Homomorphism and embedding universal structures for restricted classes

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Abstract

This paper unifies problems and results related to (embedding) universal and homomorphism universal structures. On the one side we give a new combinatorial proof of the existence of universal objects for homomorphism defined classes of structures (thus reproving a result of [6]) and on the other side this leads to the new proof of the existence of dual objects (established by [32]). Our explicite approach has further applications to special structures such as variants of the rational Urysohn space. We also solve a related extremal problem which shows the optimality (of the used lifted arities) of our construction. Our method also relates to weakly indivisible homomorphism defined classes of structures.

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1 Introduction

It is an old mathematical idea to reduce a study of a particular class of objects to a certain single "universal" object. It is hoped that this object may then be used to study the given (infinite) set of individual problems in a more systematic and perhaps even efficient way. For example the universal object may have interesting additional properties (such as symmetries and ultrahomogeneity) which in turn can be used to classify the finite problems. A good example of this is the classification of Ramsey classes of structures via the classification program of generic structures [25, 26, 17, 18]. In this paper we deal with universal objects from the homomorphism point of view. We define and study homomorphism-universal (shortly hom-universal [29]) objects and show their relationship to embedding-universal (shortly universal) objects.

Given a class $\mathcal{K}$ of countable structures, an object $\mathbf{U} \in \mathcal{K}$ is called *hom-universal* (or *universal*) if for every object $\mathbf{A} \in \mathcal{K}$ there exists a homomorphism (or an embedding) $\mathbf{A} \to \mathbf{U}$.

The characterization of those classes $\mathcal{K}$ which have an universal (or, more recently, hom-universal) object is a well known open problem which was studied intensively, see e.g. [20, 19, 7, 29, 32]. The whole area was inspired by the negative results (see [14, 10]): for example the class of graphs not containing C_l (=cycle of length l) fails to be universal for any $l > 3$. For universal structures the strongest results in the positive direction were obtained by Cherlin, Shelah and Shi in [6]. Particularly, they proved the following (as consequence of Theorem 4 [6]):

Theorem 1.1 *For every finite set $\mathbf{F}_1, \ldots, \mathbf{F}_t$ of finite connected graphs the class $Forb_h(\mathbf{F}_1, \ldots, \mathbf{F}_t)$ has an universal object.*

Here $Forb_h(\mathbf{F}_1, \ldots, \mathbf{F}_t)$ denotes the class of all graphs (and more generally relational structures) $\mathbf{G}$ for which there is no homomorphism $\mathbf{F}_i \to \mathbf{G}$ for every $i = 1, \ldots, t$. Formally $Forb_h(\mathbf{F}_1, \ldots, \mathbf{F}_t) = \{\mathbf{G}; \mathbf{F}_i \nrightarrow \mathbf{G} \text{ for } i = 1, 2, \ldots, t\}$. (In this paper the structures are denoted by bold letters. This applies also to graphs. But a cycle of length l is still denoted by C_l and complette graph on k vertices by K_k.) Similarly by $Forb_e(\mathbf{F}_1, \ldots, \mathbf{F}_t)$ we denote the class of all graphs (and more generally relational structures) $\mathbf{G}$ for which there is no embedding $\mathbf{F}_i \to \mathbf{G}$ for every $i = 1, \ldots, t$. (Note that the [6] proves a stronger result: The universal graph exists for any class with finite algebraic closure and, moreover, that the existence of ω-categorical universal graph is equivalent to this condition. This result was extended to relational structures in [8]. Our motivation comes from study of graph homomorphisms and thus in this paper we are interested in this weaker version of their result as presented by Theorem 1.1.)

The proof of Theorem 1.1 given in [6] is based on techniques of model theory and it is possible to say that no explicit universal object is constructed "in any very explicite way" ([6]). In this paper we give such a proof.

