The Fractional Chromatic Number of Zykov Products of Graphs

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Abstract

Zykov designed one of the oldest known family of triangle-free graphs with arbitrary high chromatic number. We determine the fractional chromatic number of the Zykov product of a family of graphs. As a corollary, we deduce that the fractional chromatic numbers of the Zykov graphs satisfy the same recurrence relation as those of the Mycielski graphs, that is $a_{n+1} = a_n + \frac{1}{a_n}$. This solves a conjecture of Jacobs.

1 Introduction

Since trees (connected acyclic graphs) have chromatic number 2, one could think that if G is a graph that locally looks like a tree (i.e. the size of its shorter circuit — its *girth* — is large), then it has low chromatic number. This was proven to be strongly false by Erdős [1] in 1959, who showed that a high chromatic number can emerge as a consequence of the global structure of a graph, as opposed to its local properties. Introducing what is now called the "deletion method", Erdős proved, by probabilistic means, the existence of graphs with arbitrary high girth and chromatic number. Yet, it took almost ten more years until Lovász [4] managed to design an explicit construction of such graphs. Another short constructive proof was given in 1979 by Nešetřil and Rödl [6]. Before Lovász's result, explicit

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constructions were only known for some fixed (small) girth. This is why several constructions of triangle-free graphs with arbitrary high chromatic number were designed in the 1950s. Among them, the most famous is arguably that of Mycielski [5], dating back to 1955. The *Mycielskian* $\mathscr{M}(G)$ of a graph G with vertex-set $\{v_1, \ldots, v_n\}$ is obtained by first replacing every vertex v_i by an independent set $\{v_i^1, v_i^2\}$, linking v_i^s and v_j^t if and only if v_i and v_j are adjacent in G and $(s,t) \neq (2,2)$. Next, a new vertex is added and linked to all the vertices $v_1^2, \ldots, v_n^2$. Notice that if G is triangle-free, then so is $\mathscr{M}(G)$; moreover, $\chi(\mathscr{M}(G)) = \chi(G) + 1$. In 1995, Larsen, Propp and Ullman [3] gave a short and elegant proof that the fractional chromatic number of $\mathscr{M}(G)$ satisfies the following unexpected formula.

$$\chi_f(\mathscr{M}(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}\,.$$

One of the earliest constructions of triangle-free graphs with arbitrary high chromatic number was found in 1949 by Zykov [8]. For each $n \geq 1$, the Zykov graph Z_n is triangle-free and has chromatic number n. Inspired by the relation for Mycielski's graphs, Jacobs [2] conjectured that the fractional chromatic numbers of the Zykov graphs satisfy the same recurrence relation as the Mycielski graphs.

Conjecture. *For every $n \geq 2$,*

$$\chi_f(Z_{n+1}) = \chi_f(Z_n) + \frac{1}{\chi_f(Z_n)}\,.$$

In this article, we prove Conjecture 1 by proving a more general result on a product of graphs. Let $G_1, \ldots, G_n$ be finite graphs. The *Zykov product* $\mathcal{Z}(G_1, \ldots, G_n)$ of $G_1, \ldots, G_n$ is defined as follows.

- Make a disjoint union of all the graphs G_i.
- For each possible choice of $(x_1, x_2, \ldots, x_n) \in V(G_1) \times V(G_2) \times \ldots \times V(G_n)$, add a new vertex x with neighborhood precisely $\{x_1, \ldots, x_n\}$.

Thus, $\mathcal{Z}(G_1, \ldots, G_n)$ has $\sum_{i=1}^{n} |V(G_i)| + \prod_{i=1}^{n} |V(G_i)|$ vertices. Notice also that the order in which the graphs G_i are numbered makes no difference in the construction.

It is straightforward to see that if $G_1, \ldots, G_n$ are all triangle-free, then so is $\mathcal{Z}(G_1, \ldots, G_n)$. The *Zykov graphs* is the sequence of graphs $(Z_n)_{n\geq 1}$

defined by $Z_1 := K_1$ and $Z_{n+1} := \mathscr{Z}(Z_1, \ldots, Z_n)$ for $n > 1$. Similarly to Mycielski's graphs, one can check that the chromatic number of Z_i is i.

