Maximal and supremal tolerances in multiobjective linear programming

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Abstract

Let a multiobjective linear programming problem and any efficient solution be given. Tolerance analysis aims to compute interval tolerances for (possibly all) objective function coefficients such that the efficient solution remains efficient for any perturbation of the coefficients within the computed intervals. The known methods yield tolerances that are not optimal. In this paper, we propose a method for calculating the supremal tolerance (the maximal one needn't exist). The method is exponential in the worst case. We show that the problem of determining the maximal/supremal tolerance is NP-hard, so an efficient procedure is not likely to exist.

Keywords: *Multiobjective linear programming, efficient solution, sensitivity analysis, tolerance analysis.*

1 Introduction

This paper is a contribution to postoptimal analysis in multiobjective linear programming, namely to tolerance analysis. Postoptimal analysis is a fundamental tool to study effects of various uncertainties and data perturbations

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on the model. The standard sensitivity analysis [14, 15] inspects behavior of one coefficient perturbation. Contrary, tolerance analysis was developed to handle simultaneous and independent variations of several coefficients.

Tolerance analysis was pioneered by Wendell [18, 19] in linear programming, and then investigated by many researchers; see e.g. [8, 17, 20]. In multiobjective linear programming, tolerance analysis was adapted by a few of ways. Since multiobjective linear programming problems are often solved by weighted sum scalarization, the first approach concerns tolerance analysis of the objective function weights [1, 2, 5, 11].

Another way is to adapt tolerance analysis directly on the objective function coefficients. This approach was followed by Hladík [6], who proposed an algorithm for satisfactory large, but not necessarily maximal, tolerances. Note that the maximal tolerances needn't exist, so we will speak more about *supremal* tolerances. Sitarz [16] calculates an upper bound on the supremal additive tolerance. An extension to individual tolerances was given in [7]. So far, there has been no method known for computing the supremal tolerances, only the afore-mentioned lower nad upper bounds. Herein, we present such an algorithm to compute the supremal tolerances.

The paper is organized as follows. After some preliminaries (Section 2) we propose a formula to compute the supremal tolerances in terms of an optimization problem (Section 3). In Section 3.1 we show how to practically solve the optimization problem by decomposing into orthants. A procedure to test whether the calculated supremal tolerance is also the maximal one is presented in Section 3.2. NP-hardness of determining maximal/supremal tolerance is proved in Section 3.3. Section 3.4 is concerned with an upper bound on the supremal tolerance by using edges emerging from a given vertex. Finally, we illustrate our method on examples in Section 4.

2 Preliminaries and problem statement

Let us introduce some notation first. By an interval matrix we mean a family of matrices

$$[\underline{M}, \overline{M}] = \{M \in \mathbb{R}^{m\times n};\ \underline{M} \leq M \leq \overline{M}\},$$

where $\underline{M} \leq \overline{M}$ are given matrices. The relation $x \gneqq y$ denotes in short that $x \geq y$ and $x \neq y$, $\operatorname{diag}(v)$ stands for the diagonal matrix with entries $v_1, \dots, v_n$, and $\operatorname{sgn}(v)$ for the sign vector of a vector v. The ith row of a matrix A is denoted by $A_{i\bullet}$.

Consider a multiobjective linear programming problem

$$\max Cx \ \text{ subject to } \ Ax \leq b, \tag{1}$$

where $C \in \mathbb{R}^{s\times n}$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. A feasible solution x^* is called *efficient* if there is no feasible x such that $Cx \gneqq Cx^*$.

Now, let $G \geq 0$ be an $s \times n$ matrix and consider the interval matrix $[C - \delta G, C + \delta G]$ with parameter $\delta > 0$, and x^* an efficient solution to (1). A non-negative value δ is called *admissible tolerance* if x^* remains efficient for all $C' \in [C - \delta G, C + \delta G]$. Herein, G represents perturbation scales for objective function coefficients. It is usually set up as $G_{ij} = |C_{ij}|$ for relative (percentage) tolerances, $G_{ij} = 1$ for additive tolerances, and $G_{ij} = 0$ in case when perturbation of C_{ij} is not in interest. However, they can be set up in any other way according to the decision maker preferences and importances of particular coefficients.

