Packing Chromatic Number of Distance Graphs

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that vertices of G can be partitioned into disjoint classes $X_1, ..., X_k$ where vertices in X_i have pairwise distance greater than i. We study the packing chromatic number of infinite distance graphs $G(\mathbb{Z}, D)$, i.e. graphs with the set $\mathbb{Z}$ of integers as vertex set and in which two distinct vertices $i, j \in \mathbb{Z}$ are adjacent if and only if $|i - j| \in D$.

In this paper we focus on distance graphs with $D = \{1, t\}$. We improve some results of Togni who initiated the study. It is shown that $\chi_\rho(G(\mathbb{Z}, D)) \leq 35$ for sufficiently large odd t and $\chi_\rho(G(\mathbb{Z}, D)) \leq 56$ for sufficiently large even t. We also give a lower bound 12 for $t \geq 9$ and tighten several gaps for $\chi_\rho(G(\mathbb{Z}, D))$ with small t.

Keywords: distance graph; packing coloring; packing chromatic number

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1 Introduction

In this paper we consider simple undirected graphs only. For terminology and notations not defined here we refer to [2]. Let G be a connected graph and let $\mathrm{dist}_G(u, v)$ denote the distance between vertices u and v in G. We ask for a partition of the vertex set of G into disjoint classes $X_1, ..., X_k$ according to the following constraints. Each color class X_i should be an *i-packing*, a set of vertices with property that any distinct pair $u, v \in X_i$ satisfies $\mathrm{dist}_G(u, v) > i$. Such a partition is called a *packing k-coloring*, even though it is allowed that some sets X_i may be empty. The smallest integer k for which there exists a packing k-coloring of G is called the *packing chromatic number* of G and it is denoted $\chi_\rho(G)$. The very first results about packing chromatic number were obtained by Slopper [15]. He studied an *eccentric coloring* but his results were directly translated to the packing chromatic number. The concept of packing chromatic number was introduced by Goddard et al. [9] under the name *broadcast chromatic number*. The term packing chromatic number was later proposed by Brešar et al. [3]. The determination of the packing chromatic number is computationally difficult. It was shown to be $\mathcal{NP}$-complete for general graphs in [9]. Fiala and Golovach [6] showed that the problem remains $\mathcal{NP}$-complete even for trees.

The research of the packing chromatic number was driven by investigating $\chi_\rho(\mathbb{Z}^2)$ where $\mathbb{Z}^2$ is the Cartesian product of two infinite paths - the (2-dimensional) *square lattice*. Goddard et al. [9] showed that $9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$. Fiala et al. [7] improved the lower bound to 10 and Holub and Soukal [10] improved the upper bound to 17. The lower bound was pushed further to 12 by Ekstein et al. [4]. For $\mathbb{Z}^3$ see [7, 8].

Let $D = \{d_1, d_2, ..., d_k\}$, where d_i are positive integers and $i = 1, 2, ..., k$. The (infinite) *distance graph* $G(\mathbb{Z}, D)$ with distance set D has the set $\mathbb{Z}$ of integers as a vertex set and in which two distinct vertices $i, j \in \mathbb{Z}$ are adjacent if and only if $|i-j| \in D$. We denote the graph $G(\mathbb{Z}, \{a, b\})$ by $D(a, b)$. The study of a coloring of distance graphs was initiated by Eggleton et al. [5]. In last twenty years there were more than 60 papers concerning this topic. We recall e.g. contributions by Voigt and Walter [17], Ruzsa et al. [14], Liu [12], Liu and Zhu [13] and Barajas and Serra [1].

D	$\chi_\rho \geq$	$\chi_\rho \leq$
1,2	8	8
1,3	9	9
1,4	11	16
1,5	10	12
1,6	11	23
1,7	10	15
1,8	11	25
1,9	10	18

D	$\chi_\rho \geq$	$\chi_\rho \leq$
1,2	8	8
1,3	9	9
1,4	**14**	**15**
1,5	**12**	12
1,6	**15**	23
1,7	**14**	15
1,8	**15**	25
1,9	**13**	18

Table 1: Lower and upper bounds for the packing chromatic number of $D(1, t)$. Left table contains previously known bounds and the right table contains current bounds.