We establish the following result, which implies Conjecture 1.

Theorem 1. *Let $G_1, \ldots, G_n$ be finite graphs, and set $\chi_i := \chi_f(G_i)$. Suppose also that the G_i are numbered such that $\chi_i \le \chi_{i+1}$. Then*

$$\chi_f(\mathscr{Z}(G_1, \ldots, G_n)) = \max\left(\chi_n\,,\ 2 + \sum_{i=2}^{n} \prod_{k \ge i} \left(1 - \frac{1}{\chi_i}\right)\right). \tag{1}$$

Before proving Theorem 1, let us see how it implies that Conjecture 1 is true.

Proof of Conjecture 1. For $n \ge 1$, set $f(n) := 2 + \sum_{i=2}^{n} \prod_{k \ge i} \left(1 - \frac{1}{\chi_i}\right)$. Observe that $f(n) = 2 + \left(1 - \frac{1}{\chi_n}\right) \cdot (f(n-1) - 1)$ for $n \ge 2$. We now proceed by induction on $n \ge 1$. First, notice that $\chi_2 = 2 = f(1) > \chi_1 = 1$. Now, assume that $\chi_n = f(n-1)$ for some $n \ge 2$. Then,

$$\begin{aligned} f(n) &= 2 + \left(1 - \frac{1}{\chi_n}\right) \cdot (f(n-1) - 1) \\ &= \chi_n + \frac{1}{\chi_n}\,. \end{aligned}$$

Thus, $\chi_{n+1} = f(n)$ by Theorem 1, and the conclusion follows. ∎

We define the basic concepts in the next section, and then proceed with the proof of Theorem 1.

2 Notation

If G is a graph, then $V(G)$ is its vertex-set and $E(G)$ is its edge-set. Let $\mathscr{I}(G)$ be the collection of all independent sets of the graph G. A *weighting* of a set $\mathscr{X} \subseteq \mathscr{I}(G)$ is a function $w : \mathscr{X} \to \mathbf{R}^+$. If $v \in V(G)$, then

$$w[v] := \sum_{\substack{I \in \mathscr{X} \\ v \in I}} w(I)\,.$$

A *fractional k-coloring* of G is a weighting of $\mathscr{I}(G)$ such that

- $\sum_{S\in\mathscr{I}(G)} w(S) = k$; and
- $w[v] \geq 1$ for every $v \in V(G)$.

The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of all positive real numbers k for which G has a fractional k-total coloring. In other words, the fractional chromatic number of G is the optimal value of the following linear program.

$$\text{Minimize} \sum_{S\in\mathscr{I}(G)} w(S) \quad \text{where } w \text{ is a weighting of } \mathscr{I}(G) \text{ satisfying}$$
$$\forall v \in V(G),\ w[v] \geq 1\,.$$

As is well-known, the fractional total chromatic number of a finite graph is always a rational number and the infimum is actually a minimum. Observe that for every graph G, there exists a fractional $\chi_f(G)$-coloring w of of G such that $w[v] = 1$ for every $v \in V(G)$. There are other equivalent definitions of a fractional coloring of a graph, and we refer to the book by Scheinerman and Ullman [7] for further exposition about fractional colorings (and, more generally, fractional graph theory).

3 Proof of Theorem 1: Lower Bound

We use the notations from Theorem 1. Further, we set $f(n) := 2 + \sum_{i=2}^{n} \prod_{k\geq i} \left(1 - \frac{1}{\chi_i}\right)$ for $n \geq 1$.

First, $\chi_f(G) \geq \chi_n = \chi_f(G_n)$ since G_n is a subgraph of $G = \mathcal{Z}(G_1, \ldots, G_n)$. So, we focus on proving that $\chi_f(G) \geq f(n)$. We assume that $n \geq 2$, the result being trivial if $n = 1$ (since G contains at least an edge). We start with the following observation.