Our aim is to calculate the maximal admissible tolerance. Formally, we define and denote the maximal tolerance as follows

$$\delta^{\max} := \max \delta \ \text{ subject to } \ x^* \text{ is efficient } \forall C' \in [C - \delta G, C + \delta G], \ \delta \geq 0.$$

Note that the maximal tolerance needn't exist; see Example 1. That is why we focus more on calculation of the supremal tolerance

$$\delta^{\sup} := \sup \delta \ \text{ subject to } \ x^* \text{ is efficient } \forall C' \in [C - \delta G, C + \delta G], \ \delta \geq 0.$$

Once the supremal tolerance $\delta^{\sup}$ is computed, $\delta^{\sup} - \varepsilon$ is an admissible tolerance for arbitrarily small $\varepsilon > 0$, but $\delta^{\sup}$ is not necessarily admissible.

3 Main results

Let x^* be a feasible solution and $I(x^*) = \{i; A_{i\bullet}x = b_i\}$ its active set. The tangent cone at x^* is described

$$A_{i\bullet}(x - x^*) \leq 0, \ i \in I(x^*).$$

For simplicity, we denote the system by $A^1(x - x^*) \leq 0$. It is known [3] that x^* is efficient iff there is no dominated solution within the tangent cone, that is, the system

$$A^1(x - x^*) \leq 0, \ C(x - x^*) \gneqq 0 \tag{2}$$

or

$$A^1(x - x^*) \le 0,\ C(x - x^*) = y \ge 0,\ 1^T y = 1 \tag{3}$$

has no solution. We utilize this characterization of efficiency to derive more general robust characterization of efficiency in the following lemma, and to state our main computational result in the sequel.

Lemma 1. *Let x^* be an efficient solution to* (1). *Then x^* is efficient for each $C' \in [C - \delta G, C + \delta G]$ iff the system*

$$A^1(x - x^*) \le 0,\ C(x - x^*) + \delta G|x - x^*| \gneqq 0 \tag{4}$$

has no solution.

Proof. Sufficiency. Let $C' \in [C - \delta G, C + \delta G]$ and suppose that x^* is not efficient for C', that is, there is a solution x to

$$A^1(x - x^*) \le 0,\ C'(x - x^*) \gneqq 0.$$

Then x fulfills also

$$\begin{aligned} 0 \lneqq C'(x - x^*) &= C(x - x^*) + (C' - C)(x - x^*) \\ &\le C(x - x^*) + |C' - C||x - x^*| \\ &\le C(x - x^*) + \delta G|x - x^*|. \end{aligned}$$

Necessity. Now, suppose that x solves (4). Putting

$$C' := C + \delta G \operatorname{diag}(\operatorname{sgn}(x - x^*)) \in [C - \delta G, C + \delta G]$$

we get

$$\begin{aligned} C'(x - x^*) &= C(x - x^*) + \delta G \operatorname{diag}(\operatorname{sgn}(x - x^*))(x - x^*) \\ &= C(x - x^*) + \delta G|x - x^*| \gneqq 0 \end{aligned}$$

Thus x^* is not efficient for C'. □

Theorem 1. *Let x^* be an efficient solution to* (1). *Then*

$$\begin{aligned} \delta^{\sup} = \min \delta \quad \textit{subject to} \quad & A^1(x - x^*) \le 0, && \text{(5a)} \\ & C(x - x^*) + \delta G|x - x^*| \ge 0, && \text{(5b)} \\ & 1^T G|x - x^*| = 1,\ \delta \ge 0. && \text{(5c)} \end{aligned}$$

Proof. By Lemma 1, x^* is efficient for each $C' \in [C-\delta G, C+\delta G]$ iff the system (4) has no solution. Thus we seek the supremal δ such that the system (4) has no solution. Instead, we compute minimal $\delta \geq 0$ such that the system (4) has a solution, that is

$$\inf \delta \ \text{ subject to } \ A^1(x-x^*) \leq 0,\ C(x-x^*) + \delta G|x-x^*| \gneqq 0,\ \delta \geq 0. \tag{6}$$