The study of a packing coloring of distance graphs was initiated by Togni [16]. Results for $D(1, t)$ for small values of t, obtained by Togni [16], are summarized in the left part of Table 1. Our improvements are emphasized in the right part of the table and they were obtained by a computer. We wrote two independent programs (one in Pascal and other one in C++). The source codes and the outputs of the programs can be downloaded from http://kam.mff.cuni.cz/ bernard/dist.

For larger t Togni proved the following theorem.

Theorem 1. [16] *For every $q, t \in \mathbb{N}$:*

$$\chi_\rho(D(1,t)) \leq \begin{cases} 86 & \text{if } t = 2q+1, q \geq 36, \\ 40 & \text{if } t = 2q+1, q \geq 223, \\ 173 & \text{if } t = 2q, q \geq 87, \\ 81 & \text{if } t = 2q, q \geq 224, \\ 29 & \text{if } t = 96q \pm 1, q \geq 1, \\ 59 & \text{if } t = 96q + 1 \pm 1, q \geq 1. \end{cases}$$

We improve some results of Theorem 1 as follows.

Theorem 2. *For any odd integer $t \geq 575$,*

$$\chi_\rho(D(1,t)) \leq 35.$$

For any even integer $t \geq 648$,

$$\chi_\rho(D(1,t)) \leq 56.$$

We also give a lower bound for the packing chromatic number of $D(1,t)$ for $t \geq 9$, as a corollary of the following statement.

Theorem 3. [4] *The packing chromatic number of the square lattice is at least 12.*

Corollary 4. *Let $D(1,t)$ be a distance graph, $t \geq 9$ an integer. Then*

$$\chi_\rho(D(1,t)) \geq 12.$$

Throughout the rest of the paper by a coloring we mean a packing coloring.

2 $D(1,t)$ with small t

In this section we prove new lower and upper bounds for the packing chromatic number of $D(1,t)$ which are mentioned in Table 1.

Lemma 5. $\chi_\rho(D(1,4)) \leq 15$.

Proof. We prove this lemma by exhibiting a repeating pattern for 15-packing coloring of $D(1,4)$. The pattern has period 320 and is given here:
1,3,1,2,4,1,5,1,8,2,1,3,1,10,11,1,2,1,6,4,1,3,1,2,5,1,7,1,9,2,1,3,1,12,4,
1,2,1,13,8,1,3,1,2,6,1,5,1,4,2,1,3,1,7,10,1,2,1,15,14,1,3,1,2,5,1,4,1,11,2,
1,3,1,6,9,1,2,1,8,7,1,3,1,2,4,1,5,1,12,2,1,3,1,10,13,1,2,1,4,6,1,3,1,2,5,1,
7,1,8,2,1,3,1,4,14,1,2,1,11,9,1,3,1,2,6,1,5,1,4,2,1,3,1,7,10,1,2,1,8,12,1,3,
1,2,5,1,4,1,13,2,1,3,1,6,9,1,2,1,15,7,1,3,1,2,4,1,5,1,8,2,1,3,1,10,11,1,2,1,
6,4,1,3,1,2,5,1,7,1,9,2,1,3,1,12,4,1,2,1,13,8,1,3,1,2,6,1,5,1,4,2,1,3,1,7,
10,1,2,1,14,15,1,3,1,2,5,1,4,1,11,2,1,3,1,6,9,1,2,1,8,7,1,3,1,2,4,1,5,1,12,
2,1,3,1,10,13,1,2,1,4,6,1,3,1,2,5,1,7,1,8,2,1,3,1,4,11,1,2,1,15,9,1,3,1,2,6,
1,5,1,4,2,1,3,1,7,10,1,2,1,8,12,1,3,1,2,5,1,4,1,13,2,1,3,1,6,9,1,2,1,14,7.

