Some examples of universal and generic partial orders

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Abstract

We survey structures endowed with natural partial orderings and prove their universality. These partial orders include partial orders on set of words, partial orders formed by geometric objects, grammars, polynomials and homomorphism order for various combinatorial objects.

1 Introduction

For given class $\mathcal{K}$ of countable partial orders we say that class $\mathcal{K}$ contains an *embedding-universal* (or simply *universal*) structure $(U, \leq_U)$ if every partial order $(P, \leq_P) \in \mathcal{K}$ can be found as induced suborder of $(U, \leq_U)$ (or in other words, there exists embedding from $(P, \leq_P)$ to $(U, \leq_U)$).

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Partial order $(P, \leq_P)$ is *ultrahomogeneous* (or simply *homogeneous*), if every isomorphism of finite suborders of $(P, \leq_P)$ can be extended to an automorphism of $(P, \leq_P)$.

Partial order $(P, \leq_P)$ is *generic* if it is both *ultrahomogeneous* and *universal.*

The generic objects can be obtained from the Fraïssé limit [4]. But it is important that often these generic objects (despite their apparent complexity and universality) admit a concise presentation. Thus for example the Rado graph (i.e. countable universal and homogeneous undirected graph) can be represented in various ways by elemental properties of sets or finite sequences, number theory or even probability. Similar concise representations were found for some other generic objects such as all undirected ultrahomogeneous graphs [8] or the Urysohn space [6]. The study of generic partial order motivated also this paper and we consider representation of the generic partial order in Section 3.

The notion of finite presentation we interpret here broadly as a succinct representation of an infinite set. Succinct in the sense that elements are finite models with relation induced by "compatible mapping" (such as homomorphism) between the corresponding models. This intuitive definition suffices as we are interested in the (positive) examples of such representations.

A finite presentation of the generic partial order is given in [8] however this construction is quite complicated. Here we present more streamlined constructions and relate it to Conway surreal numbers (see Section 3).

In Section 2 we present several simple constructions which yield (countably) universal partial orders. Such objects are interesting on its own and were intensively studied in the context of universal algebra and categories. For example, it is a classical result of Pultr and Trnková [19] that finite graphs with the homomorphism order are countably universal quasiorder. Extending and completing [8] we give here several constructions which yields to universal partial orders. These constructions include:

1. Order $(\mathcal{W}, \leq_{\mathcal{W}})$ on sets of words in alphabet $\{0, 1\}$.

2. Dominance order on binary tree $(\mathcal{B}, \leq_{\mathcal{B}})$.

3. Inclusion order of finite sets of finite intervals $(\mathcal{I}, \leq_{\mathcal{I}})$.

4. Inclusion order of convex hulls of finite sets of points in plane $(\mathcal{C}, \leq_{\mathcal{C}})$.

5. Order of piecewise linear functions on rationals $(\mathcal{F}, \leq_{\mathcal{F}})$.

6. Inclusion order of periodic sets $(\mathcal{S}, \subseteq)$.

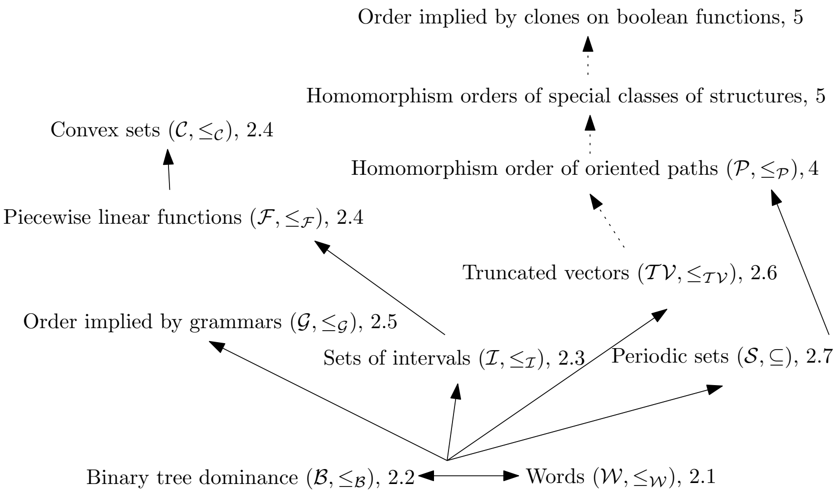
7. Order of sets of truncated vectors (generalization of orders of vectors of finite dimension) $(\mathcal{TV}, \leq_{\mathcal{TV}})$.

8. Orders implied by grammars on words $(\mathcal{G}, \leq_{\mathcal{G}})$.

9. Homomorphism order of oriented paths $(\mathcal{P}, \leq_{\mathcal{P}})$.

Note that with universal partial orders we have more freedom (than with generic one) and as a consequence we give a perhaps surprising variety of finite presentations.

We start with a simple representation by means of finite sets of binary words. This representation seems to capture properties of such universal partial order very well and it will serve as our "master" example. In most other cases we prove the universality of some particular partial order by finding a mapping from the words representation into the structure in question. This technique will be shown in several applications in the next Sections. While some of these structures are known be universal, see eg. [5, 15, 7], in several cases we can prove the universality in a new, we believe, much easier way. The embeddings of structures are presented as follows (ones denoted by dotted lines are not presented in this paper but reference is given).



At this point we would like to mention that the (countable) universality is an essentially finite problem as it can be formulated as follows: By an *on-line representation* of a class $\mathcal{K}$ of partial orders in partial order $(P, \leq_P)$, we mean that one can construct embedding $\varphi : R \to P$ of any partial order $(R, \leq_R)$ in class $\mathcal{K}$ under the circumstances that the elements of R are revealed one by one. The on-line representation of a class of partial orders can be considered as a game between two players A and B (usually Alice and Bob). Player B chooses a partial order $(P, \leq_P)$ in the class $\mathcal{K}$, and reveals the elements of P one by one to player A (B is a bad guy). Whenever an element of x of P is revealed to A, the relations among x and previously revealed elements are also revealed. Player A is required to assign vertex $\varphi(x)$ before the next element is revealed such that φ is embedding of suborder induced by $(R, \leq_R)$ on already revealed elements to $(P, \leq_P)$. Player A wins a game if he succeeds in constructing an embedding φ. The class $\mathcal{K}$ of partial orders is on-line representable in partial order $(P, \leq_P)$ if player A has a winning strategy.

On-line representation (describing winning strategy of A) is convenient way of showing universality of given partial order. In particular it transforms problem of embedding countable structures into finite problem of extending the existing partial embedding by next element.

We say that partial order $(P, \leq_P)$ has the *extension property* if the following holds: for any finite mutually disjoint subsets $L, G, U \subseteq P$ there exist vertex $v \in P$ such that $v' <_P v$ for each $v' \in L$, $v <_P v'$ for each $v' \in G$ and neither $v \leq_P v'$ nor $v' \leq_P v$ for each $v' \in U$. Extension property is stronger form of on-line representability of any partial order. Using zig-zag argument it is easy to show that partial order having extension property is homogeneous (and thus generic).

In Section 3 we show a finite representation of generic partial order related to Conway's surreal numbers. A bit surprisingly this is the only known finite presentation of the generic partial order [8]. The constructions of universal partial orders are easier, but they are often not generic. We discuss reasons why other structures fail to be homogeneous. In particular we will look for gaps in the partial order. Recall that the *gap* in partial order $(P, \leq_P)$ is a pair of elements $v, v' \in P$ such that $v <_{\mathcal{B}} v'$. Partial order having no gaps is called *dense*. We will show examples of universal partial orders both with gaps and without gaps but still failing to be generic.

2 Examples of Universal Partial Orders

To prove universality of given partially ordered set is often difficult task [5, 19, 9, 15]. The individual proofs, even if developed independently, use similar tools. We demonstrate this by isolating a "master" constructions (in Section 2.1). This constructions is then embedded into partial orders defined by other structures (as listed above). We shall see that representation of this particular order is flexible enough to simplify further embeddings.

2.1 Word representation

The set of all words over alphabet $\Sigma = \{0,1\}$ is denoted by $\{0,1\}^*$. For words W, W' we write $W \leq_w W'$ iff W' is an initial segment (left factor) of W. Thus we have, for example, $\{011000\} \leq_w \{011\}$ and $\{010111\} \not\leq_w \{011\}$.

Definition 2.1. *Denote by $\mathcal{W}$ the class of all finite subsets A of $\{0,1\}^*$ such that no distinct words W, W' in A satisfy $W \leq_w W'$. For $A, B \in \mathcal{W}$ we put $A \leq_{\mathcal{W}} B$ when for each $W \in A$ there exists $W' \in B$ such that $W \leq_w W'$.*

Obviously $(\mathcal{W}, \leq_{\mathcal{W}})$ is a partial order (antisymmetry follows from the fact that A is an antichain in the order $\leq_w$).

Definition 2.2. *For a set A of finite words denote by $\min A$ the set of all minimal words in A (i.e. we all $W \in A$ that there is no $W' \in A$ satisfying $W' <_w W$).*

Now we show an on-line embedding of any finite partial order to $(\mathcal{W}, \leq_{\mathcal{W}}$). Towards this end denote by $[n]$ set $\{1, 2, \ldots n\}$. We restrict ourselves to the partial orders whose vertex sets are sets $[n]$ (for some $n > 1$) and will always embed the vertices in natural order. For partial order $([n], \leq_P)$ denote by $([i], \leq_{P_i})$ the partial order induced by $([n], \leq_P)$ on vertices $[i]$.

Our main construction is function Ψ mapping partial orders $([n], \leq_P)$ to elements of $(\mathcal{W}, \leq_{\mathcal{W}})$ defined as follows:

Definition 2.3.

Let $L([n], \leq_P)$ be union of all $\Psi([m], \leq_{P_m})$, $m < n$, $m \leq_P n$.

Let $U([n], \leq_P)$ be the set of all words W such that W has length n, last letter is 0 and for each $m < n, n \leq_P m$ there is $W' \in \Psi([m], \leq_{P_m})$ such that W is initial segment of W'.

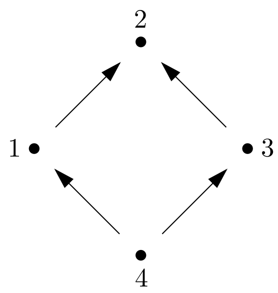


Figure 1: Partial order $([4], \leq_P)$

Finally put $\Psi([n], \leq_P)$ *as* $\min(L([n], \leq_P) \cup U([n], \leq_P))$.
In particular $L([1], \leq_P) = \emptyset, U([1], \leq_P) = \{0\}, \Psi([1], \leq_P) = \{0\}$.