Its equivalent form is (5). The reason is the following. Let (δ, x) be a feasible solution to (6). If $1^T G|x-x^*| = 0$ then $G|x-x^*| = 0$ and $C(x-x^*) \gneqq 0$, meaning that x^* cannot be efficient. Otherwise, if $1^T G|x-x^*| > 0$ then $\left(\delta, \frac{1}{1^T G|x-x^*|}(x-x^*) + x^*\right)$ solves (5). Let (δ, x) be a solution to (5). Then $(\delta+\varepsilon, x)$ is a solution to (6) for an arbitrarily small $\varepsilon > 0$. Thus the optimal values of both problems are the same. □

3.1 Decomposition procedure

The optimization problem (5) is difficult to solve, however, we can decompose it into 2^n simpler problems according to the signs of $(x-x^*)_i$, $i = 1, \ldots, n$. Let $z \in \{\pm 1\}^n$ and consider a restriction to the orthant defined by $\operatorname{diag}(z)(x-x^*) \geq 0$:

$$\delta_z = \min \delta \ \text{ subject to } \ A^1(x-x^*) \leq 0, \tag{7a}$$

$$C(x-x^*) + \delta G \operatorname{diag}(z)(x-x^*) \geq 0, \tag{7b}$$

$$1^T G \operatorname{diag}(z)(x-x^*) = 1, \tag{7c}$$

$$\delta \geq 0,\ \operatorname{diag}(z)(x-x^*) \geq 0, \tag{7d}$$

where $\min \emptyset = \infty$ by convention. The supremal tolerance is computed as $\delta^{\text{sup}} = \min_{z \in \{\pm 1\}^n} \delta_z$.

We show that the sub-problems (7) can be transformed to belong to a class of the so called generalized linear fractional programming problems, and so is efficiently solvable. Generalized linear fractional programming problems are problems of the form

$$\max \alpha \ \text{ subject to } \ Px - \alpha Qx \geq 0,\ Qx \geq 0,\ Rx \geq r,$$

or

$$\min \beta \ \text{ subject to } \ Px + \beta Qx \geq 0,\ Qx \geq 0,\ Rx \geq r.$$

These problems are polynomially solvable using an appropriate interior point method [4, 12]. By substitution $y := \operatorname{diag}(z)(x - x^*)$ we get

$$
\begin{aligned}
\delta_z = \min \delta \ \text{ subject to } \ & A^1 \operatorname{diag}(z)\, y \le 0, && \text{(8a)}\\
& C \operatorname{diag}(z)\, y + \delta G y \ge 0, && \text{(8b)}\\
& 1^T G y = 1,\ \delta \ge 0,\ y \ge 0. && \text{(8c)}
\end{aligned}
$$

Since $y \ge 0$ implies $Gy \ge 0$, the problem (8) takes the form of a generalized linear fractional program and therefore is solvable in polynomial time.

Note that not always a number of 2^n sub-problems is necessary to solve. In some cases, some components of $z \in \{\pm 1\}^n$ can be fixed to 1 or -1 and hence the cardinality is several times halved. Provided that the ith column of G is zero, we fix z_i to be 1 or -1. Another possibility is that there are some a priori bounds $l_i \le x_i \le u_i$ on the variable x_i in the constraints of (1). If $x_i^* = l_i$ then $x_i - x_i^*$ is always non-negative and we put $z_i = 1$. Similarly, $x_i^* = u_i$ implies that $x_i - x_i^*$ is non-positive and we fix $z_i = -1$.

