Contractors for flows

Delia Garijo* Andrew Goodall† Jaroslav Nešetřil‡

Abstract

We answer a question raised by Lovász and B. Szegedy [Contractors and connectors in graph algebras, *J. Graph Theory* 60:1 (2009)] asking for a contractor for the graph parameter counting the number of B-flows of a graph, where B is a subset of a finite Abelian group closed under inverses. We prove our main result using the duality between flows and tensions and finite Fourier analysis. We exhibit several examples of contractors for B-flows, which are of interest in relation to the family of B-flow conjectures formulated by Tutte, Fulkerson, Jaeger, and others.

1 Introduction

In their paper [11] Lovász and B. Szegedy introduce the notion of contractors and connectors in quantum graph algebras. Connectors and contractors for a graph parameter allow expressions for the parameter of a graph obtained by the connection of two vertices by an edge or the contraction of an edge (either operation possibly producing multiple edges or loops, respectively) in terms of values of the parameter on graphs obtained by operations that preserve simplicity. For example, a connector for the flow polynomial is

*Department of Applied Mathematics I, University of Seville, Seville, Spain,Research supported by projects O.R.I MTM2008-05866-C03-01, PAI FQM-0164 and PAI P06-FQM-01649, email:dgarijo@us.es

†Corresponding author.,Research supported by ITI 1M0545, and the Centre for Discrete Mathematics, Theoretical Computer Science and Applications (DIMATIA).,email:goodall.aj@gmail.com

‡Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Prague, Czech Republic,Research supported by ITI 1M0545, and the Centre for Discrete Mathematics, Theoretical Computer Science and Applications (DIMATIA).,email:nesetril@kam.mff.cuni.cz

any path of length two or more. Contractors for graph parameters can be thought of as a generalization of the deletion-contraction identity for the chromatic polynomial, flow polynomial, and other specializations of the Tutte polynomial.

In [11, Section 2.3] Lovász and B. Szegedy comment that "there does not seem to be a simple explicit construction for a contractor" for the number of B-flows, where B is a subset of a finite Abelian group closed under inverses. In Section 4 we give an explicit construction (Theorem 14 and its Corollary 15), followed by numerous examples of relevance to the long-standing conjectures (such as the Cycle Double Cover Conjecture and Fulkerson's Conjecture) that Jaeger reformulates as a unifying B-flow conjecture in [9].

We prove our result using the Fourier transform on finite Abelian groups. Towards this end, in Section 2 we introduce homomorphism functions, flows and tensions of a graph, connectors and contractors in graph algebras, and then describe those properties of the Fourier transform that will be used to enable us to reach the main result (Theorem 13, and its special case for B-flows, Theorem 14). The Fourier transform expresses any weighted function on the set of tensions of a graph (effectively a homomorphism function to an edge-weighted graph) as a related weighted function on the set of flows of the graph. This in particular allows the number of B-flows to be written as the number of homomorphisms to an edge-weighted graph (Lemma 3). Although the duality relationship between flows and tensions is well known, there have not been many useful applications of this correspondence. In this paper it allows us to transform a special type of instance of a problem we do not in general know yet how to solve (Problem 17 in Section 6) into one that we can solve.

For graphs G and H, the number of homomorphisms from G to H is denoted by $\hom(G,H)$, and we extend this notation to edge-weighted graphs H. (A precise definition is given in Section 2.1). In Section 3 we consider the problem of finding a connnector and contractor for the homomorphism number $\hom(\cdot\,,H)$. The minimum polynomial of the adjacency matrix of H can be used to construct a connector for $\hom(\cdot\,,H)$, and also a contractor for $\hom(\cdot\,,H)$ if the adjacency matrix of H does not have eigenvalue 0 (Theorem 7). The relevant edge-weighted graph H that gives the number of B-flows turns out to have eigenvalue 0, but its special structure as an edge-weighted Cayley graph allows us to obtain a contractor by using the minimum polynomial of the graph whose edge weights are obtained by taking the Fourier transform of the edge weights of H (Theorem 13). In Section 4 we apply the theory developed in Sections 2 and 3 to derive our main result, giving a contractor for the number of B-flows (Theorem 14).

As an application of this result, in Section 5 we give a series of examples chosen on account of their relevance to such problems as the Cycle Double Cover Conjecture, Fulkerson's Conjecture and the Petersen Flow Conjecture, where we follow Jaeger's formulation [9] of these conjectures as B-flow problems. We also consider the example $B = \{-1, +1\}$ in the additive group $\mathbb{Z}_n$ of integers modulo n, because the number of B-flows here is equal to the number of Eulerian orientations of G when the vertex degrees of G all belong to $\{0, 1, \ldots, n{-}1, n{+}1\}$. As shown in [10], the number of Eulerian orientations of a graph is known not to be a homomorphism number to a finite edge-weighted graph, and therefore does not have a contractor [11, Example 2.6]. The interesting question arises of whether there is a "limiting contractor" for Eulerian orientations.

In Section 6 we conclude by posing some further problems concerning contractors of homomorphism functions to finite edge-weighted graphs.

2 Preliminaries

2.1 Homomorphism functions

Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a *homomorphism* of G to H, written as $\psi : G \to H$, is a mapping $\psi : V(G) \to V(H)$ such that $\psi(u)\psi(v) \in E(H)$ whenever $uv \in E(G)$. The number of homomorphisms of G to H is denoted by $\hom(G, H)$. For a fixed graph H, $\hom(\,\cdot\,, H)$ is a *graph parameter*, i.e., a function defined on graphs invariant under isomorphism. This parameter can be generalized for weighted graphs H as follows.

Let H be a weighted graph with a real positive weight $\alpha_H(i)$ associated with each vertex $i \in V(H)$ and a real weight $\beta_H(ij)$ associated with each edge $ij \in E(H)$. Let G be an unweighted graph and fix $U \subseteq V(G)$. Define for every function $\phi : U \to V(H)$ the weight

$$\hom_\phi(G, H) = \sum_{\substack{\psi: V(G)\to V(H)\\ \psi_{|U}=\phi}} \prod_{u\in V(G)\setminus U} \alpha_H(\psi(u)) \prod_{uv\in E(G)} \beta_H(\psi(u), \psi(v)).$$

Then the number of homomorphisms of G to H is defined by

$$\hom(G,H) = \sum_{\phi:U\to V(H)} \left(\prod_{u\in U} \alpha_H(\phi(u))\right) \hom_\phi(G,H).$$

In this paper we assume $\alpha_H(i) = 1$ for every $i \in V(H)$ (vertex weights equal to one) as it simplifies the exposition and also because for positive integer vertex weights we can replace the vertex weight $\alpha_H(i)$ by splitting the vertex i into $\alpha_H(i)$ unweighted twin vertices. Thus, H will be considered as an *edge-weighted graph* with symmetric adjacency matrix $A = A(H) = (\beta_H(ij))$ and

$$\hom_\phi(G,H) = \sum_{\substack{\psi:V(G)\to V(H)\\ \psi_{|U}=\phi}} \prod_{uv\in E(G)} \beta_H(\psi(u)\psi(v)),$$

$$\hom(G,H) = \sum_{\phi:U\to V(H)} \hom_\phi(G,H).$$

2.2 Flows and tensions

Let $G = (V,E)$ be a graph with an arbitrary fixed orientation of its edges. Denote by $c(G)$ the number of connected components of G, and by $r(G) = |V| - c(G)$ its rank. Let D be the $V\times E$ incidence matrix of G, with (v,e)-entry given by

$$D_{v,e} = \begin{cases} +1 & v \text{ is the head of } e,\\ -1 & v \text{ is the tail of } e,\\ 0 & v \text{ is not an endpoint of } e, \text{ or } e \text{ is a loop on } v. \end{cases}$$

Suppose Γ is a finite commutative ring with unity, and let $\Gamma^V = \{f : V \to \Gamma\}$ and $\Gamma^E = \{g : E \to \Gamma\}$.

The linear transformation $D : \Gamma^E \to \Gamma^V$ of Γ-modules has kernel equal to the set of Γ-*flows* of G. This is to say a function $g : E \to \Gamma$ is a Γ-flow if and only if, for each vertex $v \in V$,

$$\sum_{\substack{e=uv\\ v \text{ head of } e}} g(e) - \sum_{\substack{e=uv\\ v \text{ tail of } e}} g(e) = 0.$$

If $B = -B$ is a subset of Γ then an B-*flow* of G is a Γ-flow taking values in B. (We have followed Jaeger's [9] usage of the term B-flow; in [11]

Lovász and B. Szegedy use the letter S instead of B, and so speak of S-flows rather than the B-flows.) In particular, when $B = \Gamma \setminus \{0\}$ an B-flow is a *nowhere-zero* Γ*-flow* of G. It is well known that the number of nowhere-zero Γ-flows of G depends only on $|\Gamma|$ and is given by the evaluation of the *flow polynomial* $F(G; z)$ at $z = |\Gamma|$.

The transpose linear transformation $D^\top : \Gamma^V \to \Gamma^E$ has image the set of Γ*-tensions* of G. A Γ-tension $g : E \to \Gamma$ arises from a vertex Γ-colouring $f : V \to \Gamma$ of G, by setting $g(e) = f(v) - f(u)$ for each edge e with tail u and head v.

