, $\Delta^-(\vec{H}_i) = \widetilde{\nabla}_0(H_i)$.

In the following, we shall prove that for $2 \leq a \leq p$ there is a polynomial N_a such that the graph T_a can be injectively embedded into a blowing $G \bullet N_a(\widetilde{\nabla}_{\frac{a-2}{2}}(G))$ of G. Observe that if this is true for $a = 2, \ldots, i$, then by using Lemma 13, we can conclude that there is a polynomial P_i such that $\widetilde{\nabla}_0(H_i) \leq P_i(\widetilde{\nabla}_{\frac{i-1}{2}}(G))$.

By definition, for $a \geq 1$, T_a is obtained from the empty graph on $V(G)$ by adding, for each arc $e = xy$ in E_a, an induced path of length a connecting x and y. Each arc $e = xy$ in E_a corresponds to a walk $W(e)$ in G of length a from x to y. By sending the induced x-y-path of length a in T_a to the corresponding walk in G connecting x and y, we obtain a homomorphism, say f, from T_a to G. However, f is not an embedding, as many vertices of T_a may have the same image. Indeed, $V(T_a)$ is the union of $V(G)$ and a set of $|E(T_a)|(a-1)$ added

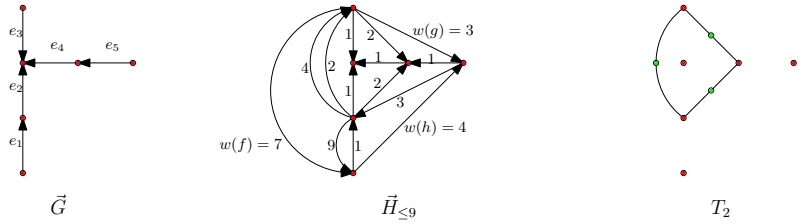

Figure 5: Example of graphs defined by a fraternal completions. On the left, the graph $\vec{G}$. In the middle, arcs of a fraternal completion of depth 9 form the multigraph $\vec{H}_{\leq 9}$; here, $\kappa(f) = (g, h)$; the walk $W(h) = (e_1 e_2 \overline{e}_4 \overline{e}_5)$ associated to h is simple but the walk $W(f) = (e_3 \overline{e}_4 \overline{e}_5 e_5 e_4 \overline{e}_2 \overline{e}_1)$ associated to f is not. On the right, the 1-subdivision of the arcs in E_2 define the undirected multigraph T_2.

vertices which are the interior vertices of the walks $W(e)$. If for some integer m, each vertex v of G is contained in at most $m-1$ of the walks $W(e)$ as an interior vertex, then T_a embeds into $G \bullet m$, as we can assign to each vertex in $f^{-1}(v)$ a distinct vertex of $\{v\} \times [m]$ in $G \bullet m$ as its image. Let us consider the case $a = 2$. By our construction, $\vec{G}$ has $\Delta^-(\vec{G}) = \lceil \widetilde{\nabla}_0(G) \rceil$ Each arc $e = xy$ in E_2 corresponds to a walk of the form (x, v, y), where (x, v) and (y, v) are arcs of $\vec{G}$. Let $d = \lceil \widetilde{\nabla}_0(G) \rceil = \Delta^-(\vec{G})$. Then v is an interior vertex of a walk $W(e)$ for $e = xy$ in E_2 if and only if (x, v) and (y, v) are arcs of $\vec{G}$. Hence v is contained in at most $\binom{d}{2}$ walks $W(e)$ as a an interior vertex. Therefore for $N_2(x) = \binom{x+1}{2} + 1$, T_2 embeds into $G \bullet N_2(\widetilde{\nabla}_0(G))$.

Assume now that the polynomial N_i is defined for $i = 2, \ldots, a-1$ and T_i embeds into $G \bullet N_i(\widetilde{\nabla}_{\frac{i-2}{2}}(G))$. Each arc $e = xy \in E_a$ corresponds to two arcs $g \in E_i$ and $g' \in E_j$ with $i + j = a$, with $g = xz$ and $g' = yz$ for some z. A vertex v is an interior vertex of the walk $W(e)$ if and only if either $v = z$ or v is an interior vertex of $W(g)$ or $W(g')$. By our definition of the fraternal completion,

v has in-degree at most $\Delta^-(\vec{H}_i) = \lceil \widetilde{\nabla}_0(H_i) \rceil$ in $\vec{H}_i$. By induction hypothesis and the observation above, $\widetilde{\nabla}_0(H_i) \leq P_i(\widetilde{\nabla}_{\frac{i-1}{2}}(G))$ for some polynomial P_i. This implies that for some polynomial Q, each vertex v has in-degree at most $d = Q(\widetilde{\nabla}_{\frac{a-2}{2}}(G))$ in $\vec{H}_{\leq(a)}$. So for each vertex v of G, there are at most $\binom{d}{2}$ pairs of arcs g, g' for which there is an edge $e = xy$ in E_a with $k(e) = (g, g')$.