Main result of this Section is the following:

Theorem 2.1. *For partial order* $([n], \leq_P)$ *we have:*

1. *For every* $i, j \in [n]$,
$$i \leq_P j \iff \Psi([i], \leq_{P_i}) \leq_{\mathcal{W}} \Psi([j], \leq_{P_j})$$
and
$$\Psi([i], \leq_{P_i}) = \Psi([j], \leq_{P_j}) \iff i = j.$$
(Or in the other words, mapping $\Phi(i) = \Psi([i], \leq_{P_i})$ *is embedding of* $([n], \leq_P)$ *into* $(\mathcal{W}, \leq_{\mathcal{W}})$*);*

2. *for every* $S \subseteq [n]$ *there is a word* W *of length* n *such that for each* $k \leq n$, $\{W\} \leq_{\mathcal{W}} \Psi([k], \leq_{P_k})$ *if and only of either* $k \in S$ *or there is* $k' \in S$ *such that* $k' \leq_P k$.

On-line embedding Φ is illustrated by following example:

Example 2.1. *Partial order* $([4], \leq_P)$ *depicted on Figure 1 has the following values of* $\Psi([k], \leq_{P_k}), k = 1, 2, 3, 4$.

$L([1], \leq_{P_1}) = \emptyset$	$U([1], \leq_{P_1}) = \{0\}$	$\Psi([1], \leq_{P_1}) = \{0\}$
$L([2], \leq_{P_2}) = \{0\}$	$U([2], \leq_{P_2}) = \{00, 10\}$	$\Psi([2], \leq_{P_2}) = \{0, 10\}$
$L([3], \leq_{P_3}) = \emptyset$	$U([3], \leq_{P_3}) = \{000, 100\}$	$\Psi([3], \leq_{P_3}) = \{000, 100\}$
$L([4], \leq_{P_4}) = \emptyset$	$U([4], \leq_{P_4}) = \{0000\}$	$\Psi([4], \leq_{P_4}) = \{0000\}$

Proof (of Theorem 2.1). We proceed by induction on n.

The Theorem obviously holds for $n = 1$.

Now assume that Theorem holds for every partial order $([i], \leq_{P_i})$, $i = 1, \dots n-1$.

We first show that 2. holds for for $([n], \leq_P)$. Fix $S \subseteq \{1, 2, \dots n\}$. Without loss of generality assume that for each $m \leq n$ such that there is $m' \in S$, $m' \leq_P m$, we also have $m \in S$ (i.e. S is closed upwards). By induction hypothesis, there is word W of length $n-1$ such that for each $n' < n$, $\{W\} \leq_{\mathcal{W}} \Psi([n'], \leq_{P_{n'}})$ if and only if $n' \in S$. Based on word W we construct word W' of length n such that $\{W'\} \leq_{\mathcal{W}} \Psi([n'], \leq_{P_{n'}})$ if and only of $n' \in S$. Consider individual cases:

1. $n \in S$
   (a) $\{W\} \leq_{\mathcal{W}} \Psi([n], \leq_P)$. Put $W' = W0$. Because $\{W'\} \leq_{\mathcal{W}} \{W\}$, W' obviously has the property.
   (b) $\{W\} \not\leq_{\mathcal{W}} \Psi([n], \leq_P)$. In this case we have for each $m < n, n \leq_P m$ also $m \in S$ and thus $\{W\} \leq_{\mathcal{W}} \Psi([m], \leq_{P_m})$. By definition of $\leq_{\mathcal{W}}$ for each such m we have $W'' \in \Psi([m], \leq_{P_m})$ such that W'' is initial segment of W. It implies that $W0$ is in $U([n], \leq_P)$ and thus $\{W\} \leq_{\mathcal{W}} \Psi([n], \leq_P)$, a contradiction.
2. $n \notin S$
   (a) $\{W\} \not\leq_{\mathcal{W}} \Psi([n], \leq_P)$. In this case we can put either $W' = W0$ or $W' = W1$.
   (b) $\{W\} \leq_{\mathcal{W}} \Psi([n], \leq_P)$. We have $\{W\} \not\leq_{\mathcal{W}} L([n], \leq_P)$, otherwise we would have $\{W\} \leq_{\mathcal{W}} \Psi([m], \leq_{P_m}) \leq_{\mathcal{W}} \Psi([n], \leq_P)$ for some $m < n$ and thus $n \in S$. Because $U([n], \leq_P)$ contains words of length n where last digit is 0, we put $W' = W1$ and we have $\{W'\} \not\leq_{\mathcal{W}} U([n], \leq_P)$ and thus also $\{W'\} \not\leq_{\mathcal{W}} \Psi([m], \leq_{P_m})$.

This finishes the proof of property 2.

Now we prove 1. We only need to verify that for $m = 1, 2, \dots n-1$ we have $\Psi([n], \leq_P) \leq_{\mathcal{W}} \Psi([m], \leq_{P_m}) \iff n \leq_P m$ and $\Psi([m], \leq_{P_m}) \leq_{\mathcal{W}} \Psi([n], \leq_P) \iff m \leq_P n$. Rest follows from the induction. Fix m and consider individual cases:

1. *$m \leq_P n$ implies $\Psi([m], \leq_{P_m}) \leq_{\mathcal{W}} \Psi([n], \leq_P)$*: This follow easily from fact that every word in $\Psi([m], \leq_{P_m})$ is in $L([n], \leq_P)$ and initial segment of each word in $L([n], \leq_P)$ is in $\Psi([n], \leq_P)$).

2. $n \leq_P m$ *implies* $\Psi([n], \leq_P) \leq_{\mathcal{W}} \Psi([m], \leq_{P_m})$: $U([n], \leq_P)$ is maximal set of words of length n with last digit 0 such that $U([n], \leq_P) \leq_{\mathcal{W}} \Psi([m'], \leq_{P_{m'}})$ for each $m' < n, n \leq_P m'$, in particular for $m' = m$. It suffices to show that $L([n], \leq_P) \leq_{\mathcal{W}} \Psi([m], \leq_{P_m})$. Consider $W \in L([n], \leq_P)$, we have some $m' \leq_P n \leq_P m$ such that $W \in \Psi([m'], \leq_{P_{m'}})$. From induction hypothesis $\Psi([m'], \leq_{P_{m'}}) \leq_{\mathcal{W}} \Psi([m], \leq_{P_m})$ and thus in particular initial segment of W is in $\Psi([m], \leq_{P_m})$.

3. $\Psi([m], \leq_{P_m}) \leq_{\mathcal{W}} \Psi([n], \leq_P)$ *implies* $m \leq_P n$: Since $U([n], \leq_P)$ contains words longer than any word of m, we have $\Psi([m], \leq_{P_m}) \leq_{\mathcal{W}} L([n], \leq_P)$. By 2. for $S = \{m\}$ there is W such that $\{W\} \leq_{\mathcal{W}} \Psi([m'], \leq_{P_{m'}})$ if and only if $m \leq_P m'$. Because $\{W\} \leq_{\mathcal{W}} L([n], \leq_P)$, we have some $m \leq_P m' \leq_P n$.

4. $\Psi([n], \leq_P) \leq_{\mathcal{W}} \Psi([m], \leq_{P_m})$ *implies* $n \leq_P m$: We have $\Psi([n], \leq_P) \leq_{\mathcal{W}} \Psi([m], \leq_{P_m})$. By 2. for $S = \{n\}$ there is W such that $\{W\} \leq_{\mathcal{W}} \Psi([m'], \leq_{P_{m'}})$ if and only if $n \leq_P m'$. Because $\{W\} \leq_{\mathcal{W}} \Psi([m], \leq_{P_m})$ we have also $n \leq_P m$.

$\square$

Corollary 2.1. *Partial order* $(\mathcal{W}, \leq_{\mathcal{W}})$ *is universal.*

Note that $\mathcal{W}$ fails to be a (ultra)-homogeneous partial order. For example the empty set is the minimal element. $\mathcal{W}$ is also not dense as shown by the following example

$$A = \{0\}, B = \{00, 01\}$$

This is not unique gap and we shall characterize all gaps in $\mathcal{W}$ after reformulating $\mathcal{W}$ in a more combinatorial setting in Section 2.2.

2.2 Dominance in the Binary tree

As it is well known the Hasse diagram of partial order $(\{0,1\}^*, \leq_w)$ forms a complete binary tree T_u of infinite depth. Let r be its root vertex (corresponding to the empty word). Using T_u we can re-formulate our universal partial order as:

Definition 2.4. *Vertices of* $(\mathcal{B}, \leq_{\mathcal{B}})$ *are finite sets* S *of vertices of* T_u *such that there is no vertex* $v \in S$ *on any path from* r *to* $v' \in S$ *except for* v'. *(Thus* S *is a finite antichain in the order of the tree* T*.)*

We say that $S' \leq_{\mathcal{B}} S$ *if and only if for each path from* r *to* $v \in S$ *there is a vertex* $v' \in S'$.

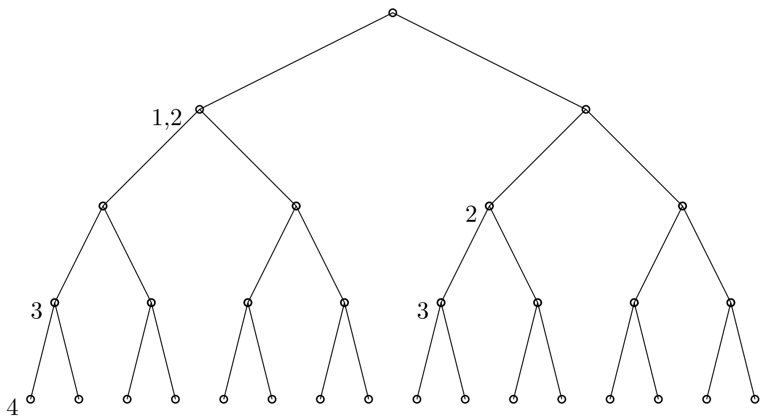


Figure 2: Tree representation of $([4], \leq_P)$ (Figure 1).