A B*-tension* of G is a Γ-tension of G taking values in B. In particular, when $B = \Gamma \setminus \{0\}$ a B-tension arises from $|\Gamma|^{c(G)}$ proper vertex Γ-colourings of G (a proper colouring by definition has the property that the difference between endpoint colours on any edge is non-zero). The number of proper vertex Γ-colourings is given by the evaluation of the *chromatic polynomial* $P(G; z)$ at $z = |\Gamma|$.

Note that to define Γ-flows and Γ-tensions all that is needed is the structure of Γ as an additive Abelian group. As we shall see in Section 2.4, it is convenient to impose the extra ring structure on Γ in order to relate the sets of tensions and flows to each other as orthogonal complements.

2.3 Connectors and contractors

For fixed positive integer k, a k*-labelled graph* (G, λ) is a finite graph G (without loops but possibly multiple edges) together with a function $\lambda : [k] \to V(G)$. Lovász and Szegedy [11] restrict λ to be an injective function, but we shall follow Schrijver [12] and allow vertices to have multiple labels. A k-labelled graph (G, λ) is *simple* if G has no parallel edges, the function λ is injective and its image is an independent set of vertices.

A k-labelled *quantum graph* is a formal linear combination of k-labelled graphs with coefficients in $\mathbb{R}$. The set of k-labelled quantum graphs is denoted by $\mathcal{G}_k$. The subset of k-labelled quantum graphs whose labelled vertices are independent is denoted by $\mathcal{G}_k^0$, and the subset of simple k-labelled quantum graphs is denoted by $\mathcal{G}_k^{\mathrm{simp}}$.

We let K_n denote the complete graph on n vertices and $\overline{K}_n$ its complement, consisting of n isolated vertices.

For two k-labelled graphs X and Y, the *product* XY is defined [11] by taking the disjoint union of X and Y and then identifying vertices which share the same label. The product is associative and commutative, and extends linearly to $\mathcal{G}_k$. The identity for this multiplication on $\mathcal{G}_k$ is $(\overline{K}_k, \lambda)$ where λ is injective, i.e., k independent vertices each with one label.

In the sequel we shall just consider the case $k = 2$.

A further binary operation on 2-labelled graphs is the *concatenation* $X \circ Y$ of X and Y, defined by identifying the vertex of X labelled 2 with the vertex labelled 1 in Y and unlabelling this merged vertex. Concatentation is associative but not commutative. It is extended linearly to $\mathcal{G}_2$. The identity for this multiplication on $\mathcal{G}_2$ is K_1, with both labels 1 and 2 on the single vertex.

Let P_ℓ, $\ell \geq 1$, denote the 2-labelled path on $\ell + 1$ vertices with one endpoint labelled 1 and the other endpoint labelled 2. In this notation, $P_1 = K_2$. We have $P_\ell \circ P_m = P_{\ell+m}$, and $P_\ell P_m = C_{\ell+m}$, where the cycle has labels spaced ℓ edges apart in one direction, m edges apart in the other. If X is a 2-labelled graph then XP_1 is the 2-labelled graph with an edge added joining the vertices labelled 1 and 2. In particular, if X is a simple 2-labelled graph then XP_1 joins non-adjacent vertices with an edge.

The graph K_1 with vertex carrying both labels 1 and 2 when applied as a product with X (giving the graph XK_1) identifies the vertices labelled 1 and 2 in X (or does nothing if both labels are on a single vertex in X). If 1 and 2 are independent in X, then XK_1 is the graph XP_1 with the edge joining 1 and 2 contracted.

The graph $\overline{K}_2$ with one vertex labelled 1 and the other labelled 2 is the identity for the product of 2-labelled graphs.

A *series-parallel* 2-labelled graph is obtained from P_1 by a sequence of products (parallel extensions) and concatenations (series extensions).

Definition 1. [11] *Let h be a graph parameter. A 2-labelled quantum graph $Z \in \mathcal{G}_2$ is a* contractor *for h if for all $X \in \mathcal{G}_2^0$ we have*

$$h(XZ) = h(XK_1).$$

A simple 2-labelled quantum graph $Z \in \mathcal{G}_k^{\mathrm{simp}}$ is a connector *for h if for all $X \in \mathcal{G}_2$ we have*

$$h(XZ) = h(XP_1).$$

Note that attaching a contractor Z at two non-adjacent vertices acts like identifying these vertices, and attaching a connector Z at two vertices acts like joining them with an edge; if these vertices are already joined by an edge then since $Z \in \mathcal{G}_2^{\mathrm{simp}}$ the graph XZ remains simple whereas XP_1 is not simple.

Example 2. *The deletion-contraction relation for the chromatic polynomial takes the form $P(XP_1;z)=P(X;z)-P(XK_1;z)$ where X has its two labels on non-adjacent vertices. Extending the chromatic polynomial linearly to $\mathcal{G}_2$, this is to say $P(XK_1;z)=P(X(\overline{K}_2-P_1);z)$. The quantum graph $\overline{K}_2-P_1$ is a contractor for the chromatic polynomial.*

For $X\in\mathcal{G}_2^{\mathrm{simp}}$, the flow polynomial satisfies $F(XP_1;q)=F(XP_\ell;q)$ for any $\ell\geq 2$; the 2-labelled graph P_2 is a simple connector for the flow polynomial.

2.4 The Fourier transform

The duality between flows and tensions is well known, especially in statistical physics, see e.g. [2], with an early example being Van der Waerden's Eulerian subgraph expansion of the Ising model partition function [14]. The material in this section is standard and can be found in [13], except for its main result, Lemma 3, for which we therefore provide a proof. We briefly recall the relevant facts here for the convenience of the reader. More examples of applications of elementary Fourier analysis on finite Abelian groups to graph theory can be found in [5].

Let Γ be a finite additive Abelian group of order n. If ℓ is the least common multiple of the orders of the elements of Γ then there is a unique sequence of positive integers $n_1,n_2\ldots,n_r$ such that $2\leq n_1\mid n_2\mid\cdots\mid n_r=\ell$ and

$$\Gamma\cong\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\cdots\oplus\mathbb{Z}_{n_r}. \tag{1}$$

We can give Γ the structure of a ring by defining multiplication of elements $x=(x_1.x_2,\ldots,x_r)$ and $y=(y_1,y_2,\ldots,y_r)$ componentwise: $xy=(x_1y_1,x_2y_2,\ldots,x_ry_r)$. Alternative multiplicative structures are possible. For example, when Γ is isomorphic to $\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\cdots\oplus\mathbb{Z}_p$ we can endow Γ with the structure of the finite field $\mathbb{F}_{p^r}$. A *character* of Γ is a homomorphism $\chi:\Gamma\to\mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group of the field of complex numbers. The set of characters of Γ form a group $\widehat{\Gamma}$ under pointwise multiplication, which is isomorphic to Γ when Γ is a finite Abelian group. For each $x\in\Gamma$, let χ_x denote the image of x under a fixed isomorphism $\Gamma\to\widehat{\Gamma}$. In particular, the trivial character χ_0 is defined by $\chi_0(y)=1$ for all $y\in\Gamma$, and $\chi_{-x}(y)=\overline{\chi_x(y)}$, the bar denoting complex conjugation.

Suppose then that the additive group Γ has been given the extra structure of a commutative ring. A character $\chi\in\widehat{\Gamma}$ is a *generating character* for Γ if $\chi_x(y)=\chi(xy)$ for each character $\chi_x\in\widehat{\Gamma}$. For example, the ring $\mathbb{Z}_n$ has a generating character defined by $\chi(x) = e^{2\pi i x/n}$, and the field $\mathbb{F}_{p^r}$ has a generating character defined by $\chi(x) = e^{2\pi \operatorname{Tr}(x)/p}$, where $\operatorname{Tr}(x) = x + x^p + \cdots + x^{p^{r-1}}$. For any additive Abelian group Γ we can use its canonical expression (1) as a direct sum of rings of the form $\mathbb{Z}_n$ to obtain a generating character.

Consider Γ^m, the m-fold direct sum of Γ, and suppose ψ is a generating character for Γ. Then χ defined by $\chi(x_1, \ldots, x_m) = \psi(x_1) \cdots \psi(x_m)$ for $(x_1, \ldots, x_m) \in \Gamma^m$ is a generating character for Γ^m. The *Euclidean inner product* (dot product) is defined for $x =(x_1, \ldots, x_m), y= (y_1, \ldots, y_m) \in \Gamma^m$ by $x \cdot y= x_1 y_1 + \cdots + x_m y_m$. Since $\psi(x_1) \cdots \psi(x_m) = \psi(x_1 + \cdots x_m)$, it follows that $\chi_x(y) = \chi(xy) = \psi(x \cdot y)$ for $x, y \in \Gamma^m$.

Denote by $\mathbb{C}^\Gamma$ the vector space over $\mathbb{C}$ of all functions from Γ to $\mathbb{C}$. This is an inner product space with Hermitian inner product $\langle \cdot \rangle$ defined for $\alpha, \beta \in \mathbb{C}^\Gamma$ by

$$\langle \alpha, \beta \rangle = \sum_{x \in \Gamma} \alpha(x) \overline{\beta(x)}.$$

The vector space $\mathbb{C}^\Gamma$ has the additional structure of an algebra under either of the following two definitions of multiplication:

(i) the *pointwise product* $\alpha \cdot \beta$ of $\alpha, \beta \in \mathbb{C}^\Gamma$, defined for $x \in \Gamma$ by $\alpha \cdot \beta(x) = \alpha(x)\beta(x)$,

(ii) the *convolution* $\alpha * \beta$ of $\alpha, \beta \in \mathbb{C}^\Gamma$, defined for $x \in \Gamma$ by

$$\alpha * \beta(x) = \sum_{y \in \Gamma} \alpha(y)\beta(x - y).$$

The set $\{\delta_x : x \in \Gamma\}$ of indicator functions defined by

$$\delta_x(y) = \begin{cases} 1 & x = y, \\ 0 & x \neq y, \end{cases}$$

form an orthonormal basis for $\mathbb{C}^\Gamma$, with $\langle \delta_x, \delta_y \rangle = \delta_x(y)$. We extend the indicator function notation to subsets B of Γ, defining $\delta_B = \sum_{x \in B} \delta_x$.