The pattern was found with help of a computer using simulated annealing heuristics [11]. □

Lemma 6.

$$14 \leq \chi_\rho(D(1,4)),$$

$$12 \leq \chi_\rho(D(1,5)),$$

$$14 \leq \chi_\rho(D(1,7)),$$

$$13 \leq \chi_\rho(D(1,9)).$$

Proof. These results were obtained by a computer using a brute force search programs. We have written two independent programs (one in Pascal and one in C++) implementing the brute force search. The programs take vertices $X = \{1,2,\ldots k\}$ from $D(1,t)$. Then they try to construct a packing coloring ϱ of X using colors from 1 up to c. First, they assign $\varrho(1) = c$ and then they try to extend ϱ to X. If the extension is not possible we conclude that $\chi_\rho(D(1,t)) > c$. The results of computations are summarized in Table 2.

D	c	k	Configurations	Time
1,4	13	81	$6.4 \cdot 10^{12}$	26 days
1,5	11	134	$8.1 \cdot 10^{9}$	25 minutes
1,7	13	229	$6.9 \cdot 10^{13}$	335 days
1,9	12	66	$6.2 \cdot 10^{12}$	28 days

Table 2: Computations from Lemma 6. Time of the computation is measured on a workstation from year 2010.

□

Let H_k denote a finite subgraph of $D(1,t)$ on vertices $1, \ldots, k$ and let H'_k denote a finite subgraph of $D(1,t)$ on vertices $-k, -k+1, \ldots, k$.

For a subset X of vertices of $D(1,t)$ we define its *density* $d(X)$ as

$$d(X) = \limsup_{k \to \infty} \frac{|X \cap V(H'_k)|}{|V(H'_k)|}.$$

For a color c we define its *density* $d(c)$ as

$$d(c) = \max_{\chi} d(X_c),$$

where χ is a packing coloring of $D(1,t)$ and X_c is a c-packing. Similarly, by $d(c_1, \ldots, c_l)$ we mean

$$d(c_1, \ldots, c_l) = \max_{\chi} d(X_{c_1} \cup \ldots \cup X_{c_l}).$$

The following statement was proved in [7].

Lemma 7. [7] *If there exists a coloring of $D(1,t)$ by k colors then, for every $1 \le l \le k$, it holds that*

$$\sum_{i=1}^{k} d(i) \ge d(1,\dots,l) + \sum_{i=l+1}^{k} d(i) \ge d(1,\dots,k) = 1.$$

Lemma 8.

$$15 \le \chi_\rho(D(1,6)),$$
$$15 \le \chi_\rho(D(1,8)).$$

Proof. To the contrary we suppose that $\chi_\rho(D(1,6)) \le 14$. Using a computer we verified that $d(1,2,3,4) \le \frac{31}{41}$ since we can color at most 31 vertices of H_{41} using colors $1,2,3,4$. The computation took about three minutes and it checked $4.6 \cdot 10^9$ configurations. Clearly, $d(i) \le \frac{1}{6i-9}$ for $i \ge 2$ since there is no pair of vertices in H_{6i-9} with distance greater than i and hence at most one vertex of H_{6i-9} can be colored by color i. By Lemma 7 we easily get

$$d(1,2,\dots,14) \le d(1,2,3,4) + \sum_{i=5}^{14} d(i) \le \frac{31}{41} + \frac{1}{21} + \cdots + \frac{1}{75} = 0.999771 < 1,$$

which is not possible since $d(1,2,\dots,14) = 1$ by the assumption that $\chi_\rho(D(1,6)) \le 14$.

Now to the contrary we suppose that $\chi_\rho(D(1,8)) \le 14$. Using a computer we verified that $d(1,\dots,6) \le \frac{50}{58}$ since we can color at most 50 vertices of H_{58} using colors $1,\dots,6$. The computation took about sixty hours and it checked $7.5 \cdot 10^{11}$ configurations. Clearly, $d(i) \le \frac{1}{8i-20}$ for $i \ge 3$ since there is no pair of vertices in H_{8i-20} with distance greater than i and hence at most one vertex of H_{8i-20} can be colored by color i. By Lemma 7 we easily get

$$d(1,2,\dots,14) \le d(1,\dots,6) + \sum_{i=7}^{14} d(i) \le \frac{50}{58} + \frac{1}{36} + \cdots + \frac{1}{92} = 0.999110 < 1,$$

which is not possible since $d(1,2,\dots,14) = 1$ by the assumption that $\chi_\rho(D(1,8)) \le 14$. □

3 $D(1,t)$ with large t

A key observation for this section is that a distance graph $D(1,t)$, for $t > 1$, can be drawn as an infinite spiral with t lines orthogonal to the spiral (e.g. $D(1,5)$ on Figure 3).