Corollary 2.2. *Partially ordered set* $(\mathcal{B}, \leq_{\mathcal{B}})$ *is universal.*

Proof. $(\mathcal{B}, \leq_{\mathcal{B}})$ is just re-formulation of $(\mathcal{W}, \leq_{\mathcal{W}})$ and thus both partial orders are isomorphic. $\square$

Figure 2 shows portion of tree T representing the same partial order as in Figure 1.

$(\mathcal{B}, \leq_{\mathcal{B}})$ brings perhaps better intuitive understanding how the universal partial order is built from very simple partial order $(\{0,1\}^*, \leq_w)$ by using sets of elements instead of single element representations. Understanding this makes it easy to find embedding of $(\mathcal{W}, \leq_{\mathcal{W}})$ (or equivalently $(\mathcal{B}, \leq_{\mathcal{B}})$) into new structure by first looking for way of representing partial order $(\{0,1\}^*, \leq_w)$ within the new structure and then way of representing sets of $\{0,1\}^*$. This idea will be exercised several times in the following Sections.

Now we characterize gaps.

Proposition 2.1. $S < S'$ *is a gap in* $(\mathcal{B}, \leq_{\mathcal{B}})$ *if and only if there exists* $s' \in S'$ *such that*

1. *there is vertex* $s \in S$ *such that both sons* s_0, s_1 *of* s *in tree* T *are in* S'.
2. $S \setminus \{s_0, s_1\} = S' \setminus \{s\}$.

This means that all gaps in $\mathcal{B}$ result from replacing a member by its twin sons.

Proof. Clearly any pair $S < S'$ satisfying 1., 2. is a gap (as any $S \leq_{\mathcal{B}} S'' \leq_{\mathcal{B}} S'$ has to contain $S' \setminus \{s\}$ and either s or both s_0, s_1).

Thus let $S \leq_{\mathcal{B}} S'$ be a gap. If there are distinct vertices s'_1 and s'_2 in S' and $s_1, s_2 \in S$ such that $s_i \leq s'_i$, i=1,2, then S'' defined as $min(S \setminus \{s_1\}) \cup \{S'_1\}$ satisfies $S <_{\mathcal{B}} S'' <_{\mathcal{B}} S'$.

Thus there is only one $S' \in S' \setminus S$ such that $s' > s$ for an $s \in S$. However then there is only one such s' (os if s_1, s_2 would be distinct then $S < S \setminus \{s_2\} < S'$). Moreover it is either $s = s'0$ or $s = s'1$ or otherwise $S < S'$ would not be a gap. □

The abundance of gaps indicates that $(\mathcal{B}, \leq_{\mathcal{B}})$ (or $(\mathcal{W}, \leq_{\mathcal{W}})$) are redundant universal partial orders. This makes them, in a way, far from being generic, since generic partial order has no gaps. In the next Section we show variant of this partial order avoiding this problem. On the other hand gaps in partial orders are interesting and are related to dualities, see [20, 18].

2.3 Geometric representation

We show that vertices of $(\mathcal{W}, \leq_{\mathcal{W}})$ can be coded by geometric objects ordered by inclusion. Since we consider only countable structures we restrict ourselves to objects in space formed by rational numbers.

While the interval on rationals ordered by inclusion can represent infinite increasing chains, decreasing chains or antichains, obviously this interval order has dimension 2 and thus fails to be universal. However considering multiple intervals overcomes this limitation:

Definition 2.5. *Vertices of $(\mathcal{I}, \leq_{\mathcal{I}})$ are finite sets S of closed disjoint intervals of form $[a, b]$ where a, b are rational numbers and $0 \leq a < b \leq 1$.*

We put $A \leq_{\mathcal{I}} B$ when every interval in A is covered by some interval of B.

In other words elements of $(\mathcal{I}, \leq_{\mathcal{I}})$ are finite sets of pairs of rational numbers. We put $A \leq_{\mathcal{I}} B$ when for every $[a, b] \in A$, there is $[a', b'] \in B$ such that $a' \leq a$ and $b \leq b'$.

Definition 2.6. *Consider any $A \in \mathcal{W}$. For each word $W \in A$. The word W on alphabet $\{0, 1\}$ is now considered as number $0 \leq n_W \leq 1$ in ternary* *expansion:*

$$n_W = \sum_{i=1}^{|W|} W_i \frac{1}{3^i}.$$

Representation of A in $\mathcal{I}$ is then set of following intervals:

$$\Phi_{\mathcal{I}}^{\mathcal{W}}(A) = \{[n_W, n_W + \frac{2}{3^{|W|+1}}]; W \in A\}$$

.

The use of ternary base might seem unnatural and indeed binary base would suffice. Main obstacle here is that embedding of $\{00, 01\}$ would be two intervals adjacent to each other overlapping in single point. This would need special care of unifying such intervals and thus we avoid such cases by using the ternary numbers.

Lemma 2.2. *$\Phi_{\mathcal{I}}^{\mathcal{W}}$ is embedding of $(\mathcal{W}, \leq_{\mathcal{W}})$ into $(\mathcal{I}, \leq_{\mathcal{I}})$.*

Proof. It is sufficient to prove that for W, W' we have interval $[n_W, n_W + \frac{1}{3^{|W|}}]$ covered by interval $[n_{W'}, n_{W'} + \frac{1}{3^{|W'|}}]$ if and only if W' is initial segment of W. This follows easily from the fact that intervals represents precisely all numbers whose ternary expansion starts with W with exception of the upper bound itself. $\square$

Example 2.2. *Representation of $([4], \leq_P)$ as defined by Figure 1 in $(\mathcal{I}, \leq_{\mathcal{I}})$ is:*

$$\begin{array}{lclcl}
\Phi_{\mathcal{I}}^{\mathcal{W}}(\Psi([1], \leq_{P_1})) & = & \Phi_{\mathcal{I}}^{\mathcal{W}}(\{0\}) & = & \{(0, \frac{2}{3^2})\} \\
\Phi_{\mathcal{I}}^{\mathcal{W}}(\Psi([2], \leq_{P_2})) & = & \Phi_{\mathcal{I}}^{\mathcal{W}}(\{0, 10\}) & = & \{(0, \frac{2}{3^2}), (\frac{1}{3}, \frac{1}{3} + \frac{2}{3^3})\} \\
\Phi_{\mathcal{I}}^{\mathcal{W}}(\Psi([3], \leq_{P_3})) & = & \Phi_{\mathcal{I}}^{\mathcal{W}}(\{000, 100\}) & = & \{(0, \frac{2}{3^4}), (\frac{1}{3}, \frac{1}{3} + \frac{2}{3^4})\} \\
\Phi_{\mathcal{I}}^{\mathcal{W}}(\Psi([4], \leq_{P_4})) & = & \Phi_{\mathcal{I}}^{\mathcal{W}}(\{0000\}) & = & \{(0, \frac{2}{3^5})\}
\end{array}$$

Corollary 2.3. *Partial order $(\mathcal{I}, \leq_{\mathcal{I}})$ is universal.*

Partial order $(\mathcal{I}, \leq_{\mathcal{I}})$ differ significantly from $(\mathcal{W}, \leq_{\mathcal{W}})$ by the following:

Proposition 2.2. *$(\mathcal{I}, \leq_{\mathcal{I}})$ has no gaps (is dense).*

Proof. Take $A, B \in \mathcal{I}$, $A <_{\mathcal{I}} B$. Because all the intervals in both A and B are closed and disjoint, there must be at least one interval I in B that is not fully covered by intervals of A (otherwise we would have $B \leq_{\mathcal{I}} A$). We may construct element C from B by shortening the interval I little bit or splitting it into two disjoint intervals in the way that $A <_{\mathcal{I}} C <_{\mathcal{I}} B$ holds. $\square$

As a consequence the presence (and abundance) of gaps in most of the universal partial orders studied is not the main obstacle when looking for representations of partial orders. It is easy to see that $(\mathcal{I}, \leq_{\mathcal{I}})$ is not generic.

By considering variant of $(\mathcal{I}, \leq_{\mathcal{I}})$ with open intervals (instead of closed) then we obtain an universal partial order $(\mathcal{I}', \leq_{\mathcal{I}'})$ with gaps. The gaps are similar to ones in $(\mathcal{B}, \leq_{\mathcal{B}})$ created by replacing interval (a, b) by two intervals (a, c) and (c, d). Half open intervals give an quasi order containing universal partial order.

2.4 Geometric representations — Convex sets

The representation as set of intervals still might be considered very artificially constructed structure. Partial orders represented by geometric objects are studied in [1]. It is shown that objects with n "degree of freedom" can not represent all $n+1$-dimensional partial orders. It follows that convex hulls in representation of generic partial order can not be defined by constant number of vertices. We show that even most simple geometric objects with unlimited "degree of freedom" represent universal partial order.

Definition 2.7. *Denote by $(\mathcal{C}, \leq_{\mathcal{C}})$ the partial order whose vertices are all convex hulls of finite sets of points in $\mathbb{Q}^2$ ordered by inclusion.*

This time we will embed $(\mathcal{I}, \leq_{\mathcal{I}})$ to $(\mathcal{C}, \leq_{\mathcal{C}})$.

Definition 2.8. *For every $A \in \mathcal{I}$ denote by $\Phi_{\mathcal{C}}^{\mathcal{I}}(A)$ convex hull generated by points:*

$$(a, a^2), (\frac{a+b}{2}, ab), (b, b^2) \text{ for every } (a, b) \in A.$$

See Figure 3 for representation of partial order from Figure 1.

Theorem 2.3. *$\Phi_{\mathcal{C}}^{\mathcal{I}}$ is embedding of $(\mathcal{I}, \leq_{\mathcal{I}})$ to $(\mathcal{C}, \leq_{\mathcal{C}})$.*

Proof. Because all points of form (x, x^2) lie on a convex parabola $y = x^2$, and points $(\frac{a+b}{2}, ab)$ are intersection of two tangents of this parabola in points (a, a^2) and (b, b^2), all points in construction of $\Phi_{\mathcal{C}}^{\mathcal{I}}(A)$ lie in convex configuration.

We have (x, x^2) in the convex hull $\Phi_{\mathcal{C}}^{\mathcal{I}}(A)$ if and only if there is $[a, b] \in A$ such that $a \leq x \leq b$. Thus for $A, B \in \mathcal{I}$ we have that $\Phi_{\mathcal{C}}^{\mathcal{I}}(A) \leq_{\mathcal{C}} \Phi_{\mathcal{C}}^{\mathcal{I}}(B)$ implies $A \leq_{\mathcal{I}} B$.