The characters of Γ are also orthogonal in this inner product space, with$\langle \chi_x, \chi_y \rangle = |\Gamma| \delta_x(y)$.

Fix an isomorphism $x \mapsto \chi_x$ of Γ with $\widehat{\Gamma}$ and let χ be a generating character for Γ such that $\chi_x(y) = \chi(xy)$.

For $\alpha \in \mathbb{C}^\Gamma$ the *Fourier transform* $\widehat{\alpha} \in \mathbb{C}^\Gamma$ is defined for $y \in \Gamma$ by

$$\widehat{\alpha}(y) = \langle \alpha, \chi_y \rangle = \sum_{x \in \Gamma} \alpha(x) \chi_y(-x). \tag{2}$$

The Fourier transform maps the basis of indicator functions to the basis of characters: $\widehat{\delta_y} = \chi_{-y}$. The *Fourier inversion formula* $\widehat{\widehat{\alpha}}(x) = |\Gamma| \alpha(-x)$, gives the inverse transform

$$\alpha(x) = \frac{1}{|\Gamma|} \langle \widehat{\alpha}, \chi_{-x} \rangle = \frac{1}{|\Gamma|} \sum_{y \in \Gamma} \widehat{\alpha}(y) \chi_x(y). \tag{3}$$

Note that $\widehat{\delta_\Gamma} = |\Gamma| \delta_0$ and $\widehat{\delta_0} = \delta_\Gamma$, since $\langle \delta_\Gamma, \chi_y \rangle = \langle \chi_0, \chi_y \rangle = |\Gamma| \delta_0(y)$.

The Fourier transform gives an isomorphism of the algebra $\mathbb{C}^\Gamma$ with multiplication pointwise product with the algebra $\mathbb{C}^\Gamma$ with multiplication convolution: for $y \in \Gamma$,

$$\widehat{\alpha * \beta}(y) = \widehat{\alpha} \cdot \widehat{\beta}(y), \tag{4}$$

$$\widehat{\alpha \cdot \beta}(y) = \frac{1}{|\Gamma|} \widehat{\alpha} * \widehat{\beta}(y). \tag{5}$$

For a subgroup C of Γ, the *annihilator* $C^\sharp$ of C is defined by

$$C^\sharp := \{ x \in R : \chi_x(y) = 1 \text{ for all } y \in C \}.$$

The annihilator $C^\sharp$ is a subgroup of Γ isomorphic to Γ / C. A second key property of the Fourier transform is that it takes indicators of subgroups to (scalar multiples of) indicators of their annihilators:

$$\widehat{\delta}_C(y) = \sum_{x \in C} \chi_x(y) = |C| 1_{C^\sharp}(y). \tag{6}$$

By (3), (4) and (6) there follows the *Poisson summation formula*,

$$\sum_{x \in C} \alpha(x + z) = \frac{1}{|C^\sharp|} \sum_{x \in C^\sharp} \widehat{\alpha}(x) \chi_z(x). \tag{7}$$

Consider again the m-fold direct sum Γ^m. The *orthogonal* to a subset C of Γ^m (with respect to the Euclidean inner product) is defined by

$$C^\perp = \{ y \in \Gamma : x \cdot y = 0 \text{ for all } y \in C \}.$$

If Γ has a generating character and C is a Γ-submodule of Γ^m, then $C^\sharp = C^\perp$.

Thus we have the following special case of the Poisson summation formula when C is a Γ-submodule of Γ^m with orthogonal complement $C^\perp$:

$$\sum_{x\in C}\alpha(x) = \frac{1}{|C^\perp|}\sum_{y\in C^\perp}\widehat{\alpha}(y). \tag{8}$$

We now apply the machinery we have set up to relate flows and tensions of a graph.

Let D be the incidence matrix of a graph G, as defined in Section 2.2. Recall that the kernel $\ker D$ of the map $D:\Gamma^E\to\Gamma^V$ consists of the Γ-flows of G, and the image $\mathrm{im} D^\top$ of the adjoint map $D^\top:\Gamma^V\to\Gamma^E$ consists of the Γ-tensions of G.

Provided Γ has a generating character, we have $(\mathrm{im}\, D^\top)^\perp = \ker D$ and $(\ker D)^\perp = \mathrm{im}\, D^\top$. We also have $|\ker D| = |\Gamma|^{|E|-r(G)}$ and $|\mathrm{im}\, D^\top| = |\Gamma|^{r(G)}$. Hence by (8),

$$\sum_{x\in\ker D}\alpha(x) = \frac{1}{|\Gamma|^{r(G)}}\sum_{y\in\mathrm{im} D^\top}\widehat{\alpha}(y). \tag{9}$$

Lemma 3. *Let $B=-B$ be a subset of an additive Abelian group Γ. Let $H=\mathrm{Cayley}(\Gamma,B)$ be the graph on vertex set Γ with an edge joining vertices i and j if and only if $j-i\in B$, and let $\widehat{H}$ be the edge-weighted Cayley graph on vertex set Γ with edge ij having weight $\widehat{\delta}_B(j-i)$.*

Then the number of B-tensions of a graph $G=(V,E)$ is equal to $|\Gamma|^{-c(G)}\hom(G,H)$ and the number of B-flows of G is equal to $|\Gamma|^{-|V|}\hom(G,\widehat{H})$.

Proof. A function $f:V\to\Gamma$ is a homomorphism contributing to $\hom(G,H)$ if and only if $f(v)-f(u)\in B$ for each edge $e=uv\in E$, which is to say that $g(e):=f(v)-f(u)$ is a B-tension of G. For a given B-tension $g:E\to\Gamma$ there are $|\Gamma|^{c(G)}$ vertex colourings $f:V\to\Gamma$ with $Df=g$. This proves that the number of B-tensions is $|\Gamma|^{-c(G)}\hom(G,H)$.

To prove that $|\Gamma|^{-|V|}\hom(G,\widehat{H})$ is the number of B-flows, take $\alpha=\delta_{B^E}$ in (9), which has Fourier transform for $g:E\to\Gamma$ given by

$$\widehat{\delta}_{B^E}(g) = \prod_{e \in E} \widehat{\delta}_B(g(e)),$$

which yields

$$|\ker D \cap B^E| = \frac{1}{|\Gamma|^{|V|}} \sum_{f : V \to \Gamma} \prod_{uv \in E} \widehat{\delta_B}(f(v) - f(u)).$$

$\square$

3 Connectors and contractors for homomorphism numbers

After the preliminaries of Section 2 we now turn to the general problem of finding a contractor for homomorphism numbers $\hom(\,\cdot\,, H)$. (In Section 4 we then consider the special case of finding a contractor for the number of B-flows.) In this section we shall also find an expression for a connector for $\hom(\,\cdot\,, H)$, although this is of less interest for the case of B-flows (where P_2 is always a connector).

Let H be an edge-weighted graph with adjacency matrix $A = (\beta(ij))$. To every $X \in \mathcal{G}_2$ assign a $V(H) \times V(H)$-matrix $M(X) = M_H(X)$ with (i, j)-entry equal to $\hom_\phi(X, H)$ where $\phi(1) = i$ and $\phi(2) = j$. As H is fixed we shall usually leave the dependence on H implicit and write $M(X)$ instead of $M_H(X)$. The function $M : \mathcal{G}_2 \to \mathbb{C}^{V(H) \times V(H)}$ is linear and satisfies

$$M(X \circ Y) = M(X)M(Y), \;\; M(XY) = M(X) \circ M(Y),$$

where $\circ$ is the entrywise (Hadamard, or Schur) product.

Note that $M(P_1) = (\beta(ij))$ is the adjacency matrix of H. Let C_k^* denote the graph on two vertices joined by k parallel edges (the dual of the k-cycle), with one vertex labelled 1 and the other vertex labelled 2. The graph P_k^* consists of one vertex with k loops attached (the dual of the path on k edges), and with labels 1 and 2 on the single vertex.

Lemma 4. *If H has adjacency matrix A then*

$$M(P_k) = A^k, \quad M(C_k^*) = A^{\circ k},$$

where $A^{\circ k}$ denotes the k-fold Hadamard product $A \circ A \circ \cdots \circ A$, and A^k the k-fold ordinary matrix product.