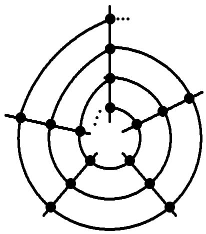

Figure 1: Distance graph $D(1,5)$.

For $i \in \{0, 1, ..., t-1\}$, the *i-band* in a distance graph $D(1,t)$, denoted by B_i, is an infinite path in $D(1,t)$ on the vertices $V(B_i) = \{i + kt, k \in \mathbb{Z}\}$. Note that the band B_i corresponds to one of t lines orthogonal to the spiral. For $i \in \{0, 1, ..., t-24\}$, the *$i$-strip* in a distance graph $D(1,t)$, $t > 23$, denoted by S_i, is a subgraph of $D(1,t)$ induced by the union of vertices of $B_i, B_{i+1}, ..., B_{i+23}$.

We use the following statement proved by Goddard et al. in [9].

Proposition 9. [9] *For every $k \in \mathbb{N}$, the infinite path can be colored by colors $k, k+1, ..., 3k+2$.*

Holub and Soukal [10] improved the upper bound for a packing coloring of the square lattice to 17 by finding a pattern on 24×24 vertices using color 1 on positions as white places on a chessboard. We use this pattern to prove the following lemma.

Lemma 10. *Let $D(1,t)$ be a distance graph, $t > 24$, and S_i its i-strip. Then $\chi_\rho(S_i) \leq 17$.*

Proof. We cyclically use the pattern on 24×24 vertices to color all the vertices of S_i. Hence it is obvious that $\chi_\rho(S_i) \leq 17$. $\square$

Lemma 11. *Let $D(1,t)$ be a distance graph and B_i its i-band. If vertices $\{i + 2jt, j \in \mathbb{Z}\}$ are colored by color 1, then it is possible to extend the coloring to all vertices of B_i using colors $k, k+1, ..., 2k-1$, for every $k \in \mathbb{N}, k > 2$.*

Proof. We color B_i by the following periodic pattern: $1, k, 1, k+1, ..., 1, 2k-1$. As the period for every color different from 1 is $2k$ and the largest used color is $2k-1$, we conclude that we get a packing coloring of B_i. $\square$

Lemma 12. *Let $D(1,t)$ be a distance graph, $t \geq 50$, and B_i, B_{i+25} its bands. Then it is possible to color B_i and B_{i+25} using colors $C = \{1, 18, 19, ..., 35\}$.*

Proof. We color the vertices of B_i and B_{i+25} repeating the pattern from the proof of Lemma 11. We start to color B_i at the vertex i and B_{i+25} at the vertex $i - kt$ for any $k \in \{11, 12, ..., 25\}$. Lemma 11 assures that the distance between two vertices colored with color c in a single band is greater than c. Let $u \in V(B_i)$ and $v \in V(B_{i+25})$ be colored by the same color. By the pattern from the proof of Lemma 11 we conclude that the distance between u and v is $\min\{k, 36-k\}+25$ which is greater than 35. Hence we have a packing coloring of B_i and B_{i+25}. $\square$

For a distance graph $D(1,t)$ we use notation $D(1,t) = S_0B_{24}S_{25}B_{49}\dots$ to express that we view $D(1,t)$ as a union of strips $S_0, S_{25}, \dots$ and bands $B_{24}, B_{49}, \dots$.