Homomorphism universal structures lead to a very different area of Constraint Satisfaction Problems. Clearly every universal object is also hom-universal. This however does not hold conversely as shown by examples of classes all with of graphs bounded chromatic numbers. Another example is provided by the class of all planar graphs: K_4 is hom-universal for the class of planar graphs by virtue of the 4-color theorem while no universal graph exists [14]. Of special interest are classes $Forb_h(\mathbf{F}_1, \ldots, \mathbf{F}_t)$ which have finite hom-universal graph. Such universal graphs are called *duals* and in this context the classes with finite hom-universal objects were characterized [32] as follows (for general finite relational structures)

Theorem 1.2 *$Forb_h(\mathbf{F}_1, \ldots, \mathbf{F}_t)$ has a finite hom-universal object (i.e. dual) if and only if all structures $\mathbf{F}_i$ are (relational) trees.*

In this paper we give a new construction of duals (Corollary 5.1).

It is the aim of this paper to investigate both universal and hom-universal objects in the context of a (seemingly much more restricted) class of generic objects. Main result (Theorem 1.3) proves Theorem 1.1 by means of *shadows (reducts)* and *lifts (expansions)* of a generic (i.e. ultrahomogeneous universal) structure. These classical model theoretic notions will be, for the sake of completeness, briefly reviewed now (see e.g. [16]):

A *structure* $\mathbf{A}$ is a pair $(A, (R^i_{\mathbf{A}}; i \in I))$ where $R^i_{\mathbf{A}} \subseteq A^{\delta_i}$ (i.e. $R^i_{\mathbf{A}}$ is a δ_i-ary relation on A). The finite family $(\delta_i; i \in I)$ is called the *type* Δ. The type is usually fixed and understood from the context. (Note that we consider relational structures only, and no function symbols.) If set A is finite we call $\mathbf{A}$ *finite structure*. A *homomorphism* $f : \mathbf{A} \to \mathbf{B} = (B, (R^i_{\mathbf{B}}; i \in I))$ is a mapping $f : A \to B$ satisfying for every $(x_1, x_2, \ldots, x_{\delta_i}) \in R^i_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_{\delta_i})) \in R^i_{\mathbf{B}}$, $i \in I$. If f is 1-1, then f is called an *embedding*. The class of all (countable) relational structures of type Δ will be denoted by $Rel(\Delta)$.

The class $Rel(\Delta)$, $\Delta = (\delta_i; i \in I)$, I finite, is fixed throughout this paper. Unless otherwise stated all structures $\mathbf{A}, \mathbf{B}, \ldots$ belong to $Rel(\Delta)$. Now let $\Delta' = (\delta'_i; i \in I')$ be a type containing type Δ. (By this we mean $I \subseteq I'$

and $\delta'_i = \delta_i$ for $i \in I$.) Then every structure $\mathbf{X} \in Rel(\Delta')$ may be viewed as structure $\mathbf{A} = (A, (R^i_{\mathbf{A}}; i \in I)) \in Rel(\Delta)$ together with some additional relations $R^i_{\mathbf{X}}$ for $i \in I' \setminus I$. To make this more explicite these additional relations will be denoted by $X^i_{\mathbf{X}}, i \in I' \setminus I$. Thus a structure $\mathbf{X} \in Rel(\Delta')$ will be written as

$$\mathbf{X} = (A, (R^i_{\mathbf{A}}; i \in I), (X^i_{\mathbf{X}}; i \in I' \setminus I))$$

and, by abuse of notation, briefly as:

$$\mathbf{X} = (\mathbf{A}, X^1_{\mathbf{X}}, X^2_{\mathbf{X}}, \ldots, X^N_{\mathbf{X}})$$

We call $\mathbf{X}$ a *lift* of $\mathbf{A}$ and $\mathbf{A}$ is called the *shadow* (or *projection*) of $\mathbf{X}$. In this sense the class $Rel(\Delta')$ is the class of all lifts of $Rel(\Delta)$. Conversely, $Rel(\Delta)$ is the class of all shadows of $Rel(\Delta')$. Note that a lift is also in the model theoretic setting called an *expansion* and a shadow is often called a *reduct*. (Our terminology is motivated by a computer science context, see [21].) We will use letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ for shadows (in $Rel(\Delta)$) and letters $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ for lifts (in $Rel(\Delta')$).

For lift $\mathbf{X} = (\mathbf{A}, X^1_{\mathbf{X}}, \ldots, X^N_{\mathbf{X}})$ we denote by $\psi(\mathbf{X})$ the relational structure $\mathbf{A}$ i.e. its shadow. (ψ is called the *forgetful functor*.) Similarly, for a class $\mathcal{K}'$ of lifted objects we denote by $\psi(\mathcal{K}')$ the class of all shadows of structures in $\mathcal{K}'$.