PROOF. Since $M(P_1) = A$, $P_k = P_1^{\circ k}$ and $C_k^* = P_1^k$ then $M(P_k) = M(P_1^{\circ k}) = M(P_1)^k = A^k$ and $M(C_k^*) = M(P_1^k) = M(P_1)^{\circ k} = A^{\circ k}$.□

Lemma 4 includes for $k = 0$ the fact that if K_1 is single vertex with labels 1 and 2 then $M(K_1) = I$. We also have $M(\overline{K}_2) = J$, where $\overline{K}_2$ comprises two isolated vertices, one labelled 1 the other 2. (The weight of a homomorphism from a graph X with no edges to H is equal to the number of maps from $V(X)$ to $V(H)$. Restricting to homomorphisms coinciding with ϕ on the labelled vertices 1 and 2 of X, we have $\hom_\phi(X, H) = 1$ when the vertices labelled 1 and 2 are distinct, and $\hom_\phi(X, H) = \delta_{\phi(1),\phi(2)}$ when the same vertex of X has labels 1 and 2.)

A generalization of Lemma 4 is that the matrices generated by A (by applying the ordinary and Hadamard products of matrices) are equal to $M(X)$ for a series-parallel graph X.

Lemma 5. *Let C_k denote the cycle on k vertices with one vertex receiving both labels* 1 *and* 2*, and P_k^* the graph comprising k loops on one vertex with both labels* 1 *and* 2*. Then*

$$M(C_k) = I \circ A^k, \quad M(P_k^*) = I \circ A^{\circ k}.$$

PROOF. The result holds by Lemma 4, since $C_k = P_k K_1$ and $P_k^* = C_k^* K_1$. □

A fundamental observation with regard to connectors and contractors for homomorphism numbers is the following:

Lemma 6. [11] *A 2-labelled quantum graph Z is a contractor for* $\hom(\, . \, , H)$ *if and only if $M(Z) = I$ and a simple 2-labelled quantum graph Z is a connector for* $\hom(\, . \, , H)$ *if and only if $M(Z) = A$.*

Using this lemma we can immediately write down a connector for $\hom(\,\cdot\,, H)$, and also a contractor when the adjacency matrix of H does not have 0 as an eigenvalue. (Compare [11, Section 4.1].)

Theorem 7. *Suppose the adjacency matrix A of H has minimum polynomial*

$$p_A(t) = p_0 + p_1 t + \cdots + p_\ell t^\ell.$$

(i) If $p_0 \neq 0$ (i.e, if 0 is not an eigenvalue of A), then a connector for $\hom(\,\cdot\,, H)$ *is given by*

$$Z = -\frac{1}{p_0} \left[p_1 P_2 + p_2 P_3 + \cdots + p_\ell P_{\ell+1} \right],$$

where $P_k = P_1^{\circ k}$ is the path with k edges, and with one endpoint vertex labelled 1, the other labelled 2. Also, a contractor for $\hom(\cdot, H)$ *is given by*

$$Z = -\frac{1}{p_0}\left[p_1P_1 + p_2P_2 + \cdots + p_\ell P_\ell\right].$$

(ii) If $p_0 = 0$, then $p_1 \neq 0$ and a connector for $\hom(\cdot, H)$ *is given by*

$$Z = -\frac{1}{p_1}\left[p_2P_2 + \cdots + p_\ell P_\ell\right].$$

Proof. Note that $P_k = P_1^{\circ k}$ and $M(P_k) = A^k$. Suppose first $p_0 \neq 0$. Then $I = -\frac{1}{p_0}\sum_{k=1}^{\ell} p_k A^k$. Hence, if $Z = -\frac{1}{p_0}\sum_{k=1}^{\ell} p_k P_k$ then $M(Z) = I$, and Z is a contractor for $\hom(\cdot, H)$. Likewise, $A = -\frac{1}{p_0}\sum_{k=1}^{\ell} p_k A^{k+1}$ so that $Z = -\frac{1}{p_0}\sum_{k=1}^{\ell} p_k P_{k+1}$ is a connector for $\hom(\cdot, H)$.

When $p_0 = 0$ and $p_1 \neq 0$, we have $A = -\frac{1}{p_1}\sum_{k=2}^{\ell} p_k A^k$ from which it follows that $Z = -\frac{1}{p_1}\sum_{k=2}^{\ell} p_k P_k$ is a connector. □

We are left with the problem of constructing a contractor for $\hom(\cdot, H)$ when the adjacency matrix of H has 0 as an eigenvalue. When H is a weighted Cayley graph we are able to find such a contractor, and the remainder of this section is devoted to showing how.

We begin with the following result of Hoffmann, for which see e.g. [3, Chapter 3].

Theorem 8. *Let A be the adjacency matrix of a graph H. The all-one matrix J is a polynomial in A if and only if H is connected and A has eigenvector $\mathbf{1}$ (i.e., H is regular).*

Corollary 9. *Suppose that the adjacency matrix A of a connected graph H on n vertices has eigenvector $\mathbf{1}$ with eigenvalue s, that its distinct eigenvalues are $s > \lambda_1 > \cdots > \lambda_{r-1}$, and that*

$$q(t) = \prod_{1 \leq i \leq r-1} (t - \lambda_i).$$

Then

$$J = \frac{n}{q(s)} q(A).$$

Proof. We have $q(A) = \ell J$ for some $\ell \neq 0$ and $q(A)$ has eigenvalues $q(s)$ and $q(\lambda_i) = 0$ for $1 \leq i \leq r-1$. Since the only non-zero eigenvalue of ℓJ is ℓn it follows that $\ell n = q(s)$. $\Box$

Note that the polynomial $q(t)$ in Corollary 9 is equal to $p_A(t)/(t-s)$, where $p_A(t)$ is the minimum polynomial of A. (Since A is symmetric it is diagonalizable and so each of its Jordan blocks has size 1, which implies the multiplicity of each root of the minimum polynomial of A is equal to one.)

Suppose now that A has rows and columns indexed by an additive Abelian group Γ of order n and has (i, j) entry equal to $(\beta(i-j))$ for all $i, j \in \Gamma$, where $\beta : \Gamma \to \mathbb{C}$ satisfies $\beta(-i) = \beta(i)$. Thus A is the adjacency matrix of a weighted Cayley graph on Γ.

Recall the definition of characters and the Fourier transform from Section 2.4. The matrix A has eigenvectors $(\chi_i(j))_{j\in\Gamma}$ for each $i \in \Gamma$ with corresponding eigenvalue $\widehat{\beta}(i)$. Define $\widehat{A}$ to be the matrix with (i, j)-entry the Fourier transform $\widehat{\beta}(i-j)$. For example, $\widehat{I} = J$ and $\widehat{J} = nI$.

Let us call a matrix Γ-*circulant* if, like the matrix A, it takes the form $(\alpha(i-j))$ for some function $\alpha \in \mathbb{C}^\Gamma$. (When Γ is cyclic these are circulant matrices as ordinarily defined.)

Lemma 10. *The map $A \mapsto \widehat{A}$ is an algebra isomorphism of the subalgebra of Γ-circulant matrices under addition and ordinary matrix multiplication with the subalgebra of Γ-circulant matrices under addition and entrywise product:*

$$\widehat{A_1A_2} = \widehat{A}_1 \circ \widehat{A}_2, \quad \widehat{A_1 \circ A_2} = \frac{1}{n}\widehat{A}_1\widehat{A}_2.$$

In particular

$$\widehat{A^k} = \widehat{A}^{\circ k}, \quad \widehat{A^{\circ k}} = \frac{1}{n^{k-1}}\widehat{A}^{\,k}.$$

Proof. This is a consequence of the identities (4) and (5) for the Fourier transform of convolutions and pointwise products. Since the Fourier transform is linear it follows that $A \to \widehat{A}$ is an algebra isomorphism. $\Box$

Lemma 11. *Given*

$$q(A) = q_0I + q_1A + q_1A^2 + \cdots + q_\ell A^\ell,$$

define

$$q^\circ(A) = q_0J + q_1A + q_2A^{\circ 2} + \cdots + q_\ell A^{\circ \ell},$$

i.e., the same polynomial except in Hadamard product powers of A instead of ordinary matrix powers of A and with the identity I for ordinary matrix multiplication replaced by the identity J for entrywise multiplication.

Then $\widehat{q(A)} = q^\circ(\widehat{A})$ *and* $\widehat{q^\circ(A)} = q(\frac{1}{n}\widehat{A})$.

Proof. This follows from Lemma 10. □

Recall that for $Z \in \mathcal{G}_2$ to be a contractor for $\hom(\,\cdot\,, H)$ we need $M(Z) = I$. The following is the key to obtaining such a contractor when H has a Γ-circulant adjacency matrix A.

Lemma 12. *Suppose A has eigenvector* $\mathbf{1}$ *with eigenvalue s, minimum polynomial $p_A(t)$ and that $q(t) = p_A(t)/(t-s)$. Then*

$$I = \frac{1}{q(s)} q^\circ(\widehat{A}).$$

Proof. By Corollary 9 we have $J = \frac{n}{q(s)} q(A)$ and by Lemma 11, $\widehat{q(A)} = q^\circ(\widehat{A})$. Since $\widehat{J} = nI$, the result follows. □

Our main theorem of this section now results:

Theorem 13. *Suppose H is a connected graph with adjacency matrix a Γ-circulant matrix A and that $\widehat{H}$ has adjacency matrix $\widehat{A}$. Suppose further that A has eigenvector* $\mathbf{1}$ *with eigenvalue s, has minimum polynomial $p_A(t)$, and that*

$$p_A(t)/(t-s) = q(t) = q_0 + q_1 t + \cdots + q_{\ell-1} t^{\ell-1}.$$

Then a contractor for $\hom(\,\cdot\,, \widehat{H})$ *is given by*

$$Z = \frac{1}{q(s)} \sum_{0 \le k \le \ell-1} q_k P_1^k,$$

where $P_1^k = C_k^*$ comprises two vertices, one labelled 1 *and the other* 2*, joined by k edges. (In particular, $P_1^0 = \overline{K}_2$.)*

Proof. This is immediate from Lemma 12 and the fact that $M_{\widehat{H}}(P_1^k) = \widehat{A}^{\circ k}$. [The matrix $\widehat{A}$ is the adjacency matrix of the edge-weighted graph $\widehat{H}$, and $M_{\widehat{H}}(X)$ the $V(\widehat{H}) \times V(\widehat{H})$ matrix whose (i,j)-entry is $\hom_\phi(X, \widehat{H})$, where $\phi(1) = i, \phi(2) = j$.] □

4 A contractor for the number of B-flows

In this section we apply Theorem 13 to the special case where $H = \mathrm{Cayley}(\Gamma, B)$ for additive Abelian group Γ, and where $B \subseteq \Gamma \setminus 0$ satisfies $-B = B$.