3.2 Checking whether δ^{sup} is the maximal tolerance

Theorem 1 gives a formula for computing the supremal tolerance δ^{sup}. A natural question arise to check whether δ^{sup} is the maximal tolerance as well. This question can be answered by using results on necessarily efficiency. The point x^* is called *necessarily efficient* with respect to an interval matrix $[C - R, C + R]$, $R \ge 0$, if it is efficient for every $C' \in [C - R, C + R]$. Clearly, δ^{sup} is the maximal tolerance if and only if x^* is necessarily efficient for $[C - \delta^{\mathrm{sup}} G, C + \delta^{\mathrm{sup}} G]$. A number of results on necessarily efficiency exists; see e.g. a survey in [13], or a sufficient condition and a necessary condition in [10].

Nevertheless, testing necessarily efficiency may be time consuming. Provided that we compute the supremal tolerance by the decomposition method, we can utilize it to derive a faster decision method. Obviously, it suffices to go through all $z \in \{\pm 1\}^n$ for which the minimum of δ_z is attained and check whether x^* is efficient for $C + \delta^{\mathrm{sup}} G \operatorname{diag}(z)$. For the others $z \in \{\pm 1\}^n$, the efficiency holds trivially. One can expect that the number of such minimizers is usually very small.

3.3 NP-hardness

The decomposition procedure described in Section 3.1 requires an exponential number of steps in the worst case. In what follows we show that the problem of determining the maximal/supremal tolerance is NP-hard, so the problem doesn't seem to be polynomially solvable.

Theorem 2. *Let x^* be an efficient solution to* (1). *Determining the supremal tolerance and checking whether it is maximal is an NP-hard problem.*

Proof. Let $R \geq 0$. In [9] it was proven that the following problem is NP-hard: Test whether x^* is efficient to for every $C \in [C-R, C+R]$, that is, whether x^* is necessarily efficient with respect to $[C-R, C+R]$. It can be easily reduced to the maximal tolerance problem. Put $G = R$ and calculate the supremal tolerance δ^{sup}. If $\delta^{\text{sup}} > 1$ then x^* is necessarily efficient, and if $\delta^{\text{sup}} < 1$ then x^* is not necessarily efficient. Eventually, $\delta^{\text{sup}} = 1$ implies that x^* is necessarily efficient if and only if δ^{sup} is also the maximal tolerance. □

3.4 Tolerances on edges

Theorem 3 below presents the opportunity of the calculation of the supremal tolerances on edges. In the case of $n = 2$ we obtain exactly supremal tolerance, but in higher dimensions we obtain only the upper bound.

Theorem 3. *Let x^* be an efficient solution to* (1)*; h_i, $i \in I$ be directions of all edges emerging from x^* and δ_i be defined for $i \in I$ as follows*

$$\delta_i = \inf \delta \quad \textit{subject to} \quad Ch_i + \delta G|h_i| \gneqq 0, \ \delta \geq 0. \tag{9}$$

Then

$$\delta^{\text{sup}} \leq \min_{i \in I} \delta_i. \tag{10}$$

Moreover, if $n = 2$, then

$$\delta^{\text{sup}} = \min_{i \in I} \delta_i. \tag{11}$$

Proof. By the earlier consideration (2) we know that x^* is an efficient solution to (1) iff two cones at x^*:

$$\{x; A^1(x - x^*) \leq 0\} \text{ and } \{x; C(x - x^*) \gneqq 0\}$$

are disjoint. Consider any $C' \in [C - \delta G, C + \delta G]$, when we grow up a cone $\{x; C'(x-x^*) \gneqq 0\}$ by growing up δ until it reaches a cone $\{x; A^1(x-x^*) \leq 0\}$, first we get one of its facets. Thus, we can consider in (6) only such x, which are on facets of $\{x; A^1(x - x^*) \leq 0\}$ at x^*. Now, we limit our consideration to directions emerging from x^*. The formula (9) is taken directly from problem (6) by changing $(x - x^*)$ into h_i. It is obvious that considering only directions emerging from x^*, we consider only the part of facets of $\{x; A^1(x-x^*) \leq 0\}$ at x^*. Thus, by taking minimal of all δ_i by (9), we obtain the upper bound of the supremal tolerance given by formula (10). Moreover, if $n = 2$, the facets of $\{x; A^1(x - x^*) \leq 0\}$ at x^* are generated by directions emerging from x^*, thus in this case, we obtain exactly the supremal tolerance; this proves the formula (11). □