Lemma 1. *Let G be a graph and w a weighting of $\mathscr{X} \subseteq \mathscr{I}(G)$. Then, for every induced subgraph H of G, there exists $x \in V(H)$ such that*

$$w[x] \leq \frac{1}{\chi_f(H)} \sum_{S\in\mathscr{X}} w(S)\,.$$

Proof. Let w_H be the weighting of $\mathscr{I}(H)$ defined by $w_H(I) := \sum_{\substack{S\in\mathscr{X}\\ S\cap V(H)=I}} w(S)$. Note that $w_H(\emptyset) = \sum_{\substack{S\in\mathscr{X}\\ S\cap V(H)=\emptyset}} w(S)$. Moreover, $w_H[v] = w[v]$ for every $v \in V(H)$. Set $m := \min_{v\in V(H)} w_H[v]$. It suffices to show that $m \leq \frac{1}{\chi_f(H)} \sum_{S\in\mathscr{X}} w(S)$.

The function $w' := \frac{1}{m} \cdot w_H$ is a fractional k-coloring of H with $k := \frac{1}{m} \sum_{I \in \mathscr{I}(H)} w_H(I)$. Thus, $k \geq \chi_f(H)$. Further,

$$\sum_{I \in \mathscr{I}(H)} w_H(I) = \sum_{S \in \mathscr{X}} w(S)$$

by the definition of w_H, and hence the conclusion follows. ∎

Let w be a fractional $\chi_f(G)$-coloring of G and let $x_1 \in V(G_1)$. Set

$$\mathcal{F}_1 := \{S \in \mathscr{I}(G) : \ x_1 \in S\}.$$

By the definition,

$$\sum_{S \in \mathcal{F}_1} w(S) = w[x] \geq 1.$$

Applying Lemma 1 with $H := G_2$ and $\mathscr{X} := \mathcal{F}_1$, we deduce that there exists $x_2 \in V_2$ such that

$$\sum_{\substack{S \in \mathcal{F}_1 \\ x_2 \in S}} w(S) \leq \frac{1}{\chi_2} \sum_{S \in \mathcal{F}_1} w(S),$$

and hence

$$\sum_{\substack{S \notin \mathcal{F}_1 \\ x_2 \in S}} w(S) \geq 1 - \frac{1}{\chi_2} \sum_{S \in \mathcal{F}_1} w(S).$$

Thus, setting

$$\mathcal{F}_2 := \{S \in \mathscr{I}(G) : \ S \cap \{x_1, x_2\} \neq \emptyset\},$$

it follows that

$$\sum_{S \in \mathcal{F}_2} w(S) \geq 1 + \left(1 - \frac{1}{\chi_2}\right) \sum_{S \in \mathcal{F}_1} w(S).$$

In a recursive way and by the exact same argument, we can construct $x_i \in V_i$ and

$$\mathcal{F}_i = \{S \in \mathscr{I}(G) : \ S \cap \{x_1, \ldots, x_i\} \neq \emptyset\}$$

for $i \leq n$, such that

$$\sum_{S \in \mathcal{F}_k} w(S) \geq 1 + \left(1 - \frac{1}{\chi_k}\right) \sum_{S \in F_{k-1}} w(S).$$

Thus, we deduce that

$$\sum_{S\in\mathcal{F}_n} w(S) \geq 1 + \sum_{i=2}^{n} \prod_{k\geq i} \left(1 - \frac{1}{\chi_i}\right) = f(n) - 1\,.$$

Now, consider the vertex $x \in V(G)$ the neighborhood of which is precisely $\{x_1, \ldots, x_n\}$. An independent set of G that contains x cannot be in $\mathcal{F}_n$. Since $w[x] \geq 1$, we infer that

$$\sum_{I\in\mathscr{I}(G)} w(I) \geq \sum_{I\in\mathcal{F}_n} w(I) + w[x] \geq f(n)\,.$$

Hence, $\chi_f(G) \geq f(n)$, as wanted.