Let δ_B be the indicator function of B. The adjacency matrix A of H has (i,j)-entry $\delta_B(i-j)$ and eigenvalues $\widehat{\delta_B}(i)$ for $i \in \Gamma$. The characteristic polynomial of A is given by

$$\phi_A(t) = \prod_{c\in\Gamma}(t - \widehat{\delta_B}(c)),$$

and has degree $|\Gamma| = |V(H)|$. The minimum polynomial of A has degree ℓ equal to the number of distinct values of $\widehat{\delta_B}(c)$ for $c \in \Gamma$, bounded below by $d(H)+1$, where $d(H)$ is the diameter of H. (This lower bound is attained when H is a distance regular graph.) The largest eigenvalue of A is $\widehat{\delta_B}(0) = |B|$, belonging to the eigenvector $\mathbf{1}$.

We reach the main theorem of this paper, giving a contractor for B-flows.

Theorem 14. *Let Γ be an additive Abelian group of order n and suppose $H = \mathrm{Cayley}(\Gamma, B)$ is a connected graph with adjacency matrix A, whose (i,j) entry is equal to $\delta_B(i-j)$. Let A have minimum polynomial $p_A(t)$ of degree ℓ equal to the number of distinct eigenvalues $\widehat{\delta_B}(i)$ of A, and set*

$$p_A(t)/(t-|B|) = q(t) = q_0 + q_1 t + \cdots + q_{\ell-1}t^{\ell-1}.$$

Define $\widehat{H}$ to be the edge-weighted graph on vertex set Γ with adjacency matrix having (i,j)-entry $\widehat{\delta_B}(i-j)$. Then

(i) A contractor for the number of B-flows is given by

$$Z = \frac{n}{q(|B|)} \sum_{0\le k\le \ell-1} q_k P_1^k,$$

where $P_1^k = C_k^*$ comprises two vertices, one labelled 1 and the other 2, joined by k parallel edges (in particular, $P_1^0 = \overline{K}_2$);

(ii) A contractor for $n^{c(G)}$ times the number of B-tensions of G is given by

$$Z = \frac{1}{q_0}\left[\frac{q(|B|)}{n}\overline{K}_2 - \sum_{1\le k\le \ell-1} q_k P_k\right].$$

Proof. Part (i) is a corollary of Theorem 13, which gives

$$\frac{1}{q(|B|)}\sum_{0\le k\le \ell-1} q_k P_1^k,$$

as a contractor for $\hom(\,\cdot\,,\widehat{H})$. The number of B-flows of G is by Lemma 3 equal to $n^{-|V(G)|}\hom(G,\widehat{H})$, and this implies the expression for the contractor as given in the theorem statement.

In order to show (ii), by Corollary 9 we have

$$J=\frac{n}{q(|B|)}\sum_{0\le k\le \ell-1} q_k A^k.$$

Since $q_0\neq 0$, we then obtain

$$I=\frac{1}{q_0}\left[\frac{q(|B|)}{n}J-\sum_{1\le k\le \ell-1} q_k A^k\right],$$

and this implies $M_H(Z)=I$, where Z is the quantum graph given in the theorem statement, i.e., Z is a contractor for $\hom(\,.\,,H)$. By Lemma 3, the number of B-tensions of G is equal to $n^{-c(G)}\hom(G,H)$. $\square$

5 Examples

In this section we exhibit some concrete instances of the contractor for B-flows given by Theorem 14.

5.1 Proper vertex colourings, nowhere-zero flows

When Γ is order n and $B=\Gamma\setminus\{0\}$, the graph $H=\mathrm{Cayley}(\Gamma,B)$ is isomorphic to K_n. The adjacency matrix $A=J-I$ has minimum polynomial $p_A(t)=(t+1)(t-n+1)$ and $q(t)=p_A(t)/(t-n+1)=1+t$. Here $J=I+A$. A contractor for $\hom(\,\cdot\,,K_n)$ is given by $\overline{K}_2-P_1$ (which amounts to the contraction–deletion identity for the chromatic polynomial). Since $A=(n-1)J-A^2$, a connector for $\hom(\,\cdot\,,K_n)$ is given by $(n-1)\overline{K}_2-P_2$. (In [11] the connector given is $\frac{1}{n-1}\left[P_3-(n-2)P_2\right]$, which follows from the equation $(n-1)A=A^3-(n-2)A^2$ obtained from $Ap_A(A)=0$.)

The graph $\widehat{H}$ has adjacency matrix $\widehat{A} = nI - J$, which satisfies $\widehat{A}^2 = n\widehat{A}$, so that $\frac{1}{n}P_2$ is a connector for $\hom(G, \widehat{H}) = n^{|V(G)|}F(G;n)$, and hence P_2 is a connector for $F(G;n)$, the number of nowhere-zero $\mathbb{Z}_n$ flows (as expected). Since $I = \frac{1}{n}(J + \widehat{A})$, a contractor is given by $\frac{1}{n}\left[\overline{K}_2 + P_1\right]$. Hence a contractor for $F(G;n)$ is given by $\overline{K}_2 + P_1$. (That this is the case amounts to the deletion–contraction identity for the flow polynomial).

5.2 Strongly regular graphs

If H is a strongly regular graph with parameters (n, k, λ, μ) then its adjacency matrix satisfies

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J.$$

In this case, a contractor Z for $\hom(\,\cdot\,, H)$ is thus given by

$$(\mu - k)Z = \mu\overline{K}_2 - (\mu - \lambda)P_1 - P_2.$$

This is the content of [6, Proposition 8].

The graph $\mathrm{Cayley}(\Gamma, B)$ is a strongly regular graph with parameters (n, k, λ, μ) precisely when B is a (n, k, λ, μ)-partial difference set (the differences $i - j$ for $i, j \in B$ give each element of B exactly λ times, each element of $\Gamma - B - \{0\}$ appears μ times, and 0 appears k times). For example, the Paley graph of order $q \equiv 1 \pmod 4$ ($\Gamma = \mathbb{F}_q$, B the set of non-zero squares) is a strongly regular graph with parameters $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$.

5.3 An intermediate result

In order to proceed with examples related to some of the B-flow conjectures described in [9], we need to supplement Theorem 14 by its statement for disconnected Cayley graphs H.

If B does not contain a set of generators for Γ then the Γ-circulant graph $H = \mathrm{Cayley}(\Gamma, B)$ is not connected. The connected components of $\mathrm{Cayley}(\Gamma, B)$ correspond to the cosets of the subgroup generated by B.

Theorem 15. *Let Γ be an additive Abelian group of order n and suppose $H = \mathrm{Cayley}(\Gamma, B)$ has adjacency matrix A. Let A have minimum polynomial $p_A(t)$ and set*

$$p_A(t)/(t - |B|) = q(t) = q_0 + q_1 t + \cdots + q_{\ell-1}t^{\ell-1}.$$

Suppose H has a isomorphic connected components, i.e., its adjacency matrix A is permutation-equivalent to a matrix of the form $I \otimes A_1$ for $a \times a$ identity matrix I and some $n/a \times n/a$ matrix A_1. Then, a contractor for the number of B-flows is given by

$$Z = \frac{n}{aq(|B|)} \sum_{0 \leq k \leq \ell - 1} q_k P_1^k,$$

where P_1^k consists of two vertices, one labelled 1 and the other 2, joined by k edges.