For an edge $e = xy \in E_a$, we say a vertex v is the *transfer vertex* of $W(e)$ if $k(e) = (g, g')$, $g = (x, v)$ and $g' = (y, v)$. An interior vertex of $W(e)$ is either a transfer vertex of $W(e)$ or an interior vertex of g for some $g \in E_i$ where $i \leq a - 1$.

By induction hypothesis, there is a polynomial Z such that a vertex v appears at most $Z(\widetilde{\nabla}_{\frac{a-3}{2}}(G))$ times as an interior vertex of a walk $W(g)$ for $g \in \cup_{i=1}^{a-1} E_i$. For each $g \in E_i$ for some $i \leq a-1$, there are at most $2d$ edges $e \in E_a$ such that $k(e) = (g, g')$ or $k(e) = (g', g)$. Therefore, each vertex v appears at most $\binom{d}{2} + 2d \times Z(\widetilde{\nabla}_{\frac{a-2}{2}}(G))$ times as an interior vertex of $W(e)$ for $e \in E_a$. As $d = Q(\widetilde{\nabla}_{\frac{a-2}{2}}(G))$ and $\widetilde{\nabla}_{\frac{a-3}{2}}(G) \leq \widetilde{\nabla}_{\frac{a-2}{2}}(G)$, with $N_a(x) = Q(x)^2 + 2Q(x)Z(x)$, each vertex appears at most $N_a(\widetilde{\nabla}_{\frac{a-2}{2}}(G)) - 1$ times as an interior vertex of $W(e)$ for some $e \in E_a$. Therefore with $m = N_a(\widetilde{\nabla}_{\frac{a-2}{2}}(G))$, T_a embeds into $G \bullet m$. This completes the proof of Lemma 14. □

6 The Dual Version

The problem addressed in this paper can be considered in the more general context of matroids:

Problem 2. Let $\mathcal{M}$ be a matroid and let p be an integer. What is the minimum number $\mathrm{Arb}_p(\mathcal{M})$ needed to colour the elements of $\mathcal{M}$ in such a way that each circuit γ gets at least $\min(|\gamma|, p+1)$ colours?

It would be interesting to find a natural class of matroids for which $\mathrm{Arb}_p^\star(\mathcal{M})$ is uniformly bounded. For graphs this leads to the following problem:

Problem 3. Let G be a graph and let p be an integer. What is the minimum integer $N = \mathrm{Arb}_p^\star(G)$ such that the edge set of G may be coloured using N-colours in such a way that each cut ω gets at least $\min(|\omega|, p+1)$ colours?

It is maybe interesting that the dual version of our problem may present different aspects. The well known theorem of Erdős [3] which asserts that there exists a graph of order at least n, girth at least g and chromatic number at least $2N+1$. As the chromatic number of a graph is bounded by $\chi(G) \leq 2\mathrm{Arb}(G)+1$ we get that there exist graphs with arbitrarily large girth and arboricity (hence arbitrarily large Arb_p). The notion dual to "G has girth at least k" (i.e. every cycle of G has length at least k) is "G is k-edge connected" (i.e. every edge cut of G has size at least k). However, there does not exist graphs with arbitrarily edge-connectivity and $\mathrm{Arb}_p^\star$. Precisely:

Proposition 16. *Let G be a graph and let p be an integer.*

- *If G is $(2p+2)$-edge connected then $\mathrm{Arb}_p^\star(G) = p+1$;*
- *if G is $(2p+1)$-edge connected then $\mathrm{Arb}_p^\star(G) \leq (p+1)(2p+1)$, and there exists infinitely many $(2p+1)$-edge connected graphs such that $\mathrm{Arb}_p^\star(G) \geq p+2$;*
- *$\mathrm{Arb}_p^\star(G)$ is not bounded for $(2p)$-edge connected graphs.*

Proof. The first item is a consequence of [6] where it is proved that a $2n$-edge connected graph has at least n pairwise edge-disjoint spanning trees. It follows that if G is $(2p+2)$-edge connected, it has at least $p+1$ edge-disjoint spanning trees $Y_1, \ldots, Y_{p+1}$. Colour i the edges of Y_i and further colour 1 the edges which are present in none of the Y_i's. As each Y_i is spanning, each cut meets all the Y_i's thus gets $p+1$ colours. It follows that $\mathrm{Arb}_p^\star(G) = p+1$.