As it is well known, an universal object may be constructed by an iterated amalgamation of finite objects and this leads to a stronger notion of *$\mathcal{K}$-generic object*: For a class $\mathcal{K}$ we say that an object $\mathbf{U}$ is $\mathcal{K}$-generic if it is both $\mathcal{K}$-(embedding) universal and it is *ultrahomogeneous*. The later notion means the following: Every isomorphism φ between two finite substructures $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{U}$ may be extended to an automorphism of $\mathbf{U}$. The notion of ultrahomogeneous structure is one of the key notions of modern model theory and it is the source of the well known classification programme [22].

The ultrahomogeneous structures are characterized by the properties of finite substructures. Denote by $Age(\mathbf{U})$ the class of all finite substructures of $\mathbf{U}$. By a following classical result [12, 16] structure $\mathbf{U}$ is ultrahomogeneous iff $Age(\mathbf{U}) = \mathcal{K}$ is a countable class with the following three properties:

(a) (Hereditary property) For every $\mathbf{A} \in \mathcal{K}$ and an induced substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$;

(b) (Joint embedding property) For every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that both $\mathbf{C}$ contains $\mathbf{A}$ and $\mathbf{B}$ as induced substructures;

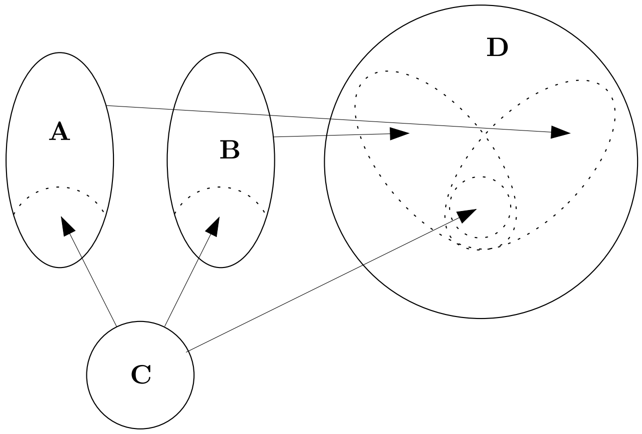


Figure 1: Amalgam of **A** and **B** over **C**.

(c) (Amalgamation property) For $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and φ embedding of $\mathbf{C}$ into $\mathbf{A}$, φ' embedding of $\mathbf{C}$ into $\mathbf{B}$, there exists $\mathbf{D} \in \mathcal{K}$ and embeddings $\psi : \mathbf{A} \to \mathbf{D}$ and $\psi' : \mathbf{B} \to \mathbf{D}$ such that $\psi \circ \varphi = \psi' \circ \varphi'$. See Figure 1.

Any class $\mathcal{K}$ satisfying the three conditions $(a), (b), (c)$ is called an *amalgamation class*. In this case we denote by $\lim \mathcal{K}$ the (up to isomorphism uniquely determined) generic object $\mathbf{U}$. $\lim \mathcal{K} = \mathbf{U}$ is called the *Fraïssé limit* of $\mathcal{K}$.

In this paper we show that universal object for classes $Forb_h(\mathcal{F})$ may be constructed as a shadow (reduct) of a generic object of an explicitly defined lifted class. The following is the principal result of this paper:

Theorem 1.3 *For every finite family $\mathcal{F}$ of finite connected relational structures of type Δ there exists a type Δ' containing Δ and a finite family $\mathcal{F}'$ of finite structures of type Δ' such that:*

- *$Forb_e(\mathcal{F}')$ is amalgamation class whose shadow is $Forb_h(\mathcal{F})$,*
- *The shadow $\mathbf{U}$ of the generic $\mathbf{U}' = \lim Forb_e(\mathcal{F}')$ is universal structure for $Forb_h(\mathcal{F})$.*

Among others this implies that $\mathbf{U}'$ is ultrahomogeneous and $\mathbf{U}$ is ω-categorical (of course ω-categoricity for universal objects for classes $Forb_h(\mathcal{F})$

was proved in [6] as well.) Note that, for lifted objects we have to consider forbidden-embedding classes $Forb_e(\mathcal{F}')$ (and not just classes $Forb_h(\mathcal{F}')$). This is necessary as we show in Lemma 3.5.