Proof. If B generates a subgroup Γ_1 of index a in Γ then $\mathrm{Cayley}(\Gamma, B)$ has a components, each isomorphic to a connected Cayley graph $H_1 = \mathrm{Cayley}(\Gamma_1, B)$ of order $n_1 = |\Gamma_1| = n/a$.
Then $\hom(G, H) = a^{c(G)} \hom(G, H_1)$ is equal to the number of B-tensions of G multiplied by $n^{c(G)}$, and $a^{c(G)} n_1^{-|V(G)|} \hom(G, \widehat{H}_1)$ is equal to the number of B-flows of G. If H_1 has adjacency matrix A_1 then H has adjacency matrix $I \otimes A_1$, where I is the $a \times a$ identity matrix. Let $q_1(t) = p_{A_1}(t)/(t - |B|)$, where $p_{A_1}(t)$ is the minimum polynomial of A_1. Then $q_1(t) = q(t)$, since the eigenvalues of A_1 coincide with those of A (only the eigenspaces differ). By Corollary 9, $J = \frac{n_1}{q_1(s)} q_1(A_1)$, whence $I = \frac{n_1}{q(|B|)} q^{\circ}(\widehat{A_1})$ by Corollary 12. The result now follows in the same fashion as for the proof of Theorem 14. □

5.4 Cycle double covers

Let Δ be an additive group of m elements, and Δ^d the d-fold Cartesian product. The Hamming weight $|x|$ of a given element $x = (x_1, \ldots, x_d) \in \Delta^d$ is defined by $|x| = \#\{i : x_i \neq 0\}$. The sets $S_r^d = \{x \in \Delta^d : |x| = r\}$ are called *shells* (of radius r). The *Krawtchouk polynomial* $K_r(w; d, m)$ of degree r is defined for $0 \leq r, w \leq d$ by

$$K_r(w; d, m) = [z^r](1+(g-1)z)^{d-w}(1-z)^w = \sum_{0 \leq i \leq r} (-1)^i (m-1)^{r-i} \binom{w}{i} \binom{d-w}{r-i}.$$

In particular, $K_0(w; d, m) = 1, K_1(w; d, m) = (m-1)d - mw$ and $K_d(w; d, m) = (-1)^w (m-1)^{d-w}$. The Fourier transform of the indicator function $\delta_{S_r^d}$ of a shell is given (see e.g. [8]) by

$$\widehat{\delta_{S_r^d}}(x) = K_r(|x|; d, m).$$

When $m = 2$ ($\Delta = \mathbb{F}_2$),

$$K_r(w; d, 2) = \sum_{0 \le i \le r} (-1)^i \binom{w}{i} \binom{d-w}{r-i}.$$

Fix $m = 2$ for now.

We begin with a trivial illustration of Theorem 15.

Let $S_d^d \subseteq \mathbb{F}_2^d$ be the single-element subset comprising the all-one vector of Hamming weight d. Then $\mathrm{Cayley}(\mathbb{F}_2^d, S_2^d)$ consists of 2^{d-1} copies of K_2. Here $\widehat{\delta_{S_d^d}}(x) = K_d(|x|; d, 2) = (-1)^{|x|}$, from which we find $q(t) = t + 1$, and a contractor for S_d^d-flows is given by Theorem 15 as

$$\frac{2^d}{2^{d-1} q(1)} \left[P_1^1 + P_1^0\right] = P_1 + \overline{K}_2.$$

This, as it should be, is also a contractor for the number of $\mathbb{F}_2$-flows.

Let $F_{S_r^d}(G)$ denote the number of S_r^d-flows of G, where S_r^d is the shell of radius r in $\mathbb{F}_2^d$, i.e., the set of vectors in $\mathbb{F}_2^d$ with exactly r non-zero coordinates.

In [9] the Double Cover Conjecture is reformulated as a B-flow problem:

Problem 16. *Is it true that for every bridgeless graph there exists $d \ge 2$ so that $F_{S_2^d}(G) \neq 0$?*

Jaeger also mentions the stronger 5-colourable Cycle Double Cover Conjecture, that in fact $d = 5$ suffices, i.e., that $F_{S_2^5}(G) \neq 0$ for every bridgeless graph G.

We have just seen that a graph G has $F_{S_2^2}(G) \neq 0$ if and only if G has a nowhere-zero $\mathbb{F}_2$-flow. It is a result due originally to Tutte that $F_{S_2^3}(G) \neq 0$ if and only if G has a nowhere-zero $\mathbb{F}_4$-flow.

5.4.1 3-colourable cycle double covers

Let $S_2^3 \subseteq \mathbb{F}_2^3$. Then $H = \mathrm{Cayley}(\mathbb{F}_2^3, S_2^3)$ consists of two connected components isomorphic to K_4. (Hence there are S_2^3-tensions of G if and only if there are $\mathbb{F}_4$-tensions of G.) The adjacency matrix of H has eigenvalues $\widehat{\delta}_{S_2^3}(x) \in \{3, -1\}$, giving $q(t) = t + 1$. A contractor for S_2^3-flows is by Theorem 15 given by

$$\frac{8}{2 \cdot q(3)} [P_1^1 + P_1^0] = P_1 + \overline{K}_2.$$

Again, this is as it should be as S_2^3-flows are in one-to-one correspondence with nowhere-zero $\mathbb{F}_4$-flows (truncate the elements of $S_2^3 \subseteq \mathbb{F}_2^3$ in the last position; conversely, to a nowhere-zero $\mathbb{F}_2^2$-flow add a parity check digit to make a S_2^3-flow).

5.4.2 4-colourable cycle double covers

Let $S_2^4 \subseteq \mathbb{F}_2^4$. Then $H = \mathrm{Cayley}(\mathbb{F}_2^4, S_2^4)$ consists of two isomorphic connected components, each a regular graph of degree $\binom{4}{2} = 6$ on 8 vertices (i.e., K_8 minus a perfect matching). The adjacency matrix of H has eigenvalues $\widehat{\delta}_{S_2^4}(x) \in \{6, 0, -2\}$, giving $q(t) = t(t+2)$. A contractor for S_2^4-flows is by Theorem 15 given by

$$\frac{16}{2 \cdot q(6)}[P_1^2 + 2P_1^1] = \frac{1}{6}[C_2 + 2P_1].$$

5.4.3 d-colourable cycle double covers

We now consider the general case of d-colourable cycle double covers. In this case we have

$$\begin{aligned}
\widehat{\delta_{S_2^d}}(x) &= K_2(|x|; d, 2) \\
&= \binom{d-|x|}{2} - |x|(d-|x|) + \binom{|x|}{2} \\
&= \binom{d}{2} + 2|x|(|x|-d).
\end{aligned}$$

The adjacency matrix A of the Cayley graph $H = \mathrm{Cayley}(\mathbb{F}_2^d, S_2^d)$ has $\lceil \frac{d+1}{2} \rceil$ distinct eigenvalues, namely $\binom{d}{2} + 2w(w-d)$ for $0 \le w \le \lfloor \frac{d}{2} \rfloor$. The largest eigenvalue $\binom{d}{2} = |S_2^d|$ belongs to the eigenvector $\mathbf{1}$. In the notation of Theorem 15,

$$q(t) = \prod_{1 \le w \le \lfloor d/2 \rfloor} \left[t - \binom{d}{2} + 2w(d-w) \right],$$

and

$$q(\binom{d}{2}) = \begin{cases} 2^{d/2-1} d! & d \text{ even,} \\ 2^{(d-1)/2}(d-1)! & d \text{ odd.} \end{cases}$$

Since the elements of S_2^d generate precisely the even-weight vectors in $\mathbb{F}_2^d$, the graph $H = \mathrm{Cayley}(\mathbb{F}_2^d, S_2^d)$ has two connected components. This allows us to explicitly construct a contractor for $F_{S_2^d}(G)$, as we have done for $d \in \{2,3,4\}$ and we now do for the case $d = 5$.

5.4.4 5-colourable cycle double covers

Let A be the adjacency matrix of $\mathrm{Cayley}(\mathbb{F}_2^5, S_2^5)$, which has $a = 2$ connected components, eigenvalues $10, 2, -2$, and $q(t) = (t-2)(t+2) = t^2 - 4$, i.e., $q_0 = -4, q_1 = 0, q_2 = 1$, and $q(10) = 96$. We have $n = 2^5 = 32$ and $F_{S_2^5}(G) = 32^{-|V(G)|}\hom(G, \widehat{H})$. Hence, by Theorem 15 a contractor for $F_{S_2^5}(G)$ is given by

$$\frac{32}{2 \cdot q(10)}[P_1^2 - 4P_1^0] = \frac{1}{6}\left[C_2 - 4\overline{K}_2\right].$$

If we let e be an edge with endpoints labelled 1 and 2 and we write $G = XP_1$, then $X = G\backslash e$ and $XK_1 = G/e$. Suppose we write $G\|e$ for the operation that inserts an extra edge parallel to $e \in E(G)$. Then $XP_1^2 = G\|e$. In this notation,

$$F_{S_2^5}(G\|e) = 4F_{S_2^5}(G\backslash e) + 6F_{S_2^5}(G/e).$$

5.5 Fulkerson flows

Fulkerson's Conjecture is that in every bridgeless cubic graph G there is a family of six perfect matchings such that each edge appears in exactly two of them. Jaeger [9, Theorem 6.1] shows that Fulkerson's Conjecture is equivalent to the assertion that a bridgeless cubic graph has a S_4^6-flow, where $S_4^6 \subseteq \mathbb{F}_2^6$ comprises those vectors containing exactly four 1s. We follow [4] and call these types of flow "Fulkerson flows".

Let A be the adjacency matrix of $\mathrm{Cayley}(\mathbb{F}_2^6, S_4^6)$, which has $a = 2$ connected components, eigenvalues $15, -5, -1, 3$, and $q(t) = (t+5)(t+1)(t-3) = t^3 + 3t^2 - 13t - 15$, i.e., $q_0 = -15, q_1 = -13, q_2 = 3$, $q_3 = 1$ and $q(15) = 20 \cdot 16 \cdot 12$, $n = 2^6$. We have $F_{S_4^6}(G) = 64^{-|V(G)|}\hom(G, \widehat{H})$. Hence, by Theorem 15 a contractor for $F_{S_4^6}(G)$ is given by

$$\frac{64}{2 \cdot q(15)}[P_1^3 + 3P_1^2 - 13P_1^1 - 15P_1^0] = \frac{1}{120}\left[P_1^3 + 3C_2 - 13P_1 - 15\overline{K}_2\right],$$

where $P_1^3 = C_3^*$ consists of two vertices joined by 3 parallel edges.