Remark 1. Let us comment on Theorem 1 in the case of $n = 2$. If x^* belongs to one of facets of $\{x; A^1(x - x^*) \leq 0\}$, there are two directions emerging from it. Thus, we have $I = \{1, 2\}$. Moreover, to obtain the supremal tolerance we do not need to solve the optimization problems (5) or (7), but only some linear inequalities generated by (11) need to be solved. The number δ_i can be interpreted as the tolerance connected with direction h_i. The illustration of the above consideration is presented in Example 1.

The formula (9) doesn't seem simple at the first sight, but it is an easy univariate optimization problem.

Theorem 4. *The value δ_i from* (9) *is computable as follows. If $G|h_i| = 0$ or there is $k \in \{1, \dots, s\}$ such that $G_{k\bullet}|h_i| = 0$ and $C_{k\bullet}h_i < 0$ then $\delta_i = \infty$. Otherwise*

$$\delta_i = \max_{k: G_{k\bullet}|h_i|>0} \frac{C_{k\bullet}h_i}{G_{k\bullet}|h_i|}. \tag{12}$$

Proof. When $G|h_i| = 0$ then there is no feasible solution to (9) due to efficiency of x^*. Thus there is no feasible solution for any $\delta \geq 0$. Similar considerations hold true when $G_{k\bullet}|h_i| = 0$ and $C_{k\bullet}h_i < 0$ then $\delta_i = \infty$ for certain $k \in \{1, \dots, s\}$. For otherwise, there is always some $k \in \{1, \dots, s\}$ such that $G_{k\bullet}|h_i| > 0$ and $C_{k\bullet}h_i \geq 0$. By elimination of δ from the inequality $Ch_i + \delta G|h_i| \gneqq 0$ we obtain (12), and the value is non-negative. □

We have proven that the formula (9) is easy to calculate. Thus the upper bound computation (10) is efficient as long as the number of edges is mild. This is satisfied e.g. when the efficient solution x^* is non-degenerate. In this case, there are exactly n edges emerging from x^* and can be explicitly characterized. Denote by B the optimal basis and by A_B the matrix consisting of the basic rows of A. Then the directions of edges are given as columns of the matrix $-A_B^{-1}$.

If x^* is degenerate, then the number of outcoming edges may be exponential in the worst case. However, we can choose a moderate number of them and the statement of Theorem 3 remains true.

4 Examples

Example 1. We adopt the example from [6]

$$\begin{aligned} &\max(2.5x_1 + 2x_2,\, 3.5x_1 + 0.65x_2) \\ &\text{ subject to } \; 3x_1 + 4x_2 \le 42,\; 3x_1 + x_2 \le 24,\; x_2 \le 9,\; x \ge 0, \end{aligned}$$

that is,

$$C = \begin{pmatrix} 2.5 & 2 \\ 3.5 & 0.65 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 4 \\ 3 & 1 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 42 \\ 24 \\ 9 \\ 0 \\ 0 \end{pmatrix}.$$

Consider the efficient solution $x^* = (6,6)^T$. First, put $G_{ij} = 1$ for all i,j. In [6], an admissible tolerance $\delta = 0.7161$ was calculated, but it is not the maximal one. Here, we will proceed along the presented algorithm to compute the maximal/supremal tolerance. The tangent cone to x^* reads

$$3x_1 + 4x_2 \le 42,\; 3x_1 + x_2 \le 24,$$

so we have

$$A^1 = \begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix}.$$

The sets $A^1(x - x^*) \le 0$ and $C(x - x^*) \gneq 0$ are illustrated in Figure 1, where one can see that these sets are disjoint, so it illustrates x^* to be an efficient solution as well. Now, we call (8) for all $z \in \{\pm 1\}^2$, and we obtain the following results