To see the other implication, observe that convex hull of (a, a^2), $(\frac{a+b}{2}, ab)$, (b, b^2) is subset of convex hull of (a', a'^2), $(\frac{a'+b'}{2}, a'b')$, (b', b'^2) for every $[a, b]$ subinterval of $[a', b']$. $\square$

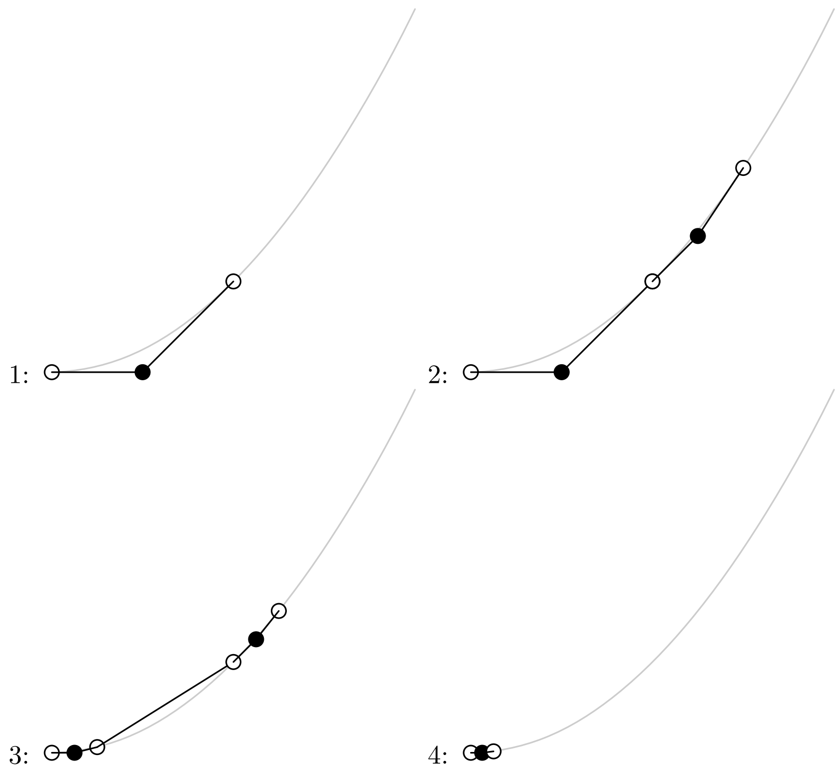


Figure 3: Representation of partial order $([4], \leq_P)$ in $(\mathcal{C}, \leq_{\mathcal{C}})$.

We have

Corollary 2.4. *Partial order* $(\mathcal{C}, \leq_{\mathcal{C}})$ *is universal*

Remark 2.1. *Our construction is related to Venn diagrams. Consider partial order* $([n], \leq_P)$*. For empty relation* $\leq_P$ *the representation constructed by* $\Phi_{\mathcal{C}}^{\mathcal{I}}(\Phi_{\mathcal{I}}^{\mathcal{W}}(\Psi([n], \emptyset)))$ *is Venn diagram as follows from Theorem 2.1* (2.)*. Statement 2. of Theorem 2.1 can be seen as Venn diagram condition under the constrains imposed by* $\leq_P$*.*

The same construction can be applied to functions so we have perhaps more precise and technical:

Corollary 2.5. *Consider class $\mathcal{F}$ of all convex piecewise linear functions on interval $(0,1)$ consisting of finite set of segments each with rational boundaries. Put $f \leq_{\mathcal{F}} g$ if and only if $f(x) \leq g(x)$ for every $0 \leq x \leq 1$. Then the partial order $(\mathcal{F}, \leq_{\mathcal{F}})$ is universal.*

Similarly following holds

Theorem 2.4. *Denote by $\mathcal{O}$ class of all finite polynomials with rational coefficients. For $p, q \in \mathcal{O}$, put $p \leq_{\mathcal{O}} q$ if and only if $p(x) \leq q(x)$ for $x \in (0,1)$. Partial order $(\mathcal{O}, \leq_{\mathcal{O}})$ is universal.*

Proof of this theorem needs more involved tools of mathematical analysis and it will appear elsewhere (jointly with Robert Šámal).

2.5 Grammars

The rewriting rules used in context free grammar can be also used to define an universal partially ordered set.

Definition 2.9. *Vertices of $(\mathcal{G}, \leq_{\mathcal{G}})$ are all words over alphabet $\{\downarrow, \uparrow, 0, 1\}$ created from word 1 by the following rules:*

$$\begin{array}{ccc} 1 & \to & \downarrow 11 \uparrow \\ 1 & \to & 0 \end{array}$$

$W \leq_{\mathcal{G}} W'$ if and only if W can be constructed from W' by:

$$\begin{array}{rcl} 1 & \to & \downarrow 11 \uparrow \\ 1 & \to & 0 \\ \downarrow 00 \uparrow & \to & 0 \end{array}$$

$(\mathcal{G}, \leq_{\mathcal{G}})$ is quasi order: transitivity of $\leq_{\mathcal{G}}$ follows from composition of lexical transformations.

Definition 2.10. *Given $A \in \mathcal{W}$ construct $\Phi_{\mathcal{G}}^{\mathcal{W}}$ as follows:*

1. $\Phi_{\mathcal{G}}^{\mathcal{W}}(\emptyset) = 0$.
2. $\Phi_{\mathcal{G}}^{\mathcal{W}}(\{\textit{empty word}\}) = 1$.
3. *$\Phi_{\mathcal{G}}^{\mathcal{W}}(A)$ is defined as concatenation $\downarrow \Phi_{\mathcal{G}}^{\mathcal{W}}(A_0)\Phi_{\mathcal{G}}^{\mathcal{W}}(A_1) \uparrow$. Where A_0 is created from all words of A starting with 0 with the first digit removed and A_1 is created from all words of A starting with 1 with first digit removed.*

Example 2.3. *Representation of* $([4], \leq_P)$ *as defined by Figure 1 in* $(\mathcal{G}, \leq_{\mathcal{G}})$ *is as follows (see also correspondence with* $\mathcal{B}$ *representation at Figure 2):*

$$\begin{array}{lcl l l}
\Phi^{\mathcal{W}}_{\mathcal{G}}(\Psi([1], \leq_{P_1})) & = & \Phi^{\mathcal{W}}_{\mathcal{G}}(\{0\}) & = & \downarrow 10 \uparrow \\
\Phi^{\mathcal{W}}_{\mathcal{G}}(\Psi([2], \leq_{P_2})) & = & \Phi^{\mathcal{W}}_{\mathcal{G}}(\{0, 10\}) & = & \downarrow 1 \downarrow 10 \uparrow\ \uparrow \\
\Phi^{\mathcal{W}}_{\mathcal{G}}(\Psi([3], \leq_{P_3})) & = & \Phi^{\mathcal{W}}_{\mathcal{G}}(\{000, 100\}) & = & \downarrow\ \downarrow\ \downarrow 10 \uparrow 0 \uparrow\ \downarrow\ \downarrow 10 \uparrow 0 \uparrow\ \uparrow \\
\Phi^{\mathcal{W}}_{\mathcal{G}}(\Psi([4], \leq_{P_4})) & = & \Phi^{\mathcal{W}}_{\mathcal{G}}(\{0000\}) & = & \downarrow\ \downarrow\ \downarrow\ \downarrow 10 \uparrow 0 \uparrow 0 \uparrow 0 \uparrow
\end{array}$$

We state without proof as it follows straightforwardly from definitions

Proposition 2.3. *For* $A, B \in \mathcal{W}$ *inequality* $A \leq_{\mathcal{W}} B$ *holds if and only if* $\Phi^{\mathcal{W}}_{\mathcal{G}}(A) \leq_{\mathcal{G}} \Phi^{\mathcal{W}}_{\mathcal{G}}(B)$.

Corollary 2.6. *Quasi order* $(\mathcal{G}, \leq_{\mathcal{G}})$ *contains universal partial order.*

2.6 Multicuts and Truncated Vectors

Universal partially ordered structure similar to $(\mathcal{W}, \leq_{\mathcal{W}})$ but less fitting for further embeddings was already studied in [5, 15, 9]. While all three structures are easily shown to be equivalent, their definition and motivations was different. [15] first used notion of on-line embeddings to prove universality of the structure and also used it as intermediate structure to prove universality of homomorphism order of multigraphs. The motivation for this structure came from analogy with Dedekind cuts and thus its members was called *multicuts*. In [9] essentially equivalent structure with inequality reversed was used as intermediate structure for stronger result showing universality of oriented paths. This time the structure arrises in the context of orders of vectors (as simple extension of orders of finite dimension represented by finite vectors of rationals) resulting in name *truncated vectors*.

We follow presentation of [9].

Definition 2.11. *Let* $\vec{v} = (v_1, \ldots, v_t)$, $\vec{v}' = (v'_1, \ldots, v'_{t'})$ *be* 0–1 *vectors. We put:*

$$\vec{v} \leq_{\vec{v}} \vec{v}' \text{ if and only if } t \geq t' \text{ and } v_i \geq v'_i \text{ for } i = 1, \ldots, t'.$$

Thus we have e.g. $(1, 0, 1, 1, 1) <_{\vec{v}} (1, 0, 0, 1)$ and $(1, 0, 0, 1) >_{\vec{v}} (1, 1, 1, 1)$. An example of infinite descending chain is e.g.

$$(1) >_{\vec{v}} (1, 1) >_{\vec{v}} (1, 1, 1) >_{\vec{v}} \ldots$$

Any finite partially ordered set is representable by vectors with this ordering: for vectors of a fixed length we have just reverse ordering used in the (Dushnik-Miller) dimension of partially ordered sets, see e.g. [20].

Definition 2.12. *We denote by $\mathcal{TV}$ the class of all finite vector-sets. Let $\vec{V}$ and $\vec{V}'$ be two finite set of* 0–1 *vectors. We put $\vec{V} \leq_{\mathcal{TV}} \vec{V}'$ if and only if for every $\vec{v} \in \vec{V}$ there exists $\vec{v}' \in \vec{V}'$ such that $\vec{v} \leq \vec{v}'$.*

For word W on alphabet $\{0, 1\}$ we construct vector $\vec{v}(W)$ of length $2|W|$ such that $2n$-th element of vector $\vec{v}(W)$ is 0 if and only if n-th character of W is 0 and $(2n + 1)$-th element of vector $\vec{v}(W)$ is 1 if and only if n-th character of W is 0.