After similar calculations, a contractor for $F_{S_2^6}(G)$ (the next case in the Double Cover Conjecture series of examples begun above) is given by

$$\frac{1}{90}\left[P_1^3 - C_2 - 17P_1 - 15\overline{K}_2\right].$$

5.6 Petersen flows

The Petersen Flow Conjecture of Jaeger [9] states that every bridgeless graph has a Petersen flow. As explained in e.g. [4], this conjecture implies the 5-colourable cycle double conjecture and other long-standing conjectures. Petersen flows are defined as follows.

Take $\Gamma = \mathbb{F}_2^6$. Let $C^{(1)}, \ldots, C^{(6)}$ be a basis of the cycle space of the Petersen graph $P = (V, E)$ and let $y^{(1)}, \ldots, y^{(6)}$ be the corresponding indicator vectors in $\mathbb{F}_2^E$ (where $|E| = 15$). Let $B \subseteq \mathbb{F}_2^6$ be the set of vectors $(y_e^{(1)}, \ldots, y_e^{(6)})$ for each $e \in E$. Then $|B| = 15$ since for every pair of distinct edges there is a cycle in the basis which contains exactly one of them.

The indicator function δ_B has Fourier transform

$$\begin{aligned}
&\widehat{\delta_B}(a_1 \ldots a_6) = \sum_{b_1 \ldots b_6 \in B} (-1)^{a_1 b_1 + \cdots + a_6 b_6} \\
&= \#\{e : (a_1 y^{(1)} + \cdots + a_6 y^{(6)})_e = 0\} - \#\{e : (a_1 y^{(1)} + \cdots + a_6 y^{(6)})_e = 1\} \\
&= |E| - 2|C|,
\end{aligned}$$

where C is the cycle with indicator vector $a_1 y^{(1)} + \cdots + a_6 y^{(6)}$. Since the Petersen graph has cycles of lengths $0, 5, 6, 8$ and 9, it follows that $\mathrm{Cayley}(\mathbb{F}_2^6, B)$ has eigenvalues $15, 5, 3, -1$ and -3. Thus

$$q(t) = (t+1)(t^2-9)(t-5) = t^4 - 4t^3 - 14t^2 + 36t + 45,$$

and $n = 64, q(15) = 34560$. Since B contains a set of generators for $\mathbb{F}_2^6$ the graph $\mathrm{Cayley}(\mathbb{F}_2^6, B)$ is connected. Hence a contractor for the number of B-flows is given by

$$\frac{64}{34560}\left[P_1^4 - 4P_1^3 - 14P_1^2 + 36P_1 + 45\overline{K}_2\right],$$

the fraction simplifying to $\frac{1}{540}$.

This construction of a contractor for Petersen flows extends more generally to B-flows when B is defined in a similar way via a basis for the cycle space of a graph $Q = (V, E)$, the eigenvalues of $\mathrm{Cayley}(\mathbb{F}_2^E, B)$ equal to $|E| - 2|C|$ for Eulerian subgraphs C of Q. (B-flows of G here are cycle-continuous maps from G to Q. See e.g. [4].)

5.7 Summary

In Figure 1 we display contractors for those B-flows that we have encountered so far.

In general, to find a contractor for the number of B-flows we need to find the ℓ distinct eigenvalues of the adjacency matrix of Cayley(Γ, B) ($\widehat{\delta_B}(x)$ for $x \in \Gamma$), discard the largest one ($|B|$), and then the kth elementary symmetric functions of the remaining $\ell - 1$ eigenvalues form the coefficients in the expression for a contractor as a linear combination of graphs $P_1^{\ell-1-k}$ (two vertices joined by $\ell - 1 - k$ edges).

We finish our selection of examples with one that is of interest because of its connection with the graph parameter counting Eulerian orientations.

5.8 Nowhere-zero $n, 1$-flows

Take $\Gamma = \mathbb{Z}_n$, the integers modulo n, and $B = \{-1, +1\}$. A B-flow in a graph G is an orientation of G in which the indegree is congruent to the outdegree modulo n (also known as a nowhere-zero $n, 1-$flow). When $n = 3$ we have B-flows equal to nowhere-zero flows, and a contractor is given by $\overline{K}_2 + K_2$. In case G is a 4-regular graph, B-flows of G are Eulerian orientations of G. More generally, if we take B-flows modulo n and all the vertex degrees of G belong to $\{0, 1, 2, \ldots, n-1, n+1\}$ then B-flows modulo n correspond to Eulerian orientations of G.

The indicator function of B has Fourier transform $\widehat{\delta_B}(a) = \zeta^a + \zeta^{-a}$, where $\zeta = e^{2\pi i/n}$. When $n = 4$ these eigenvalues are $2, 0, -1$ and we find that $q(t) = t(t+2)$, so that $P_1^2 + 2P_1$ is a contractor for B-flows modulo 4 (where we recall that P_1^k denotes the graph with two vertices joined by k edges). When $n = 5$ we have

$$q(t) = (t - \zeta - \zeta^{-1})(t - \zeta^2 - \zeta^{-2}) = t^2 + t - 1,$$

so $P_2 + P_1 - \overline{K}_2$ is a contractor for B-flows modulo 5.

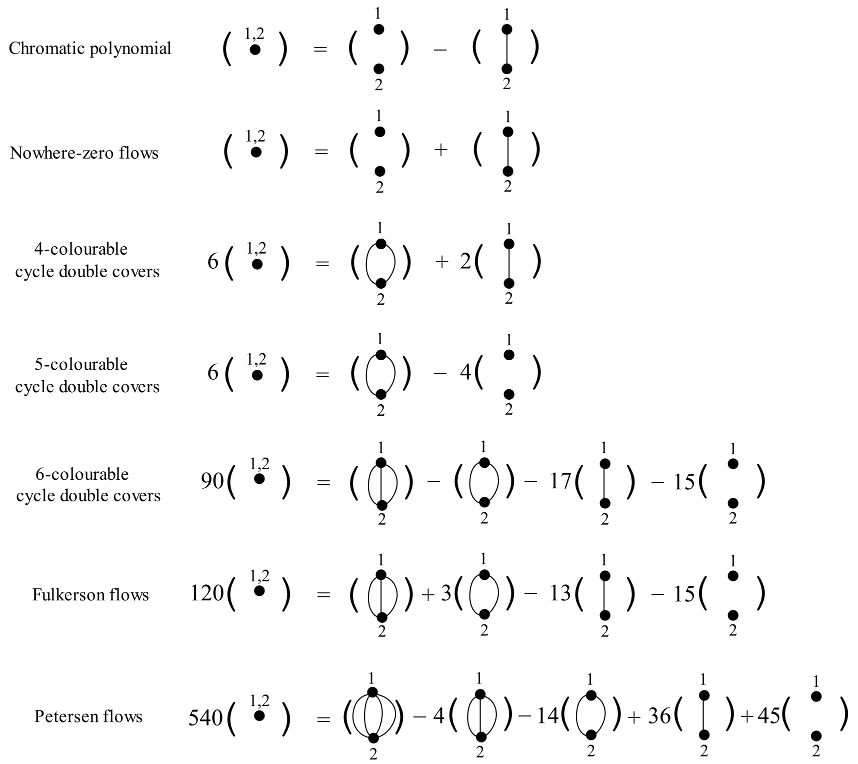
Generally then, for B-flows modulo n we have

$$q(t) = \prod_{a=1}^{\lfloor n/2 \rfloor} (t - \zeta^a - \zeta^{-a}).$$

When n is odd,

$$q(2) = \prod_{a=1}^{\frac{n-1}{2}} (1 - \zeta^a)(1 - \zeta^{-a}) = n,$$

Figure 1: Examples of contractors for a selection of graph invariants. The identities are as seen modulo the graph parameter in question. The first example is to be understood as saying that $\overline{K}_2 - K_2$ is a contractor for the chromatic polynomial, i.e., $P(XK_1; q) = P(X\overline{K}_2; q) - P(XK_2; q)$ for every graph X with a vertex labelled 1 and a vertex labelled 2, where K_1 has labels 1 and 2 on its single vertex, K_2 has one vertex label 1 and the other label 2, and similarly for $\overline{K}_2$. The graph XK_1 is the result of identifying the vertices labelled 1 and 2 in X, XK_2 adds an edge between 1 and 2, etc.



and when n is even $q(2)=2n$.