We prove Theorem 1.3 in three steps: First we prove that there is a class $\mathcal{L}$ of lifted objects which is an amalgamation class (Theorem 2.2) and then we define $\mathcal{L}$ by a set of forbidden rooted substructures (Theorem 3.2) and finally by the set of forbidden embeddings (Theorem 3.4).

Thus we can get universal objects for classes $Forb_h(\mathcal{F})$ in a particularly efficient way as shadow of a Fraïssé limit of certain (explicitly defined) lifted class $Forb_e(\mathcal{F}')$.

Theorem 1.3 and its combinatorial proof have several other consequences. For example, we can prove, that many classes $Forb_h(\mathcal{F})$ are weakly indivisible. In the another direction we show that in the case that all $\mathbf{F} \in \mathcal{F}$ are relational trees it suffices to consider only a class of monadic lifts (i.e. all relations $X^i_{\mathbf{A}}$ for $i \in I' \setminus I$ are monadic; this case corresponds to structures endowed with a coloring of its vertices). This is stated in Section 5. Consequently, in this case the generic structure $\mathbf{U}'$ has not only the universal shadow $\mathbf{U}$ but in fact this shadow $\mathbf{U}$ has a finite core $\mathbf{D}$ which is thus a finite hom-universal structure and thus a dual. This is stated as Corollary 5.1. In this way we reprove the finite duality theorem [32]. This is particularly pleasing: (in this sense) duals are generic objects and this indicates yet another context of hom-dualities.

In Section 5.2 we include further corollaries of our method to metric spaces (which can be treated analogously to forbidding cycles). In Section 4 we solve a related extremal problem: we show that our method lead to the optimal arities of added lifts. This is based on a non-trivial Ramsey-type result (Lemma 4.1).

Let us add a few more definitions. For a structure $\mathbf{A} = (A, R^i_{\mathbf{A}}, i \in I)$ the *Gaifman graph* (in combinatorics often called *2-section*) is the graph $\mathbf{G_A}$ with vertices A and all those edges which are a subset of a tuple of a relation of $\mathbf{A}$:

$$\mathbf{G_A} = (V, E)$$

where $\{x, y\} \in E$ iff $x \neq y$ and there exists tuple $\vec{v} \in R^i_{\mathbf{A}}, i \in I$ such that $x, y \in \vec{v}$.

A *cut* in $\mathbf{A}$ is a subset C of A such that the Gaifman graph $\mathbf{G_A}$ is disconnected by removing set C (i.e. if C is graph theoretic cut of $\mathbf{G_A}$). By a *minimal cut* we always mean an inclusion minimal cut. If C is a set of vertices then $\overrightarrow{C}$ will denote a tuple (of length $|C|$) from all elements of

C. Alternatively, $\overrightarrow{C}$ is arbitrary linear ordering of C.

2 Basic construction ($Forb_h(\mathcal{F})$ classes)

The following is our basic notion which resembles decomposition techniques standard in graph theory and thus we adopted similar terminology.

Definition 2.1 *For a relational structure* $\mathbf{A}$ *and minimal cut* R *in* $\mathbf{A}$*, a* piece *of relational structure* $\mathbf{A}$ *is pair* $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$*. Here* $\mathbf{P}$ *is structure induced on* $\mathbf{A}$ *by union of* R *and vertices of some connected component of* $\mathbf{A} \setminus R$*. Tuple* $\overrightarrow{R}$ *consist of the vertices of cut* R *in a (fixed) linear order.*

Note that from the inclusion-minimality of the cut R follows that all pieces of a connected structure are connected structures. It follows that for connected structure $\mathbf{A}$ the pieces of $\mathbf{A}$ determined by any minimal cut C cover the whole $\mathbf{A}$.

All pieces are considered as rooted structures: a piece $\mathcal{P}$ is a structure $\mathbf{P}$ rooted at $\overrightarrow{R}$. Accordingly, we say that pieces $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ and $\mathcal{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ are *isomorphic* if there is function $\varphi : P_1 \to P_2$ that is isomorphism of structures $\mathbf{P}_1$ and $\mathbf{P}_2$ and φ restricted to $\overrightarrow{R}_1$ is the monotone bijection between $\overrightarrow{R}_1$ and $\overrightarrow{R}_2$ (we denote this $\varphi(\overrightarrow{R}_1) = \overrightarrow{R}_2$).