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Case 1: t is odd.
Let r, s be positive integers such that $t = 24s + r$, where $r < 24$ is also odd. Since $t \geq 575$, we get $s \geq r$ (for $r = 23$ we have $24s \geq 552$). Thus we have s disjoint strips and r disjoint bands such that $D(1,t) = S_0B_{24}S_{25}B_{49}...S_{24(r-1)+r-1}B_{24r+r-1}S_{24r+r}...S_{24(s-1)+r}$.

For odd $j = 1, 3, ..., r$, we color the strips $S_{24(j-1)+j-1}$ cyclically with the pattern on 24×24 vertices starting at the vertex $24(j-1)+j-1$. For even $j = 2, 4, ..., r-1$, we color $S_{24(j-1)+j-1}$ cyclically with the pattern on 24×24 vertices starting at the vertex $24(j-1)+j-1-t$. For $j = r+1, r+2, ..., s$, we color $S_{24(j-1)+r}$ cyclically with the pattern on 24×24 vertices starting at the vertex $24(j-1)+r-t$. Hence we have a packing coloring of all s disjoint strips of $D(1,t)$ using the same principle as in the proof of Lemma 10.

For odd $j = 1, 3, ..., r-2$, we color the bands $B_{24j+j-1}$ cyclically with the sequence of colors $1, 18, 1, 19, ..., 1, 35$ starting at the vertex $24j + j - 1$. For even $j = 2, 4, ..., r-3$, we color $B_{24j+j-1}$ cyclically with the sequence of colors $1, 18, 1, 19, ..., 1, 35$ starting at the vertex $24j+j-1-17t$. We color $B_{24(r-1)+r-2}$, $B_{24r+r-1}$ cyclically with the sequence of colors $1, 18, 1, 19, ..., 1, 35$ starting at the vertex $24(r-1) + r - 2 - 13t$, $24r + r - 1 - 24t$, respectively. Hence we have a packing coloring of all r disjoint bands of $D(1,t)$ using the same principle as in the proof of Lemma 12.

Note that the bands are colored by colors $1, 18, 19, ..., 35$ and the strips are colored by colors $1, 2, ..., 17$ such that no pair of adjacent vertices is colored with color 1. Then we conclude that we have a packing coloring of $D(1,t)$, hence $\chi_\rho(D(1,t)) \leq 35$.

We illustrate this situation on Figure 2. The black vertices are colored by 1 and we color bands cyclically only with the sequence of colors of length 6 instead of 36 and a strip consists of only 4 bands instead of 24. Note that this decomposition is equivalent to our situation.

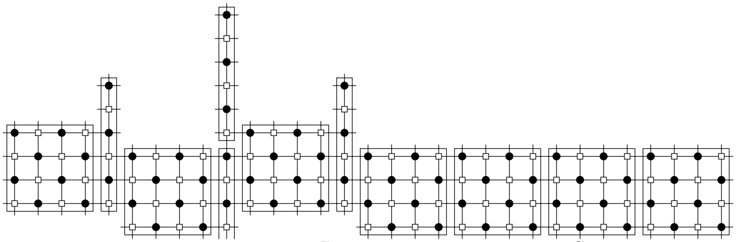

Figure 2: Distance graph $D(1,t)$ for odd t.

Case 2: t is even.
Let r,s be positive integers such that $t=24(s+2)+r$, where $0<r\leq 24$ is also even. Since $t\geq 648$, we get $s\geq r$ (for $r=24$ we have $24s\geq 576$). Thus we have now $s+2$ disjoint strips and r disjoint bands such that $D(1,t)=S_0S_{24}B_{48}S_{49}B_{73}...S_{24(r-1)+r-2}B_{24r+r-2}S_{24r+r-1}$ $S_{24(r+1)+r-1}...S_{24(s+1)+r-1}B_{24(s+2)+r-1}$.