4 Proof of Theorem 1 : Upper Bound

Again, we follow the notation of Theorem 1 and let $f(n) :=$ $2 + \sum_{i=2}^{n} \prod_{k\geq i} \left(1 - \frac{1}{\chi_i}\right)$. For convenience, we set $V_i := V(G_i)$ for $i \in \{1, 2, \ldots, n\}$. Further, let V_0 be the vertices of G not in $\cup_{i=1}^{n} V_i$. Recall that V_0 is an independent set and there are no edges between V_i and V_j if $i \neq j$ and $i, j \in \{1, \ldots, n\}$. Therefore, every maximal independent set S of G is determined by its intersection with the sets V_i for $i \in \{1, \ldots, n\}$, since $S \cap V_0$ is then composed of the vertices of V_0 with no neighbors in any set $S \cap V_i$ for $i \in \{1, \ldots, n\}$.

To prove the upper bound, we define a weighting w of $\mathscr{I}(G)$ of total weight $\max(\chi_n, f(n))$. To this end, only a subfamily of the maximal stable sets of G will be assigned a positive weight by w.

For $i = \{1, \ldots, n\}$ we define the collection $\mathcal{F}_i$ of stable sets of G by

$$\mathcal{F}_i := \{S \in \mathscr{I}(G) : \quad \forall\, j \in \{1, \ldots, n\}, \quad S \cap V_j = \emptyset \text{ if and only if } j < i\}\,.$$

Let

$$\mathcal{F} := \bigcup_{i=1}^{n} \mathcal{F}_i\,.$$

We first define a weighting p of $\mathcal{F}$ as a product of weightings of each graph G_i. For each $i \in \{1, 2, \ldots, n\}$, set $\mathscr{I}_i := \mathscr{I}(G_i)$ and let w_i be a fractional χ_i-coloring of G_i such that $w_i[v] = 1$ for every $v \in V_i$. (Note that

$w_i(\emptyset) = 0$.) We define $p_i : \mathscr{I}_i \to \mathbf{R}^+$ by $p_i(S) := c_i(S)/\chi_i$ if $S \neq \emptyset$ and $p_i(\emptyset) := 1$. Thus,

$$\sum_{S \in \mathscr{I}_i \setminus \{\emptyset\}} p_i(S) = 1$$

and

$$\forall x \in V_i, \quad p_i[x] = \frac{1}{\chi_i}\,.$$

We now define p as

$$\begin{array}{rccl} p: & \mathcal{F} & \longrightarrow & \mathbb{R}_+ \\ & S & \longmapsto & \displaystyle\prod_{i=1}^{n} p_i(S \cap V_i)\,. \end{array}$$

The next lemma states some useful properties of p.

Lemma 2. *Let $i, j \in \{1, \dots, n\}$. The weighting p satisfies the following.*

(i)

$$\sum_{S \in \mathcal{F}_i} p(S) = 1\,.$$

(ii) For each $x \in V_j$, if $i \leq j$ then

$$\sum_{\substack{S \in \mathcal{F}_i \\ x \in S}} p(S) = \frac{1}{\chi_j}\,.$$

If $i > j$, this sum is equal to zero since none of the elements of $\mathcal{F}_i$ intersects V_j.

(iii) For each $(x_1, x_2, \dots, x_n) \in V_1 \times V_2 \times, \dots \times V_n$,

$$\sum_{\substack{S \in \mathcal{F}_i \\ S \cap \{x_1, x_2, \dots, x_n\} = \emptyset}} p(S) = \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right)\,.$$

Proof of Lemma 2. First,

$$\begin{aligned} \sum_{S \in \mathcal{F}_i} p(S) &= \sum_{S \in \mathcal{F}_i} \prod_{k \geq i} p_k(S \cap V_i) \\ &= \prod_{k \geq i} \sum_{S \in \mathscr{I}_k \setminus \emptyset} p_k(S) \\ &= 1\,, \end{aligned}$$

which proves (i).