Observe that for relational trees, pieces are equivalent to rooted branches. Fig 2 shows all pieces of the Petersen graph.

Lemma 2.1 *Let* $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ *be a piece of structure* $\mathbf{A}$ *and* $\mathcal{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ *a piece of* $\mathbf{P}_1$*. If* $R_1 \cap P_2 \subseteq R_2$*, then* $\mathcal{P}_2$ *is also a piece of* $\mathbf{A}$*.*

Proof. Denote by $\mathbf{C}_1$ connected component of $\mathbf{A} \setminus R_1$ that produces $\mathcal{P}_1$ (i.e. $C_1 \cup R_1 = P_1$). Denote by $\mathbf{C}_2$ component of $\mathbf{P}_1 \setminus R_2$ that produces $\mathcal{P}_2$ (i.e. $C_2 \cup R_2 = P_2$). As $R_1 \cap P_2 \subseteq R_2$ one can check that then $\mathbf{C}_2$ is contained in $\mathbf{C}_1$ and every vertex of $\mathbf{A}$ connected by tuple to any vertex of $\mathbf{C}_2$ is contained in $\mathbf{P}_1$. Thus $\mathbf{C}_2$ is also connected component of $\mathbf{A}$ created after removing vertices of R_2. $\square$

Let $\mathcal{F}$ be a fixed countable set of finite relational structures of (finite) type Δ. For construction of universal structure of $Forb_h(\mathcal{F})$ we use special lifts, called $\mathcal{F}$-lifts.

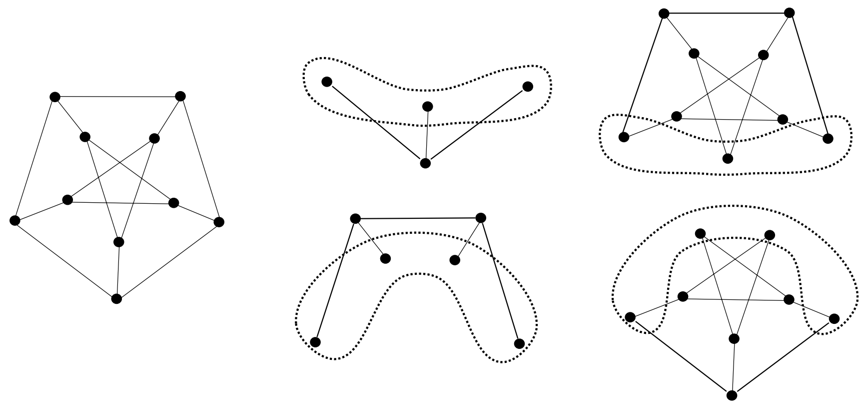


Figure 2: Pieces of Petersen graph up to isomorphisms (and a permutations of roots).

Let $\mathcal{P}_i = (\mathbf{P}_i, \overrightarrow{R}_i), i \in I'$ be all pieces of all relational structures $\mathbf{F} \in \mathcal{F}$. Notice that there are only finitely many pieces. This enumeration of pieces is fixed throughout this section.

Relational structure $\mathbf{X} = (\mathbf{A}, (X^i_{\mathbf{X}}, i \in I'))$ is called *$\mathcal{F}$-lift* of relational structure $\mathbf{A}$ when the arities of relations $X^i_{\mathbf{X}}, i \in I'$, correspond to $|\overrightarrow{R}_i|$.

For relational structure $\mathbf{A}$ we define *canonical lift* $\mathbf{X} = L(\mathbf{A})$ by putting $(v_1, v_2, \ldots, v_l) \in X^i_{\mathbf{X}}$ iff there is homomorphism φ from $\mathbf{P}_i$ to A such that $\varphi(\overrightarrow{R}_i) = (v_1, v_2, \ldots, v_l)$

Denote by $\mathcal{L}$ the class of all induced substructures of lift $L(\mathbf{A})$, $\mathbf{A} \in Forb_h(\mathcal{F})$.