For odd $j=1,3,...,r-1$, we color the strips S_0, $S_{24j+j-1}$ cyclically with the pattern on 24×24 vertices starting at the vertex 0, $24j+j-1$, respectively. For even $j=2,...,r-2$, we color $S_{24j+j-1}$ cyclically with the pattern on 24×24 vertices starting at the vertex $24j+j-1-t$. For $j=r,r+1,...,s+2$, we color $S_{24j+r-1}$ cyclically with the pattern on 24×24 vertices starting at the vertex $24j+r-1-t$. Hence we have a packing coloring of all $s+2$ disjoint strips of $D(1,t)$ using the same principle as in the proof of Lemma 10.

For odd $j = 1, 3, ..., r-1$, we color the bands $B_{24(j+1)+j-1}$ cyclically with the sequence of colors $1, 18, 1, 19, ..., 1, 35$ starting at the vertex $24(j+1)+j-1$. For even $j = 2, 4, ..., r-2$, we color $B_{24(j+1)+j-1}$ cyclically with the sequence of colors $1, 18, 1, 19, ..., 1, 35$ starting at the vertex $24(j+1)+j-1-17t$. We color $B_{24(s+2)+r-1}$ with sequence of colors $18, 19, ..., 56$ starting at the vertex $24(s+2)+r-1$ by Proposition 9 for $k = 18$. Note the band $B_{24(s+2)+r-1}$ is the only one with colors greater than 35. We have a packing coloring of all r disjoint bands of $D(1,t)$ by the fact that the distance between an arbitrary vertex of $B_{24(s+2)+r-1}$ and a vertex of any other band is at least 49 and using the same principle as in the proof of Lemma 12.

Note that the bands are colored by colors $1, 18, 19, ..., 56$ and the strips are colored by colors $1, 2, ..., 17$ such that no pair of adjacent vertices is colored with color 1. Then we conclude that we have a packing coloring of $D(1,t)$, hence $\chi_\rho(D(1,t)) \leq 56$.

We illustrate this situation on Figure 3. Note that this decomposition is equivalent to our situation as in Case 1.

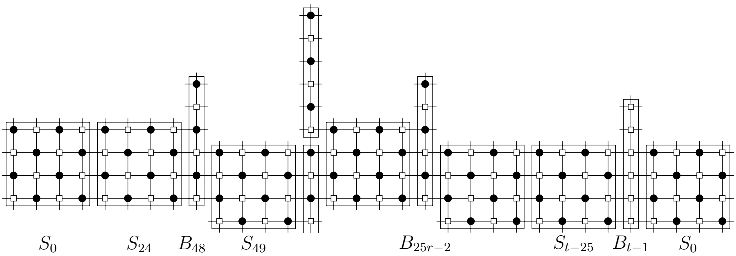

Figure 3: Distance graph $D(1,t)$ for even t.

□

r	$t \geq$		r	$t \geq$
1	25		2	98
3	75		4	148
5	125		6	198
7	175		8	248
9	225		10	298
11	275		12	348
13	325		14	398
15	375		16	448
17	425		18	498
19	475		20	548
21	525		22	598
23	575		24	648

Table 3: Table for t depending on r.

Note that in some cases we can decrease t for which Theorem 2 is true. It depends on r from the proof of Theorem 2. We have $t \geq 24r + r$ for odd t and $t \geq 24r + r + 48$ for even t (see Table 3).

4 Lower bound from square lattice

In this section we give a proof of the lower bound for $\chi_\rho(D(1,t))$.

Proof of Corollary 4. By the proof of Theorem 3, a finite square lattice 15×9 cannot be colored using 11 colors. Clearly $D(1,t)$ contains a finite square grid as a subgraph and $t \geq 9$ assures existence of the square lattice 15×9 in $D(1,t)$. Therefore, $\chi_\rho(D(1,t)) \geq 12$ for every $t \geq 9$. $\square$

5 Conclusion

We have shown that the packing chromatic number of an infinite distance graph $D(1,t)$ is at least 12 for $t \geq 9$ and at most 35 for odd t greater or equal than 575 or at most 56 for even t greater or equal than 648. Moreover, we have found some smaller values of t for which Theorem 2 holds. The next research in this area can be focused on finding better bounds for $D(1,t)$. In particular, obtaining a lower bound for $D(1,t)$ which would exceed the upper bound for the square lattice would be an interesting result.

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