It is easy to see that $W \leq_{\mathcal{W}} W'$ if and only if $\vec{v}(W) \leq \vec{v}(W')$. The embedding $\Phi^{\mathcal{W}}_{\mathcal{TV}} : (\mathcal{W}, \leq_{\mathcal{W}}) \to (\mathcal{TV}, \leq_{\mathcal{TV}})$ is constructed as follows:

$$\Phi^{\mathcal{W}}_{\mathcal{TV}} = \{\vec{v}(W), W \in A\}$$

For our example $([4], \leq_P)$ at Figure 1 we have embedding:

$$\begin{array}{lcl}
\Phi^{\mathcal{W}}_{\mathcal{TV}}(\Psi([1], \leq_{P_1})) & = & \Phi^{\mathcal{W}}_{\mathcal{TV}}(\{0\}) & = & \{(0, 1)\} \\
\Phi^{\mathcal{W}}_{\mathcal{TV}}(\Psi([2], \leq_{P_2})) & = & \Phi^{\mathcal{W}}_{\mathcal{TV}}(\{0, 10\}) & = & \{(0, 1), (1, 0, 0, 1)\} \\
\Phi^{\mathcal{W}}_{\mathcal{TV}}(\Psi([3], \leq_{P_3})) & = & \Phi^{\mathcal{W}}_{\mathcal{TV}}(\{000, 100\}) & = & \{(0, 1, 0, 1, 0, 1), \\
 & & & = & \ (1, 0, 0, 1, 0, 1)\} \\
\Phi^{\mathcal{W}}_{\mathcal{TV}}(\Psi([4], \leq_{P_4})) & = & \Phi^{\mathcal{W}}_{\mathcal{TV}}(\{0000\}) & = & \{(0, 1, 0, 1, 0, 1, 0, 1)\}
\end{array}$$

Corollary 2.7. *Quasi order $(\mathcal{TV}, \leq_{\mathcal{TV}})$ contains universal partial order.*

The structure $(\mathcal{TV}, \leq_{\mathcal{TV}})$ compared to $(\mathcal{W}, \leq_{\mathcal{W}})$ is more complex for further embeddings: the partial order of vectors is already complex finite-universal partial order. The reason why the structure was noticed earlier is that it allows remarkably simple on-line embedding we outline now.

Again we restrict ourselves to the partial orders whose vertex sets are sets $[n]$ (for some $n > 1$) and will always embed the vertices in natural order. Function Ψ' mapping partial orders $([n], \leq_P)$ to elements of $(\mathcal{TV}, \leq_{\mathcal{TV}})$ defined as follows:

Definition 2.13. *Let $\vec{v}([n], \leq_P) = (v_1,\ v_2,\ \ldots,\ v_n)$ where $v_m = 1$ if and only if $n \leq_{\mathcal{P}} m$, $m \leq n$, otherwise $v_m = 0$.*

Let

$$\Psi'([n], \leq_P) = \{\vec{v}([m], \leq_{P_m}); m \in P, m \leq n, m \leq_{\mathcal{P}} n\}.$$

For our example at Figure 1 we get different (and more compact) embedding:

$$\begin{array}{rcl@{\quad}rcl}
\vec{v}(1) & = & (1), & \Psi'([1], \leq_{P_1}) & = & \{(1)\} \\
\vec{v}(2) & = & (0,1), & \Psi'([2], \leq_{P_2}) & = & \{(1),(0,1)\} \\
\vec{v}(3) & = & (1,0,1), & \Psi'([3], \leq_{P_3}) & = & \{(1,0,1)\} \\
\vec{v}(4) & = & (1,1,1,1), & \Psi'([4], \leq_{P}) & = & \{(1,1,1,1)\}
\end{array}$$

Theorem 2.5. *Fix partial order $([n], \leq_P)$. For every $i, j \in [n]$,*

$$i \leq_P j \iff \Psi'([i], \leq_{P_i}) \leq_{\mathcal{TV}} \Psi'([j], \leq_{P_j})$$

and

$$\Psi'([i], \leq_{P_i}) = \Psi'([j], \leq_{P_j}) \iff i = j.$$

(Or in the other words, mapping $\Phi'(i) = \Psi'([i], \leq_{P_i})$ is embedding of $([n], \leq_P$ $)$ into $(\mathcal{TV}, \leq_{\mathcal{TV}})$);

The proof can be done via induction analogously as in second part of proof of Theorem 2.1. See [9]. Main advantage of this embedding is that size of answer is $O(n^2)$ instead of $O(2^n)$.

2.7 Periodic sets

Consider partial order induced by inclusion on sets of whole numbers. This partial order in uncountable and contains every countable partial order. We however can show perhaps surprising fact that it does include countably universal subset of all periodic subsets which has a very simple (and finite) description:

Definition 2.14. *$S \subseteq \mathbb{Z}$ is p-*periodic *if for every $x \in S$ we have also $x + p \in S$ and $x - p \in S$.*

For periodic set S with period p denote by signature $s(p, S)$ *a word at alphabet $\{0, 1\}$ of length p such that n-th letter is 1 if and only if $n \in S$.*

By $\mathcal{S}$ we denote class of all sets $S \subseteq \mathbb{Z}$ such that S is 2^n-periodic for some n.

Clearly every periodic set is determined by its signature and thus $(\mathcal{S}, \leq_{\mathcal{S}})$ is finite presentation. We consider the ordering of periodic sets by inclusion and prove:

Theorem 2.6. *Partial order $(\mathcal{S}, \subseteq)$ is universal.*

Proof. We embed $(\mathcal{W}, \leq_{\mathcal{W}})$ to $(\mathcal{S}, \subseteq)$ as follows: For $A \in \mathcal{W}$ denote by $\Phi^{\mathcal{W}}_{\mathcal{S}}(A)$ set of whole numbers such that $n \in \Phi^{\mathcal{W}}_{\mathcal{S}}(A)$ if and only if there is $W \in A$ and $|A|$ least significant digits of binary expansion of n forms a reversed word W (when binary expansion has fewer than $|W|$ digits, add 0 as needed).

It is easy to see that $\Phi^{\mathcal{W}}_{\mathcal{S}}(A)$ is 2^n-periodic, where n is the length of longest word in W and $\Phi^{\mathcal{W}}_{\mathcal{S}}(A) \subseteq \Phi^{\mathcal{W}}_{\mathcal{S}}(A')$ if and only if $A \leq_{\mathcal{W}} A'$. $\square$

$(\mathcal{S}, \subseteq)$ is dense, but it fails to have 3-extension property: there is no set strictly smaller than set with signature 01 and greater than both sets with signature 0100 and 0010.

3 Generic Poset and Conway numbers

One of the striking (and concise) incarnations of the generic Rado graph is provided by set theory: vertices of $\mathcal{R}$ are all sets in a fixed countable model $\mathfrak{M}$ of the theory of finite sets and edges correspond to pairs $\{A, B\}$ for which either $A \in B$ or $B \in A$. In [8] we aimed for similarly concise representation of generic partial order. That appeared to be a difficult task and we had to settle for a weaker notion a 'finite presentation". At present [8] is the only finite presentation of generic partial order. This is related to Conway surreal numbers [11, 2].

In this section, for the completeness, we give finite presentation of the generic partial order as shown in [8]. This construction is out of independent interest as one can give finite presentation of the rational Urysohn space along the same lines [6]. We work in a fixed countable model $\mathfrak{M}$ of theory of finite sets extended by a single atomic set $\Apple$. To represent ordered pairs (M_L, M_R), we use following notation:

$$M_L = \{A; A \in M, \Apple \notin A\};$$

$$M_R = \{A; (A \cup \{\Apple\}) \in M, \Apple \notin A\}.$$

Definition 3.1. *Define the partially ordered set $(\mathcal{P}_\in, \leq_\in)$ as follows:*
The elements of $\mathcal{P}_\in$ are all sets M with the following properties:

1. *(correctness)*
   (a) $\Apple \notin M$;
   (b) $M_L \cup M_R \subset \mathcal{P}_\in$;

(c) $M_L \cap M_R = \emptyset$.

2. *(ordering property)* $(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset$ *for each* $A \in M_L, B \in M_R$*;*

3. *(left completeness)* $A_L \subseteq M_L$ *for each* $A \in M_L$*;*

4. *(right completeness)* $B_R \subseteq M_R$ *for each* $B \in M_R$*;*

The relation of $\mathcal{P}_\in$ *is denoted by* $\leq_\in$ *and it is defined as follows: We put* $M <_\in N$ *if:*

$$(\{M\} \cup M_R) \cap (\{N\} \cup N_L) \neq \emptyset$$

We write $M \leq_\in N$ *if either* $M <_\in N$ *or* $M = N$*.*

The class $\mathcal{P}_\in$ is nonempty (as it is $M = \emptyset = (\,\emptyset \mid \emptyset\,) \in \mathcal{P}_\in$). (Obviously correctness property holds. Because $M_L = \emptyset$, $M_R = \emptyset$, ordering property and completeness properties follow trivially.)

Here are a few examples of non-empty elements of the structure $\mathcal{P}_\in$ are:

$$(\,\emptyset \mid \emptyset\,)$$
$$(\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)$$
$$(\,\{(\,\emptyset \mid \emptyset\,), (\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)\} \mid \emptyset\,)$$

It is a non-trivial fact that $(\mathcal{P}_\in, \leq_\in)$ is a partially ordered set. This will be proved after introducing some auxiliary notions:

Definition 3.2. *Any element* $W \in (A \cup A_R) \cap (B \cup B_L)$ *is called a* witness *of the inequality* $A <_\in B$*.*

The level *of* $A \in \mathcal{P}_\in$ *is defined as follows:*

$$\begin{aligned} l(\emptyset) &= 0; \\ l(A) &= max(l(B); B \in A_L \cup A_R) + 1 \text{ for } A \neq \emptyset. \end{aligned}$$

We observe the following facts (which follow directly from the definition of $\mathcal{P}_\in$):

Fact 1. $X <_\in A <_\in Y$ *for every* $A \in \mathcal{P}_\in$*,* $X \in A_L$ *and* $Y \in A_R$*.*

Fact 2. $A \leq_\in W^{AB} \leq_\in B$ *for any* $A <_\in B$ *and witness* W^{AB} *of* $A <_\in B$*.*

Fact 3. *Let* $A <_\in B$ *and let* W^{AB} *to be witness of* $A <_\in B$*. Then* $l(W^{AB}) \leq \min(l(A), l(B))$ *and either* $l(W^{AB}) < l(A)$ *or* $l(W^{AB}) < l(B)$*.*

First we prove transitivity.