For $\mathbf{X} \in \mathcal{L}$ we denote by $W(\mathbf{X})$ a structure $\mathbf{A} \in Forb_h(\mathcal{F})$ such that structure $\mathbf{X}$ is induced on X by $L(\mathbf{A})$. $W(\mathbf{X})$ is called a *witness* of the fact that $\mathbf{X}$ belongs to $\mathcal{L}$.

Theorem 2.2 *Let $\mathcal{F}$ be a finite set of finite connected relational structures. The class $\mathcal{L}$ is an amalgamation class.*

Consequently, there is a generic structure $\mathbf{U}$ in $\mathcal{L}$ and its shadow $\psi(\mathbf{U})$ is a universal structure for class $Forb_h(\mathcal{F})$.

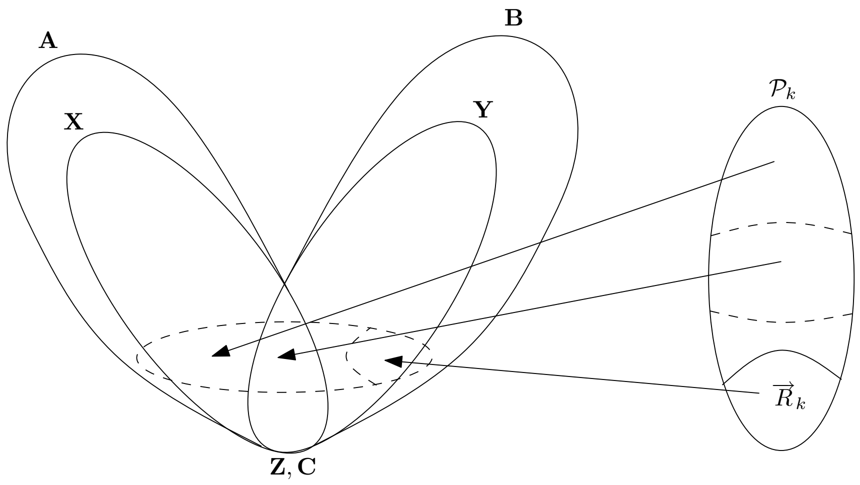


Figure 3: Construction of amalgam.

Proof. By definition the class $\mathcal{L}$ is hereditary, isomorphism closed, has a joint embedding property. $\mathcal{L}$ is countable, because there are only countably many structures in $Forb_h(\mathcal{F})$ (because type Δ is finite) and thus also countably many lifts. To show that $\mathcal{L}$ is amalgamation class it remains to verify that $\mathcal{L}$ has the amalgamation property.

Consider $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{L}$. Assume that structure $\mathbf{Z}$ is substructure induced by both $\mathbf{X}$ and $\mathbf{Y}$ on Z and without loss of generality assume that $X \cap Y = Z$.

Put

$$\mathbf{A} = W(\mathbf{X})$$

$$\mathbf{B} = W(\mathbf{Y})$$

$$\mathbf{C} = \psi(\mathbf{Z})$$

Because $\mathcal{L}$ is closed under isomorphism, we can still assume that $\mathbf{A}$ and $\mathbf{B}$ are vertex disjoint with exception of vertices of $\mathbf{C}$.

Let $\mathbf{D}$ be free amalgam of $\mathbf{A}$ and $\mathbf{B}$ over vertices of $\mathbf{C}$: vertices of $\mathbf{D}$ are $A \cup B$ and there is $\vec{v} \in R^i_{\mathbf{D}}$ iff $\vec{v} \in R^i_{\mathbf{A}}$ or $\vec{v} \in R^i_{\mathbf{B}}$.

We claim that the structure

$$\mathbf{V} = L(\mathbf{D})$$

is (not necessarily free) amalgam of $L(\mathbf{A}), L(\mathbf{B})$ over $\mathbf{Z}$ and thus also amalgam of $\mathbf{X}, \mathbf{Y}$ over $\mathbf{Z}$.

First we show that the substructure induced by $\mathbf{V}$ on A is $L(\mathbf{A})$ and that the substructure induced by $\mathbf{V}$ on B is $L(\mathbf{B})$. In the other words no new tuples to $L(\mathbf{A}), L(\mathbf{B})$ (and thus also to $\mathbf{X}$ and to $\mathbf{Y}$) were introduced.