z	δ_z
$(1,1)$	∞
$(1,-1)$	0.875
$(-1,1)$	1.7214
$(-1,-1)$	2

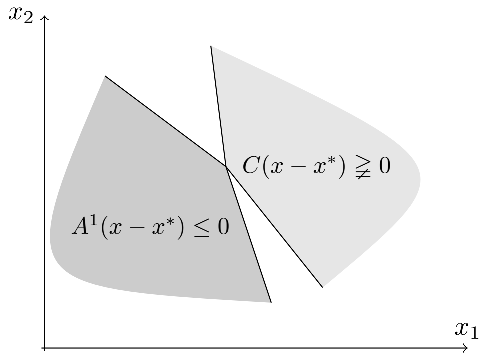

Figure 1: The considered cones connected with x^* in Example 1.

Observe the minimal value is 0.875, so the supremal tolerance reads $\delta^{\sup} = 0.875$. That is, all objective function coefficients may perturb up to almost 0.875 and x^* remains efficient. We check if it is also the maximal tolerance. The minimal value of δ_z is attained for only one minimizer $z = (1, -1)$, so it suffices to verify whether x^* is efficient for

$$C' = C + \delta^{\sup} G \operatorname{diag}(z) = \begin{pmatrix} 3.375 & 1.125 \\ 4.375 & -0.225 \end{pmatrix}. \tag{13}$$

It is easy to calculate that x^* is not efficient in this setting, and it is dominated by the feasible points on the edge emerging from x^* in the direction of $(1, -3)$. See Figure 2, where C' is given by (13).

Analyse the geometrical interpretation of the obtained results. The constraint $y = \operatorname{diag}(z)(x - x^*) \geq 0$ (see (8)) divides our space $\mathbb{R}^2$ into four orthants with origin in x^*. By solving problem (8) we find an optimal solution over these orthants. The illustration of obtaining δ_z is given in Figure 3. Notice that in the case of $z = (1, 1)$ we obtain the empty feasible set, thus $\delta_{(1,1)} = \infty$ (see Figure 3(a)). In the cases of $z = (1, -1)$ and $z = (-1, 1)$ we obtain δ_z on the edges emerging from x^* (see Figures 3(b) and 3(c)). For $z = (-1, -1)$ we obtain δ_z on the edge of the corresponding orthant (see Figure 3(d)).

Now, we illustrate Theorem 3 by using Figure 3. We have $I = 1, 2$ and $h_1 = (1, -3)$, $h_2 = (-4, 3)$. By using (9) we obtain $\delta_1 = 0.875$ and $\delta_2 = 1.7214$. Figure 3(b) presents obtaining δ_1 on edge h_1 and Figure 3(c) shows obtaining δ_2 on edge h_2. Moreover, notice that $\delta_1 = \delta_{(1,-1)}$ and

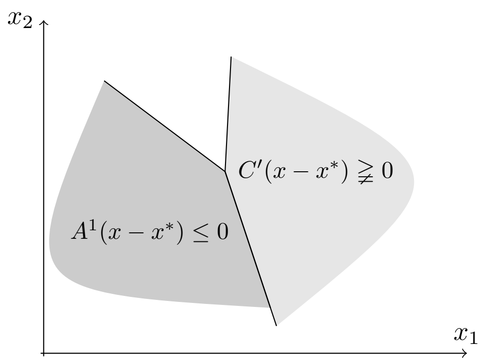

Figure 2: Checking whether $\delta^{\sup}$ is the maximal tolerance in Example 1.

$\delta_2 = \delta_{(-1,1)}$, because the considered edges belong to the orthants generated by $z = (1, -1)$ and $z = (-1, 1)$ respectively.

Example 2. We consider Example 1 with a new matrix $G = |C|$. In [6], an admissible tolerance $\delta = \frac{1}{3}$ was calculated, but it is not the maximal one, either. By (8), we calculate

z	δ_z
$(1, 1)$	∞
$(1, -1)$	0.4118
$(-1, 1)$	0.7555
$(-1, -1)$	1

Thus, the supremal tolerance is $\delta^{\sup} = 0.4118$. In other words, the objective function coefficients may perturb up to almost 41.18% while preserving efficiency of x^*. In a similar manner as above we verify that the maximal tolerance is not attained.