Similarly, if $x \in V_j$ and $i \leq j$, then

$$\begin{aligned}
\sum_{\substack{S\in\mathcal{F}_i\\ x\in S}} p(S) &= \sum_{\substack{S\in\mathcal{F}_i\\ x\in S}} \prod_{k\geq i} p_k(S\cap V_i)\\
&= \left(\sum_{\substack{S\in\mathscr{I}_j\\ x\in S}} p_j(S)\right)\left(\prod_{\substack{k\geq i\\ k\neq j}}\sum_{S\in\mathscr{I}_k\setminus\emptyset} p_k(S)\right)\\
&= \frac{1}{\chi_j}\,.
\end{aligned}$$

Further, if $i > j$ then no element of $\mathcal{F}_i$ intersects V_j, and hence (ii) holds. We omit the proof of (iii), which can be established similarly by again switching the sum and the product signs and using (ii). ■

We are ready to define our final weighting w of $\mathscr{I}(G)$. For convenience, set $\chi_0 := 0$. For every $S \in \mathcal{F}$, we define $w(S)$ to be $(\chi_i - \chi_{i-1}) \cdot p(S)$ where $i \in \{1, \ldots, n\}$ such that $S \in \mathcal{F}_i$. Further, we set $w(V_0) := \max(0, f(n) - \chi_n)$; all the other independent sets are assigned weight 0 by w. Recall that the graphs G_i are ordered such that $\chi_i \geq \chi_{i-1}$.

Lemma 2(i) implies that

$$\begin{aligned}
\sum_{S\in\mathscr{I}(G)} w(S) &= \sum_{S\in\mathcal{F}} w(S) + w(V_0)\\
&= \sum_{i=1}^{n}(\chi_i - \chi_{i-1}) \cdot \sum_{S\in\mathcal{F}_i} p(S) + \max(0, f(n) - \chi_n)\\
&= \chi_n + \max(0, f(n) - \chi_n)\\
&= \max(\chi_n, f(n))\,.
\end{aligned}$$

By Lemma 2(ii), for each $x \in V_j$

$$\begin{aligned} w[x] &= \sum_{i=1}^{j} (\chi_i - \chi_{i-1}) \cdot \sum_{\substack{S \in F_i \\ x \in S}} p(S) \\ &= \frac{1}{\chi_j} \cdot \sum_{i=1}^{j} (\chi_i - \chi_{i-1}) \\ &= 1\,. \end{aligned}$$

It remains to show that $w[x] \geq 1$ if $x \in V_0$. Let $x \in V_0$, and let $(x_1, x_2, \ldots, x_n) \in V_1 \times V_2 \times \ldots \times V_n$ be the n-uple of its neighbors in G. Then, using Lemma 2(iii),

$$\begin{aligned} w[x] &= w(V_0) + \sum_{\substack{S \in \mathcal{F} \\ S \cap \{x_1, x_2, \ldots, x_n\} = \emptyset}} w(S) = w(V_0) + \sum_{i=1}^{n} \sum_{\substack{S \in \mathcal{F}_i \\ S \cap \{x_1, \ldots, x_n\} = \emptyset}} w(S) \\ &= w(V_0) + \sum_{i=1}^{n} \left((\chi_i - \chi_{i-1}) \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) \right) \\ &= w(V_0) + \sum_{i=1}^{n} \left(\chi_i \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) \right) \\ &\quad - \sum_{i=1}^{n} \left((\chi_{i-1} - 1) \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) \right) - \sum_{i=1}^{n} \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) \\ &= w(V_0) + \sum_{i=1}^{n} \left((\chi_i - 1) \prod_{k > i} \left(1 - \frac{1}{\chi_k}\right) \right) \\ &\quad - \sum_{i=1}^{n} \left((\chi_{i-1} - 1) \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) \right) - \sum_{i=1}^{n} \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) \\ &= w(V_0) + \chi_n - 1 - \sum_{i=2}^{n} \prod_{k \geq i} \left(1 - \frac{1}{\chi_i}\right) \\ &= \max(0, f(n) - \chi_n) + \chi_n + 1 - f(n) \\ &\geq 1\,. \end{aligned}$$

Hence, w is a fractional $\max(\chi_n, f(n))$-coloring of G, which concludes the proof. ■

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