Assume the contrary that there is a new tuple $(v_1, \ldots, v_t) \in X^k_{\mathbf{V}}$ and among all tuples and possible choices of k choose one with the minimal number of vertices of the corresponding piece $\mathcal{P}_k$. Such $\mathcal{P}_k$ is fixed throughout this proof. By symmetry we can assume that $v_i \in A, i = 1, \ldots, t$. Explicitely, we assume that there is homomorphism φ from $\mathbf{P}_k$ to $\mathbf{D}$ such that $\varphi(\overrightarrow{R_k}) = (v_1, v_2, \ldots v_t) \notin X^k_{L(\mathbf{A})}$.

The set $\varphi^{-1}(A)$ of vertices of $\mathbf{P}_k$ mapped to $L(\mathbf{A})$ is nonempty, because it contains all vertices of $\overrightarrow{R_k}$. $\varphi^{-1}(B)$ is nonempty because there is no homomorphism φ' from $\mathbf{P}_k$ to $\mathbf{A}$ such that $\varphi'(\overrightarrow{R_k}) = (v_1, v_2, \ldots v_t)$ (otherwise we would have $(v_1, v_2, \ldots v_t) \in X^k_{L(\mathbf{A})}$).

$\varphi^{-1}(C)$ is also non empty as there are no edges from vertices $A \setminus C$ to vertices $B \setminus C$ in $\mathbf{D}$ and because pieces are connected. Moreover the vertices of $\varphi^{-1}(C)$ form a cut of $\mathbf{P}_k$.

Denote by $\mathbf{K}_1, \mathbf{K}_2, \ldots \mathbf{K}_l$ all connected components of substructure induced on $P_k \setminus \varphi^{-1}(A)$ by $\mathbf{P}_k$. For each component $\mathbf{K}_i$, $1 \leq i \leq l$ there is vertex cut K'_i of $\mathbf{P}_k$ constructed by all vertices of $\varphi^{-1}(A)$ connected to K_i. This cut is always contained in $\varphi^{-1}(C)$.

Because $\mathcal{P}_k$ is a piece of some $\mathbf{F} \in \mathcal{F}$ and because $(\mathbf{K}_i, \overrightarrow{K'_i})$ are pieces of $\mathbf{P}_k$, by Lemma 2.1, they are also pieces of $\mathbf{F}$. We denote by $\mathcal{P}_{k_1}, \mathcal{P}_{k_2}, \ldots \mathcal{P}_{k_l}$ all pieces isomorphic to pieces $(\mathbf{K}_1, \overrightarrow{K'_1}), (\mathbf{K}_2, \overrightarrow{K'_2}), \ldots (\mathbf{K}_l, \overrightarrow{K'_l})$ via isomorphisms $\varphi_1, \varphi_2, \ldots \varphi_l$.

Now we use the minimality of the piece $\mathcal{P}_k$. All the pieces $\mathcal{P}_{k_i}, i = 1, \ldots, l$ have smaller size than $\mathcal{P}_k$ (as $\varphi^{-1}(C)$ is a cut of $\mathcal{P}_k$). Thus we have that tuple $\varphi(K_i)$ of $L(\mathbf{D})$ is as well a tuple of $L(\mathbf{A})$. Thus there exists homomorphism φ'_i from $\mathbf{K}_i$ to $\mathbf{D}$ such that $\varphi'_i(\overrightarrow{K'_i}) = \varphi(\overrightarrow{K'_i})$ for every $i = 1, 2, \ldots l$.

In this situation we define $\varphi'(x) : P_k \to A$ as follows:

1. $\varphi'(x) = \varphi'_i(x)$ when $x \in K_i$ for some $i = 1, 2, \ldots l$.

2. $\varphi'(x) = \varphi(x)$.

It is easy to see that $\varphi'(x)$ is homomorphism from $\mathbf{P}_k$ to $L(\mathbf{A})$. This is a contradiction.