Example 3. It might seem that the maximal tolerance is never attained. This is not true, as observed by the following example. Consider the feasible set from Example 1 with new objective and tolerance scale matrices respectively

$$C = \begin{pmatrix} 3.5 & 3.5 \\ 6 & 7.5 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

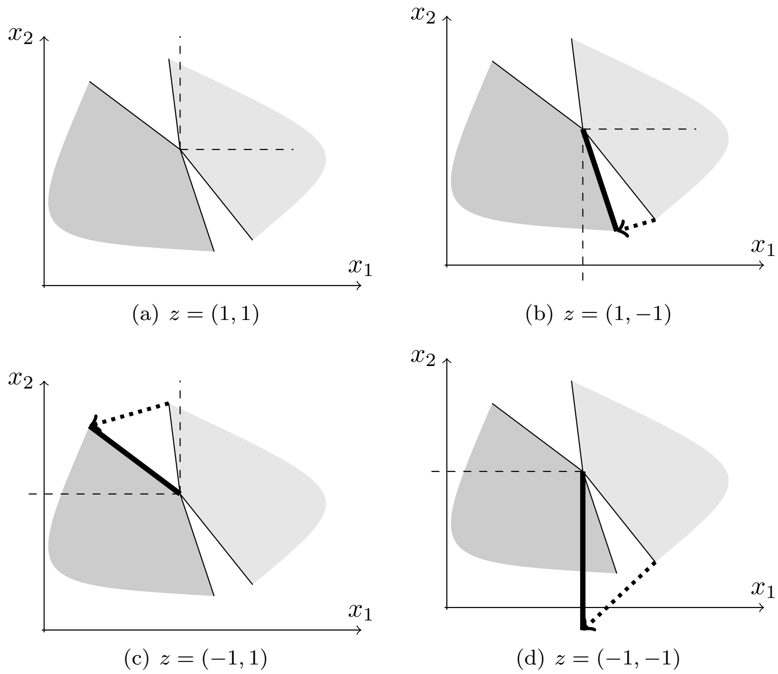

Figure 3: Obtaining δ_z in Example 1, the filled sets are the same as in Figure 1.

Herein, $x^* = (6, 6)^T$ is an efficient solution, too. The supremal tolerance computed $\delta^{\text{sup}} = 0.5$, and it corresponds to a unique minimizer $z = (-1, 1)$. Testing efficiency for

$$C' = C + \delta^{\text{sup}} G \operatorname{diag}(z) = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}. \tag{14}$$

becomes successful, see Figure 4, where C' is given by (14). Therefore, the maximal tolerance is attained, i.e., $\delta^{\text{sup}} = \delta^{\max}$. Checking whether δ^{sup} is the maximal tolerance in Examples 1 and 3 gives us some geometrical intuition, but characterization of existence of the maximal tolerance is a more complex question in general, and needs further research.

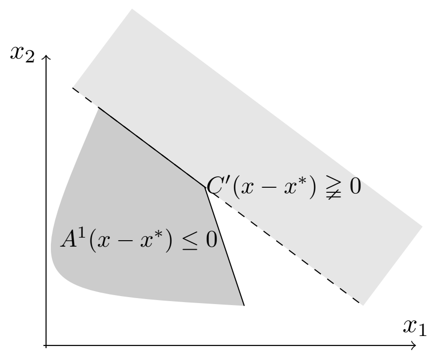

Figure 4: Checking whether δ^{sup} is the maximal tolerance in Example 3.

5 Conclusion

We proposed an algorithm for determining the supremal tolerance interval in which objective function coefficients may simultaneously vary while a given point remains efficient. The algorithm is exponential in the worst case, which is not surprising with respect to NP-hardness of the problem. Moreover, we propose a procedure to check whether the supremal tolerance is the maximal tolerance as well. What we leave as an open problem is any theoretical characterization of the situation when the maximal tolerance is attained.

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