Lemma 3.1. *Relation $\leq_\in$ is transitive for the class $\mathcal{P}_\in$.*

Proof. Assume that three elements A, B, C of $\mathcal{P}_\in$ satisfy $A <_\in B <_\in C$. We prove that $A <_\in C$ holds. Let W^{AB} and W^{BC} to be witnesses of the inequalities $A <_\in B$ and $B <_\in C$ respectively. First we prove that $W^{AB} \leq_\in W^{BC}$. We distinguish four cases (according to the definition of the witness):

1. $W^{AB} \in B_L$ and $W^{BC} \in B_R$.

   In this case it follows from Fact 1 that $W^{AB} <_\in W^{BC}$.

2. $W^{AB} = B$ and $W^{BC} \in B_R$.

   Then W^{BC} is witness of the inequality $B <_\in W^{BC}$ and thus $W^{AB} <_\in W^{BC}$.

3. $W^{AB} \in B_L$ and $W^{BC} = B$.

   Inequality $W^{AB} \leq_\in W^{BC}$ follows analogously to the previous case.

4. $W^{AB} = W^{BC} = B$ (and thus $W^{AB} \leq_\in W^{BC}$).

In the last case B is the witness of the inequality $A <_\in C$. Thus we may assume that $W^{AB} \neq_\in W^{BC}$. Let W^{AC} be a witness of the inequality $W^{AB} <_\in W^{BC}$. Finally we prove that W^{AC} is a witness of the inequality $A <_\in C$. We distinguish three possibilities:

1. $W^{AC} = W^{AB} = A$.
2. $W^{AC} = W^{AB}$ and $W^{AC} \in A_R$.
3. $W^{AC} \in W^{AB}_R$, then also $W^{AC} \in A_R$ from the completeness property.

It follows that either $W^{AC} = A$ or $W^{AC} \in A_R$. Analogously either $W^{AC} = C$ or $W^{AC} \in C_L$ and thus W^{AC} is the witness of inequality $A <_\in C$. $\square$

Lemma 3.2. *Relation $<_\in$ is strongly antisymmetric on the class of elements of $\mathcal{P}_\in$.*

Proof. Assume that A and B, $A <_\in B <_\in A$, is a counterexample with minimal $l(A) + l(B)$. Let W^{AB} be a witness of the inequality $A <_\in B$ and W^{BA} a witness of reverse inequality. From Fact 2 it follows that $A \leq_\in W^{AB} \leq_\in B \leq_\in W^{BA} \leq_\in A \leq_\in W^{AB}$. From the transitivity we know that $W^{AB} \leq_\in W^{BA}$ and $W^{BA} \leq_\in W^{AB}$.

Again we shall consider 4 possible cases:

1. $W^{AB} = W^{BA}$.

   From the disjointness of the sets A_L and A_R it follows that $W^{AB} = W^{BA} = A$. Analogously we obtain $W^{AB} = W^{BA} = B$ which is a contradiction.

2. Either $W^{AB} = A$ and $W^{BA} = B$ or $W^{AB} = B$ and $W^{BA} = A$.

   Then contradiction follows in both cases from the fact that $l(A) < l(B)$ and $l(B) < l(A)$ (by Fact 3).

3. $W^{AB} \neq A$, $W^{AB} \neq B$, $W^{AB} \neq W^{BA}$.

   Then $l(W^{AB}) < l(A)$ and $l(W^{AB}) < l(B)$. Additionally we have $l(W^{BA}) \leq l(A)$ and $l(W^{BA}) \leq l(B)$ and thus A and B is not the minimal counter example.

4. $W^{BA} \neq A$, $W^{BA} \neq B$, $W^{AB} \neq W^{BA}$.

   The contradiction follows symmetrically to the previous case from minimality of $l(A) + l(B)$.

□

Theorem 3.3. $(\mathcal{P}_\in, \leq_\in)$ *is partially ordered set.*

Proof. Reflexivity of the relation follows directly from the definition, transitivity and antisymmetry follows from Lemmas 3.1 and 3.2. □

Now we are ready to prove the main result of this section:

Theorem 3.4. $(\mathcal{P}_\in, \leq_\in)$ *is the universal and homogeneous partially ordered class.*

First we show the following Lemma:

Lemma 3.5. $(\mathcal{P}_\in, \leq_{\mathcal{P}_\in})$ *has the extension property.*

Proof. Let M be a finite subset of the elements of $\mathcal{P}_\in$. We want to extend the partially ordered set induced by M by the new element X. This extension can be described by three subsets of M: M_- containing elements smaller than X, M_+ containing elements greater than X and M_0 containing elements incomparable with X. Since the extended relation is a partial order we have the following properties of these sets:

I. Any element of M_- is strictly smaller than any element of M_+;

II. $B \leq_\in A$ for no $A \in M_-$, $B \in M_0$;

III. $A \leq_\in B$ for no $A \in M_+$, $B \in M_0$;

IV. M_-, M_+ and M_0 form a partition of M.

Put

$$\overline{M_-} = \bigcup_{B \in M_-} B_L \cup M_-;$$

$$\overline{M_+} = \bigcup_{B \in M_+} B_R \cup M_+.$$

We verify that the properties I., II., III., IV. still hold for sets $\overline{M_-}$, $\overline{M_+}$, M_0.

ad I. We prove that any element of $\overline{M_-}$ is strictly smaller than any element of $\overline{M_+}$:

Let $A \in \overline{M_-}, A' \in \overline{M_+}$. We prove $A <_\in A'$. By the definition of $\overline{M_-}$ there exists $B \in M_-$ such that either $A = B$ or $A \in B_L$. By the definition of $\overline{M_+}$ there exists $B' \in M_+$ such that either $A' = B'$ or $A' \in B'_R$. By the definition of $<_\in$ we have $A \leq_\in B$, $B <_\in B'$ (by I.) and $B' \leq_\in A'$ again by the definition of $<_\in$. It follows $A <_\in A'$.

ad II. We prove that $B \leq_\in A$ for no $A \in \overline{M_-}$, $B \in M_0$:

Let $A \in \overline{M_-}, B \in M_0$ and let $A' \in M_-$ satisfies either $A = A'$ or $A \in A'_L$. We know that $B \not\leq_\in A'$ and as $A \leq_\in A'$ we have also $B \not\leq_\in A$.

ad III. To prove that $A \leq_\in B$ for no $A \in \overline{M_+}$, $B \in M_0$ we can proceed similarly to ad II.

ad IV. We prove that $\overline{M_-}$, $\overline{M_+}$ and M_0 are pairwise disjoint:

$\overline{M_-} \cap \overline{M_+} = \emptyset$ follows from I. $\overline{M_-} \cap M_0 = \emptyset$ follows from II. $\overline{M_+} \cap M_0 = \emptyset$ follows from III.

It follows, that $A = (\, \overline{M_-} \mid \overline{M_+} \,)$ is an element of $\mathcal{P}_\in$ with the desired inequalities to the elements in the sets M_- and M_+.

Obviously each element of M_- is smaller than A and each element of M_+ greater than A.

It remains to be shown that each $N \in M_0$ is incomparable with A. However we run into a problem here: it is possible that $A = N$. We can avoid this problem by first considering the set:

$$M' = \bigcup_{B \in M} B_R \cup M.$$

It is then easy to show that $B = (\emptyset \mid M')$ is an element of $\mathcal{P}_\in$ strictly smaller than all elements of M.

Finally we construct the set $A' = (A_L \cup \{B\} \mid A_R)$. The set A' has the same properties with respect to the elements of the sets M_- and M_+ and differs from any set in M_0. It remains to be shown that A' is incomparable with N.

For contrary, assume for example, that $N <_\in A'$ and $W^{NA'}$ is the witness of the inequality. Then $W^{NA'} \in \overline{M_-}$ and $N \leq_\in W^{NA'}$. Recall that $N \in M_0$. From IV. above and definition of A' follows that $N <_\in W^{NA'}$. From *ad* III. above follows that there is no choice of elements such as $N <_\in W^{NA'}$, a contradiction.

The case $N >_\in A'$ is analogous. □

Proof. Proof of Theorem 3.4 follows by combining Lemma 3.5 and fact that extension property imply both universality and homogeneity of the partial order. □

3.0.1 Remark on Conway's surreal numbers

Recall the definition of surreal numbers, see [11]. (For a recent generalization see [3]). Surreal numbers are defined recursively together with their linear order. We briefly indicate how the partial order $(\mathcal{P}_\in, \leq_{\mathcal{P}_\in})$ fits to this scheme.

Definition 3.3. *A surreal number is a pair $x = \{x^L | x^R\}$, where every member of the sets x^L and x^R is a surreal number and every member of x^L is strictly smaller than every member of x^R.*

We say that a surreal number x is less than or equal to the surreal number y if and only if y is not less than or equal to any member of x^L and any member of y^R is not less than or equal to x.

We will denote the class of surreal numbers by $\mathbb{S}$.

$\mathcal{P}_\in$ may be thought as a subset of $\mathbb{S}$ (we recursively add $\acute{\emptyset}$ to express pairs x^L, x^R). The recursive definition of $A \in \mathcal{P}_\in$ leads to the following order which we define explicitly:

Definition 3.4. *For elements $A, B \in \mathcal{P}_\in$ we write $A \leq_{\mathbb{S}} B$, when there is no $l \in A_L$, $B \leq_{\mathbb{S}} l$ and no $r \in B_R$, $r \leq_{\mathbb{S}} X$.*

$\leq_{\mathbb{S}}$ is a linear order of $\mathcal{P}_\in$ and it is the restriction of Conway's order. It is in fact linear extension of partial order $(\mathcal{P}_\in, \leq_\in)$:

Theorem 3.6. *For any $A, B \in \mathcal{P}_\in$ $A <_\in B$ implies $A <_{\mathbb{S}} B$.*

Proof. We proceed by induction on $l(A) + l(B)$.

For empty A and B the theorem holds as they are not comparable by $<_\in$.