The upper bound of the second item is similarly obtained by doubling each edge of G (thus obtaining a $(4p+2)$-edge connected multigraph) and considering $2p+1$ edge-disjoint spanning trees of this new multigraph G', and colouring each edge e of G by the set of (at

most two) colours assigned to the two edges of G' corresponding to e. The lower bound is obtained by considering non-complete $(2p+1)$-regular $(2p+1)$-edge connected graphs G: if $\mathrm{Arb}_p(G) = p+1$ would hold then each colour class would include a spanning tree and hence $\|G\| \geq (p+1)(|G|-1)$ would hold.

The last item follows from the following construction. For integers L, p let $C_L^{(p)}$ be the multigraph obtained from a cycle of length L by replacing each edge by p parallel edges (see Fig. 6). The graph $C_L^{(p)}$ is $(2p)$-edge connected. However, $\mathrm{Arb}_p(C_L^{(p)}) \geq L^{1/p}$ as if each cut gets at least $p+1$ colours then no two group of parallel edges can be coloured by the same set of colours.

□

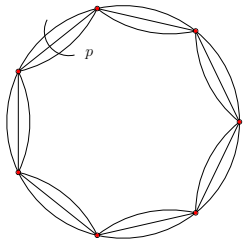

Figure 6: The multigraph $C_L^{(p)}$ is $(2p)$-edge connected and $\mathrm{Arb}_p(C_L^{(p)}) \geq L^{1/p}$

References

References

[1] Z. Dvořák, *Asymptotical structure of combinatorial objects*, Ph.D. thesis, Charles University, Faculty of Mathematics and Physics, 2007.

[2] Z. Dvořák, *On forbidden subdivision characterizations of graph classes*, European J. Combin. **29** (2008), no. 5, 1321–1332.

[3] P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** (1959), no. 1, 34–38.

[4] F. Gavril and J. Urrutia, *An algorithm for fraternal orientation of graphs*, Inform. Process. Lett. **41** (1992), 271–274.

[5] S. Kenkre and S. Vishwanathan, *A bound on the chromatic number using the longest odd cycle length*, Journal of Graph Theory **54** (2007), 267–276.

[6] S. Kundu, *Bounds on the number of disjoint spanning trees*, Journal of Combinatorial Theory, Series B **17** (1974), no. 2, 199 – 203.

[7] C. St J. A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. London Math. Soc. **36** (1961), 445–450.

[8] ———, *Decomposition of finite graphs into forests*, J. London Math. Soc. **39** (1964), 12.

[9] J. Nešetřil and P. Ossona de Mendez, *The grad of a graph and classes with bounded expansion*, 7th International Colloquium on Graph Theory (A. Raspaud and O. Delmas, eds.), Electronic Notes in Discrete Mathematics, vol. 22, Elsevier, 2005, pp. 101–106.

[10] ———, *Linear time low tree-width partitions and algorithmic consequences*, STOC'06. Proceedings of the 38th Annual ACM Symposium on Theory of Computing, ACM Press, 2006, pp. 391–400.

[11] ———, *Tree depth, subgraph coloring and homomorphism bounds*, European Journal of Combinatorics **27** (2006), no. 6, 1022–1041.

[12] ———, *Grad and classes with bounded expansion I. decompositions*, European Journal of Combinatorics **29** (2008), no. 3, 760–776.

[13] ———, *From sparse graphs to nowhere dense structures: Decompositions, independence, dualities and limits*, European Congress of Mathematics, European Mathematical Society, 2010, pp. 135–165.

[14] D.J. Skrien, *A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular arc graphs, and nested interval graphs*, J. Graph Theory **6** (1982), no. 3, 309–316.

[15] X. Zhu, *Colouring graphs with bounded generalized colouring number*, Discrete Math. **309** (2009), no. 18, 5562–5568.