Take n odd and let $(-1)^j e_j$ denote the coefficient of $t^{\frac{n-1}{2}-j}$ in $q(t)$. Let

$$s_k=\sum_{a=1}^{\frac{n-1}{2}}(\zeta^a+\zeta^{-a})^k,$$

denote the kth power sum symmetric function in the roots of $q(t)$. By the Newton-Girard identities the elementary symemtric functions in the roots of $q(t)$ are given by

$$e_j=\sum_{k_1+2k_2+\cdots+jk_j=j}\frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{s_1}{1}\right)^{k_1}\left(\frac{s_2}{2}\right)^{k_2}\cdots\left(\frac{s_j}{j}\right)^{k_j},$$

the sum over all partitions of j with k_i parts equal to i. We find that

$$\begin{aligned}s_k &= \sum_{b=1}^{n-1}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{j}\zeta^{(k-2j)b}\\ &\begin{cases}=-\sum_{j=0}^{\frac{k-1}{2}}\binom{k}{j}=-2^{k-1} & k\text{ odd},\\ =-\sum_{j=0}^{\frac{k}{2}-1}\binom{k}{j}+\frac{n-1}{2}\binom{k}{k/2}=\frac{n}{2}\binom{k}{k/2}-2^{k-1} & k\text{ even}.\end{cases}\end{aligned}$$

So $s_1=-1, s_2=n-2, s_3=-4, s_4=3n-8, s_5=-16$ etc., giving $e_1=-1$, $e_2=-\frac{n-3}{2}$ for odd $n\geq 3$, $e_3=\frac{n-5}{2}$ for odd $n\geq 5$, $e_4=\frac{(n-5)(n-7)}{8}$ for odd $n\geq 5$. Thus for odd n the polynomial $q(t)$ begins

$$q(t)=t^{\frac{n-1}{2}}+t^{\frac{n-1}{2}-1}-\frac{n-3}{2}t^{\frac{n-1}{2}-2}-\frac{n-5}{2}t^{\frac{n-1}{2}-3}+\frac{(n-5)(n-7)}{8}t^{\frac{n-1}{2}-4}+\cdots$$

and the contractor for B-flows modulo odd n accordingly begins

$$P_1^{\frac{n-1}{2}}+P_1^{\frac{n-1}{2}-1}-\frac{n-3}{2}P_1^{\frac{n-1}{2}-2}-\frac{n-5}{2}P_1^{\frac{n-1}{2}-3}+\frac{(n-5)(n-7)}{8}P_1^{\frac{n-1}{2}-4}+\cdots$$

Is there a simple formula for each of the coefficients of $q(t)$?

In Table 1 we list contractors for $\{\pm1\}$-flows modulo n for $3\leq n\leq 9$. For graphs whose vertex degrees all belong to $\{0,1,2,\ldots,n-1,n+1\}$, the number of $\{\pm1\}$-flows is equal to the number of Eulerian orientations of G.

Table 1:

n	$q(t)$	Contractor for $\{\pm 1\}$-flows modulo n
3	$1+t$	$\overline{K}_2 + P_1$
4	$2t+t^2$	$P_1 + \frac{1}{2}P_1^2$
5	$-1+t+t^2$	$-\overline{K}_2 + P_1 + P_1^2$
6	$-2-t+2t^2+t^3$	$-\overline{K}_2 - \frac{1}{2}P_1 + P_1^2 + \frac{1}{2}P_1^3$
7	$-1-2t+t^2+t^3$	$-\overline{K}_2 - 2P_1 + P_1^2 + P_1^3$
8	$-4t-2t^2+2t^3+t^4$	$-2P_1 - P_1^2 + P_1^3 + \frac{1}{2}P_1^4$
9	$1-2t-3t^2+t^3+t^4$	$\overline{K}_2 - 2P_1 - 3P_1^2 + P_1^3 + P_1^4$

We have seen that the number of Eulerian orientations is expressible as the limit of the graph parameters counting the number of $\{\pm 1\}$-flows in $\mathbb{Z}_n$ as $n \to \infty$. Equivalently, Eulerian orientations are counted by the number of homomorphisms to the Fourier dual of the infinite graph Cayley$(\mathbb{Z}, \{\pm 1\})$, which is the graph on vertex set the real interval $[0,1]$ with edges ij weighted $2\cos 2\pi(i-j)$, equal to the Fourier transform of the edge weights on Cayley$(\mathbb{Z}, \{\pm 1\})$. (The dual group of $\mathbb{Z}$ is the circle group, which in the usual way can be identifed with the additive group of reals in $[0,1]$ modulo 1.) This is the limiting graph density for the graph parameter counting Eulerian orientations that is described in [10, 11]. (In the notation of these two papers, in terms of "graphons" the number of Eulerian orientations of G can be expressed as $t(G,W)$, where $W(x,y) = 2\cos(2\pi(x-y))$.) Is there any sense in which we can say these contractors for $\{\pm 1\}$-flows modulo n converge to a "limit contractor" for Eulerian orientations?

6 Conclusion

In this paper we provide a new and constructive proof for the existence of a contractor for B-flows. (Lovász and B. Szegedy [11] do not seem to provide an explicit construction in their proof of the existence of a series-parallel contractor for homomorphism numbers.)

Theorem 13 provides a construction of a contractor for the number of homomorphisms to an edge-weighted Cayley graph. Specifically, it gives a contractor for $\hom(\,.\,,\widehat{H})$ when H is an edge-weighted Cayley graph of a finite Abelian group Γ, and $\widehat{H}$ is the edge-weighted Cayley graph with weights given by the Fourier transform of the weights in H; Theorem 14 applies this to the special case of B-flows, where $H = \mathrm{Cayley}(\Gamma, B)$.

Let us finish this paper with a few problems.

Left open by Theorem 7, and only partially covered by Theorem 14, is the following:

Problem 17. *Given an edge-weighted connected graph H with adjacency matrix A that has* 0 *as an eigenvalue, find a contractor for* $\hom(\,\cdot\,, H)$.

By the result of Lovász and B. Szegedy, in Problem 17 it suffices to restrict the search to series-parallel graphs.

Recall that the quantum graph Z is a contractor for the graph parameter $\hom(\,\cdot\,, H)$ if $\hom_\phi(G(Z - K_1), H) = 0$ for each $G \in \mathcal{G}_2$ and $\phi : [2] \to V(G)$. This equality has (with a switch of homomorphisms from the left and from the right) a similar form to the inequality in the following problem:

$$\text{Given a } k\text{-labelled quantum graph } Y\text{, is it the case that } \hom_\phi(Y, G) \geq 0 \text{ for all } k\text{-labelled graphs } G \text{ and maps } \phi : [k] \to V(G)? \tag{10}$$

Lovász's Seventeenth Problem asks whether such a positive quantum graph Y can be expressed as the sum of squares of labelled quantum graphs (with multiplication being the disjoint union of labelled graphs followed by identification of like-labelled vertices). This has been recently answered in the negative by Hatami and Norine [7]: Problem (10) is in general undecidable. It would be interesting to know what the complexity of this problem is when Y is restricted to a subclass of quantum graphs, such as series-parallel graphs, and likewise the computational complexity of Problem 17.

Finally, we highlight a particular case of Problem 17, since it may be of interest to compare its solution to the three-term deletion–contraction–elimination recursion for the polynomial of Averbouch–Godlin–Makowsky [1].

Problem 18. *Find a contractor for the graph parameter* $\hom(G, K_{q-p}^1 + K_p^y)$*, where K_p^y denotes the complete graph on p vertices with a loop of weight y on each vertex (and similarly for K_{q-p}^1), and the $+$ denotes graph join. (This homomorphism number is equal to $\xi_G(q, y-1, (p-q)(y-1))$ in the notation of* [1]*).*

References

[1] I. Averbouch, B. Godlin and J. Makowsky, An extension of the bivariate chromatic polynomial, *European J. Combin.* **31** (2010), 1–17.

[2] N.L. Biggs, *Interaction Models*, London Mathematical Society Lecture Notes Series **30**, Cambridge Univ. Press, Cambridge, 1977.

[3] N.L. Biggs. *Algebraic Graph Theory.* Cambridge Univ. Press, 2nd ed., 1994.

[4] M. DeVos, J. Nešetřil, and A. Raspaud. On edge-maps whose inverse preserves flows or tensions. In: *Graph Theory in Paris (Proceedings of a Conference in Memory of Claude Berge)*, Trends in Mathematics, pp. 109-138. Birkhäuser, Basel, 2006.

[5] A.J. Goodall. Fourier analysis on finite Abelian groups: some graphical applications. In: G. Grimmett and C. McDiarmid, eds., *Combinatorics, Complexity, Chance: A Tribute to Dominic Welsh*, pp. 103–129. Oxford Univ. Press, Oxford, 2007.

[6] P. de la Harpe and F. Jaeger. Chromatic invariants for finite graphs: theme and polynomial variations. *Linear Algebra Appl.* **226-228** (1995), 226–228.

[7] H. Hatami and S. Norine, Undecidability of linear inequalities in graph homomorphism densities, arXiv:1005.2382v3 [math.CO], 2010.

[8] W.C. Huffman and V. Pless. *Fundamentals of Error-correcting Codes.* Cambridge Univ. Press, Cambridge, 2003.

[9] F. Jaeger, Nowhere-zero flow problems. In: L.W. Beineke and R.J. Wilson, eds., *Selected Topics in Graph Theory 3*, pp. 71–95. Academic Press, New York, 1988.

[10] L. Lovász and B. Szegedy, Limits of dense graph sequences, *J. Combin. Theory Ser. B* **96**:6 (2006), 933–957.

[11] L. Lovász and B. Szegedy, Contractors and connectors in graph algebras, *J. Graph Theory* **60**:1 (2009), 11–30.

[12] A. Schrijver, Graph invariants in the spin model, *J. Combin. Theory Ser. B* **99** (2009), 502–511.

[13] A. Terras, *Fourier analysis on finite groups and applications*, London Math. Soc. Stud. Texts **43**, Cambridge Univ. Press, Cambridge, 1999.

[14] B.L. van der Waerden, Die lange Reichweite der regelmässigen Atomanordnung in Mischkristallen, *Z. Physik*, **118** (1941), 473–488.