Let $A <_\in B$ and W^{AB} be the witness. In the case $W^{AB} \neq A, B$, then $A <_{\mathbb{S}} W^{AB} <_{\mathbb{S}} B$ by induction. In the case $A \in B_L$, then $A <_{\mathbb{S}} B$ from definition of $<_{\mathbb{S}}$. $\square$

4 Universality of Graph Homomorphisms

Perhaps the most natural order between finite models is induced by homomorphisms. Universality of homomorphism order at class of all finite graphs was first shown by [19].

Numerous other classes followed (see e. g. [19]) but planar graphs (and other topologically restricted classes) presented a problem. This has been resolved in [7, 9] by showing that finite oriented paths with homomorphism order are universal. In this section we give a new proof of this result. The proof is simpler and yields a stronger result (see Theorem 4.7).

Recall that an *oriented path* P of length n is any oriented graph (V, E) where $V = \{v_0, v_1, \ldots, v_n\}$ and for every $i = 1, 2, \ldots, n$ either $(v_{i-1}, v_i) \in E$ or $(v_i, v_{i-1}) \in E$ (but not both), and there are no other edges. Thus an oriented path is any orientation of an undirected path.

Denote by $\mathcal{P}$ the class of all finite paths ordered it by homomorphism order. Explicitely given paths $P = (V, E)$, $P' = (V', E')$ a *homomorphism* is a mapping $\varphi : V \to V'$ which preserves edges:

$$(x, y) \in E \implies (\varphi(x), \varphi(y)) \in E'.$$

For path P and P' we write $P \leq_{\mathcal{P}} P'$ if and only if there is homomorphism $\varphi : P \to P'$

To show universality of oriented paths, we will construct a mapping from $(\mathcal{S}, \subseteq)$ to paths. Recall that the class $\mathcal{S}$ denotes the class of all periodic subsets of $\mathbb{Z}$ (see Section 2.7). This is new feature, which gives new, more streamlined and shorter proof of the [7]. The main difference of the proof in [7, 9] and here is use of $(\mathcal{S}, \subseteq)$ as the base of representation instead of $(\mathcal{TV}, \leq_{\mathcal{TV}})$. Linear nature of graph homomorphisms among oriented paths make it very difficult to adapt many to one mapping involved in $\leq_{\mathcal{TV}}$. The cyclic mappings of $(\mathcal{S}, \subseteq)$ are easier to use.

Let us introduce more terms and notations useful when speaking of homomorphisms in between paths (We follow standard notations as e.g. in [18]).

While oriented paths do not make difference between initial and terminal vertices, we will always consider paths in a specific order of vertices from initial to terminal vertex. We denote the initial vertex v_0 and the terminal vertex v_n of P by $in(P)$ and $term(P)$. For path P we will denote by $\overleftarrow{P}$ the flipped path P with order of vertices $v_n, v_{n-1}, \ldots, v_0$. For paths P and P' we denote by PP' path created as concatenation of P and P' (i.e. disjoint union of P and P' with $term(P)$ identified with $in(P')$).

The *length* of a path P is the number of edges in P. The *algebraic length* of a path P is the number of forwarding minus the number of backwarding edges in P. Thus the algebraic length of a path could be negative. The *level* $l_P(v_i)$ of v_i is the algebraic length of the subpath $(p_0, p_1, \ldots, p_i)$ of P. By *distance* of vertex p_i and p_j, $d_P(p_i, p_j)$ we denote value $|j - i|$. By *algebraic distance,* $a_P(p_i, p_j)$ level of $l_P(v_j) - l_P(v_i)$.

Denote by $\varphi : P \to P'$ a homomorphism from path P to P'. Observe that we always have $d_P(p_i, p_j) \leq d_{P'}(\varphi(p_i), \varphi(p_j))$ and $a_P(p_i, p_j) = a_{P'}(\varphi(p_i), \varphi(p_j))$. We will construct paths in such a way that every homomorphism φ between path P and P' must map the initial vertex of P to the initial vertex of P' and thus preserve levels of vertices (see Lemma 4.1 bellow).

The basic building blocks if our construction are paths shown in Figure 4 (H stands for *head*, T for *tail*, B for *body* and S for *šipka*—arrow in Czech language). Their initial vertices appear on left, terminal vertices on right. Except for H and T the paths are balanced (i.e. their algebraic lenght is 0). We will construct paths by concatenating of copies these blocks. H will be always first path, T always last. (The dotted line in Figure 4 and Figure 5 determine vertices with level -3.)

Definition 4.1. *Given word W on alphabet $\{0, 1\}$ of length 2^n, we assign path $p(W)$ recursively as follows:*

1. $p(0) = B_0$.

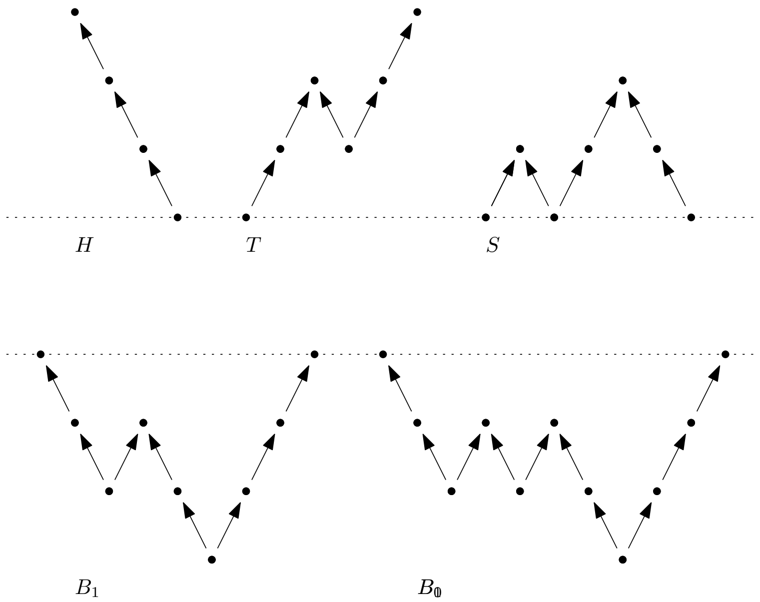


Figure 4: Building blocks of $p(W)$.

2. $p(1) = B_1$.

3. $p(W) = p(W_1)S\overleftarrow{p(W_2)}$ *where* W_1 *and* W_2 *words of length* 2^{n-1} *such that* $W = W_1W_2$.

Put $\overline{p}(W) = Hp(W)T$.

Example 4.1. *For periodic set* S*,* $s(4, S) = 0110$*, we construct* $\overline{p}(s(4, S))$ *in the following way:*

$$p(0) = B_0$$
$$p(1) = B_1$$
$$p(01) = B_0S\overleftarrow{B_1}$$
$$p(10) = B_1S\overleftarrow{B_0}$$

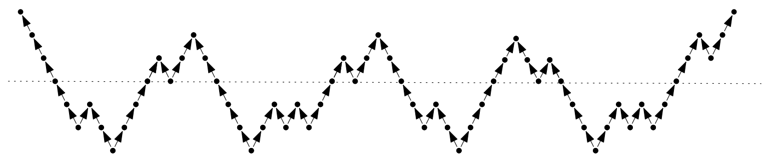


Figure 5: $\overline{p}(0110)$.

$$p(0110) = B_0 S \overleftarrow{B_1} S B_0 \overleftarrow{S}\,\overleftarrow{B_1}$$
$$\overline{p}(0110) = H B_0 S \overleftarrow{B_1} S B_0 \overleftarrow{S}\,\overleftarrow{B_1} T$$

See Figure 5.

The key result of our construction is given by the following:

Proposition 4.1. *Fix periodic set S of period 2^k and periodic set S' of period $2^{k'}$. There is a homomorphism $\varphi : \overline{p}(s(2^k, S)) \to \overline{p}(s(2^{k'}, S'))$ if and only if $S \subseteq S'$ and $k' \leq k$.*

If homomorphism φ exists, then φ maps the initial vertex of $\overline{p}(s(2^k, S))$ to the initial vertex of $\overline{p}(s(2^{k'}, S'))$. If $k' = k$ then φ maps the terminal vertex of $\overline{p}(s(2^k, S))$ to the terminal vertex of $\overline{p}(s(2^{k'}, S'))$. If $k' < k$ then φ maps the terminal vertex of $\overline{p}(s(2^k, S))$ to the initial vertex of $\overline{p}(s(2^{k'}, S'))$.

Advancing proof of the Proposition 4.1 we start with observations about homomorphisms in between our special paths.

Lemma 4.1. *Any homomorphism $\varphi : \overline{p}(W) \to \overline{p}(W')$ must map the initial vertex of $\overline{p}(W)$ to the initial vertex of $\overline{p}(W')$.*

Proof. $\overline{p}(W)$ starts by the monotone path of 5 edges. φ must map this path to monotone path in $\overline{p}(W')$. The only such subpath of $\overline{p}(W')$ is formed by first 6 vertices of $\overline{p}(W')$.

It is easy to see that φ can not flip the path: If φ maps the initial vertex of $\overline{p}(W)$ to the 6th vertex of $\overline{p}(W')$ then $\overline{p}(W)$ has vertices at level -7 and because homomorphisms must preserve algebraic distances, they must map to vertex of level 1 in $\overline{p}(W')$ and there is no such vertex in $\overline{p}(W')$. $\square$

Lemma 4.2. *Fix words W, W' of the same length 2^k. Let φ be homomorphism $\varphi : p(W) \to p(W')$. Then φ maps the initial vertex of $p(W)$ to the initial vertex of $p(W')$ if and only if φ maps the terminal vertex of $p(W)$ to the terminal vertex of $p(W')$.*

Proof. We proceed by induction on length of W:

For $W = i$ and $W' = j$, $i, j \in \{0, 1\}$ we have $p(W) = B_i$ and $p(W') = B_j$. There is no homomorphism $B_1 \to B_0$. The unique homomorphism $B_0 \to B_1$ has the desired properties and there is only isomorphism $B_0 \to B_0$.

In the induction step put $W = W_0 W_1$ and $W' = W_0' W_1'$ where W_0, W_1, W_0', W_1' are words of length 2^{k-1}. We have $p(W) = p(W_0) S \overleftarrow{p(W_1)}$ and $p(W') = p(W_0') S \overleftarrow{p(W_1')}$.

First assume that φ maps $in(p(W))$ to $in(p(W'))$. Then φ clearly maps $p(W_0)$ to $p(W_0')$ and thus by the induction hypothesis φ maps $term(p(W_0))$ to $term(p(W_0'))$. Because vertices of S are at different levels than vertices of final blocks B_0 or B_1 of $p(W_0')$, copy of S that follows in $p(W)$ after $p(W_0)$ must map to copy if S that follows in $p(W')$ after $p(W_0')$. Further φ can not flip S and thus φ maps $term(S)$ to $term(S)$. By same argument φ maps $p(W_1)$ to $p(W_1')$. Initial vertex of $p(W_1)$ is terminal vertex of $p(W)$ and it must map to initial vertex of $p(W_1')$ and thus also terminal vertex of $p(W')$.

The second possibility is that φ maps $term(p(W))$ to $term(p(W'))$. This can be handled similarly (starting from the terminal vertex of paths in the reverse order). $\square$

Lemma 4.3. *Fix periodic sets S, S' of the same period 2^k. There is a homomorphism $\varphi : p(s(2^k, S)) \to p(s(2^k, S'))$ mapping $in(p(s(2^k, S)))$ to $in(p(s(2^k, S')))$ if and only $S \subseteq S'$.*

Proof. When $S \subseteq S'$ Lemma follows from construction of $p(s(2^k, S))$. Every digit 1 of $s(2^k, S)$ has corresponding copy of B_1 in $p(s(2^k, S))$ and every digit 0 has corresponding copy of B_0 in $p(s(2^k, S))$. It is easy to build homomorphism φ by concatenating homomorphism $B_0 \to B_1$ and identical maps of S, B_0 and B_1.

In the opposite direction, assume that there is homomorphism φ from $p(s(2^k, S))$ to $p(s(2^k, S'))$. By the assumption and Lemma 4.2, φ must map $term(p(s(2^k, S)))$ to $term(p(s(2^k, S')))$. Because S use vertices at different levels than B_0 and B_1, all copies of S must mapped to copies of S. Similarly copies of B_0 and B_1 must be mapped to copies of B_0 or B_1. If $S \not\subseteq S'$ then there is position i such that i-th letter of $s(2^k, S)$ is 1 and i-th letter of $s(2^k, S')$ is 0. It follows that the copy of B_1 corresponding to this letter would have to map to a copy of B_0. This contradicts with fact that there is no homomorphism $B_1 \to B_0$. $\square$

Lemma 4.4 (folding). *For word W of length 2^k. There is homomorphism $\varphi : \overline{p}(WW) \to \overline{p}(W)$ mapping $in(\overline{p}(WW))$ to $in(\overline{p}(W))$ and $term(\overline{p}(WW))$ to $in(\overline{p}(W))$.*

Proof. By definition $\overline{p}(WW) = Hp(W)S\overleftarrow{p(W)}T$ and $\overline{p}(W) = Hp(W)T$. Homomorphism φ maps first copy of $p(W)$ in $\overline{p}(WW)$ to copy of $p(W)$ in $\overline{p}(W)$, copy of S is mapped to T such that terminal vertex of S maps to initial vertex of T and thus it is possible to map copy of $\overleftarrow{p(W)}$ in $\overline{p}(WW)$ to the same copy of $p(W)$ in $\overline{p}(W)$. □

We will use the folding Lemma iteratively. By composition of homomorphisms there is also homomorphism $p(WWWW) \to p(WW) \to p(W)$ i.e. from path constructed from 2^k copies of W to $p(W)$.

Proof (of Proposition 4.1). Assume the existence of homomorphism φ as in Proposition 4.1. First observe that $k' \leq k$ (if $k < k'$ then copy of T in $\overline{p}(s(2^k, S))$ would have to map into middle of $\overline{p}(s(2^{k'}, S'))$, but there are no vertices at the level 0 in $\overline{p}(s(2^{k'}, S'))$ except of initial and terminal vertex).

For $k = k'$ the statement follows directly from Lemma 4.3.

For $k' < k$ denote by W'' word that consist of $2^{k-k'}$ concatenations of W'. Consider homomorphism φ' from $p(W)$ to $p(W'')$ mapping $in(p(W)$ to $in(p(W''))$. W and W'' has the same length and such homomorphism exists by Lemma 4.3 if and only if $S \subseteq S'$. Applying 4.4 there is homomorphism $\varphi'' : p(W'') \to p(W')$. Homomorphism φ can be obtained by composing φ' and φ''. It is easy to see that any homomorphism $\overline{p}(W) \to \overline{p}(W')$ must follow same scheme of "folding" the longer path $\overline{p}(W)$ into $\overline{p}(W')$ and thus there is homomorphism φ if and only if $S \subseteq S'$. We omit the details. □

For periodic set S denote by $S^{(i)}$ inclusion maximal periodic subset of S with period i. (For example for $s(4, S) = 0111$ we have $s(2, S^{(2)}) = 01$)

Definition 4.2. *For $S \in \mathcal{S}$ denote i minimal integer such that S has period 2^i. Put $\Phi^{\mathcal{S}}_{\mathcal{P}}(S)$ to be concatenation of paths*

$$H,$$

$$\overline{p}(s(1, S^{(1)}))\overleftarrow{\overline{p}(s(1, S^{(1)}))},$$

$$\overline{p}(s(2, S^{(2)}))\overleftarrow{\overline{p}(s(2, S^{(2)}))},$$

$$\overline{p}(s(4, S^{(4)}))\overleftarrow{\overline{p}(s(4, S^{(4)}))},$$

$$\dots$$

$$\overline{p}(s(2^{i-1},S^{(2^{i-1})}))\overleftarrow{\overline{p}(s(2^{i-1},S^{(2^{i-1})}))},$$

$$\overline{p}(s(2^{i},S))\overleftarrow{\overline{p}(s(2^{i},S))}.$$

Theorem 4.5. *$\Phi^{\mathcal{S}}_{\mathcal{P}}(v)$ is embedding of $(\mathcal{S},\subseteq)$ to $(\mathcal{P},\leq_{\mathcal{P}})$.*

Proof. Fix S and S' in $\mathcal{S}$ of periods 2^i and $2^{i'}$.

Assume that $S\subseteq S', i>i'$. Then the homomorphism $\varphi:\Phi^{\mathcal{S}}_{\mathcal{P}}(S)\to\Phi^{\mathcal{S}}_{\mathcal{P}}(S')$ can be constructed via concatenation of homomorphisms:

$$H\to H,$$

$$\overline{p}(s(1,S^{(1)}))\overleftarrow{\overline{p}(s(1,S^{(1)}))}\to\overline{p}(s(1,S'^{(1)}))\overleftarrow{\overline{p}(s(1,S'^{(1)}))},$$

$$\overline{p}(s(1,S^{(2)}))\overleftarrow{\overline{p}(s(1,S^{(2)}))}\to\overline{p}(s(2,S'^{(2)}))\overleftarrow{\overline{p}(s(2,S'^{(2)}))},$$

$$\dots$$

$$\overline{p}(s(2^{i'-1},S^{2^{(i'-1)}})\overleftarrow{\overline{p}(s(2^{i'-1},S^{(2^{i'-1})}))}\to\overline{p}(s(2^{i'-1},S'^{(i'-1)}))\overleftarrow{\overline{p}(s(2^{i'-1},S'^{(i'-1)}))}$$

$$\overline{p}(s(2^{i'},S^{2^{(i')}})\overleftarrow{\overline{p}(s(2^{i'},S^{(2^{i'})}))}\to\overline{p}(s(2^{i'},S'))$$

$$\overline{p}(s(2^{i'+1},S^{2^{(i'+1)}})\overleftarrow{\overline{p}(s(2^{i'+1},S^{(2^{i'+1})}))}\to\overline{p}(s(2^{i'},S'))$$

$$\dots$$

$$\overline{p}(s(2^{i},S))\overleftarrow{\overline{p}(s(2^{i'},S))}\to\overline{p}(s(2^{i'},S'))$$

The homomorphisms exists by Proposition 4.1. For $i\leq i'$ the construction is even easier.

In the opposite direction assume that there is homomorphism $\varphi:\Phi^{\mathcal{S}}_{\mathcal{P}}(S)\to\Phi^{\mathcal{S}}_{\mathcal{P}}(S')$. $\Phi^{\mathcal{S}}_{\mathcal{P}}(S)$ starts by two concatenations of H and thus by long monotone path and by same argument as in Lemma 4.1, φ must map the initial vertex of $\Phi^{\mathcal{S}}_{\mathcal{P}}(S)$ to the initial vertex of $\Phi^{\mathcal{S}}_{\mathcal{P}}(S')$. It follows that φ preserves levels of vertices. It follows that for every $k=1,2,4,\dots 2^i$, φ must map $\overline{p}(s(k,S^{(k)})$ to $\overline{p}(s(k',S'^{(k')}))$ for some $k'\leq k, k'=1,2,4,\dots 2^{i'}$. By application of Proposition 4.1 it follows that $S^{(k)}\subseteq S'^{(k')}$. In particular $S\subseteq S'^{(k')}$. This holds only if $S\subseteq S'$. $\square$

Theorem 4.6 ([7])**.** *Quasi order $(\mathcal{P},\leq_{\mathcal{P}})$ contains universal partial order.*

In fact our new proof of Corollary 4.6 gives the following strengthening for rooted homomorphisms of paths. A *plank* (P, r) is a oriented path rooted at the initial vertex $r = in(P)$. Given planks (P, r) and (P', r'), a homomorphism $\varphi : (P, r) \to (P', r')$ is homomorphism $P \to P'$ such that $\varphi(r) = r'$.

Theorem 4.7. *Quasi order formed by all planks ordered by existence of homomorphisms contains universal partial order.*

5 Related results

Universality of oriented paths imply universality of homomorphism order of many naturally defined classes of structures (such as undirected planar or series-parallel graphs) ordered by homomorphism via indicator construction (see [9], [17]). By similar techniques universality of homomorphism order on labelled partial orders is shown by [12].

Lehtonen and Nešetřil [13] consider also partial order defined on boolean functions in the following way. Each clone $\mathcal{C}$ on a fixed base set A determines a quasiorder on the set of all operations on A by the following rule: f is a $\mathcal{C}$-minor of g if f can be obtained by substituting operations from $\mathcal{C}$ for the variables of g. By embedding homomorphism order on hypergraphs, it can be shown that a clone $\mathcal{C}$ on $\{0, 1\}$ has the property that the corresponding $\mathcal{C}$ minor partial order is universal if and only if $\mathcal{C}$ is one of the countably many clones of clique functions or the clone of self-dual monotone functions (using classification of Post classes).

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