It remains to verify that $\mathbf{D} \in Forb_h(\mathcal{F})$. We proceed analogously. Assume that φ is homomorphism of a $\mathbf{F} \in \mathcal{F}$ to $\mathbf{D}$. Because $\mathbf{A}, \mathbf{B} \in Forb_h(\mathcal{F})$, φ must use vertices of $\mathbf{C}$ and $\varphi^{-1}(C)$ forms cut of $\mathbf{F}$. Denote by E a minimal cut contained in $\varphi^{-1}(C)$. $\varphi(E)$ must contain tuples corresponding to all pieces of $\mathbf{F}$ having E as roots in $\mathbf{Z}$. This is a contradiction with $\mathbf{Z} \in \mathcal{L}$. $\square$

3 Forbidden lifts ($Forb_e(\mathcal{F}')$ classes)

In Theorem 2.2 we found an amalgamation class $\mathcal{L} \in Rel(\Delta')$ of lifted objects such that the shadow of the Fraïssé limit of $\mathcal{L}$ is universal object of $Forb_h(\mathcal{F})$. In this section we further refine this result by giving an explicit description of the amalgamation class $\mathcal{L}$ in terms of forbidden substructures. We prove that $\mathcal{L}$ is equivalent to a class $Forb_e(\mathcal{F}')$ for an explicitly defined class of lifts $\mathcal{F}'$ (derived from the class $\mathcal{F}$). First we show a more explicit construction of a witness.

Definition 3.1 *For a piece $\mathcal{P}_i$ such that $\overrightarrow{R_i}$ is n-tuple and for an n-tuple $\vec{x}$ of vertices of $\mathbf{A}$ we denote by $A +_{\vec{x}} \mathcal{P}_i$ relational structure created as disjoint union of $\mathbf{A}$ and $\mathbf{P}_i$ identifying vertices of $\overrightarrow{R}_i$ along $\vec{x}$ (i.e. $A + \mathcal{P}_i$ is free amalgamation over $\overrightarrow{x}$).*

We put $\mathbf{X} +_{X^i_{\mathbf{X}}} \mathcal{P}_i = \psi(\mathbf{X}) +_{\vec{x}_1} \mathcal{P}_i +_{\vec{x}_2} \mathcal{P}_i +_{\vec{x}_3} \dots +_{\vec{x}_k} \mathcal{P}_i$ where $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} = X^i_{\mathbf{X}}$. Finally we put $UW(\mathbf{X}) = \psi(\mathbf{X}) +_{X^1_{\mathbf{X}}} \mathcal{P}_1 +_{X^2_{\mathbf{X}}} \mathcal{P}_2 +_{X^3_{\mathbf{X}}} \dots +_{X^N_{\mathbf{X}}} \mathcal{P}_N$. We will call $UW(\mathbf{X})$ the universal witness *of $\mathbf{X}$.*

Now we develop alternative and more explicite description of the class $\mathcal{L}$ (introduced in Section 2). We preserve all the notation introduced there.

Lemma 3.1 *Lift $\mathbf{X}$ belongs to $\mathcal{L}$ iff $UW(\mathbf{X}) \in Forb_h(\mathcal{F})$ and $\mathbf{X}$ is induced on X by $L(UW(\mathbf{X}))$ (in the other words $UW(\mathbf{X}) \in Forb_h(\mathcal{F})$ is a witness of $\mathbf{X}$).*

Proof. Assume that $\mathbf{X} \in \mathcal{L}$.

$$\mathbf{A} = W(\mathbf{X})$$

$$\mathbf{B} = UW(\mathbf{X})$$

It follows from the construction that there exists a homomorphism $\varphi : \mathbf{B} \to \mathbf{A}$ which is identity on $\mathbf{X}$.

If there was homomorphism φ' from some $\mathbf{F}$ to $UW(\mathbf{X})$ then by composing with φ, there exists also a homomorphism from $\mathbf{F}$ to $W(\mathbf{X})$. This is not possible, since $\mathbf{A}$ is witness.

Let us assume now that $\mathbf{X}$ is not induced by $L(\mathbf{B})$ on X. From the construction of $L(\mathbf{B})$ we have trivially that for each $\vec{v} \in X^i_{\mathbf{X}}$, there is also $\vec{v} \in X^i_{L(\mathbf{B})}$. Assume that there is some $\vec{v} \in X^i_{L(\mathbf{B})}$ consisting only from vertices of $\mathbf{X}$ such that $\vec{v} \notin X^i_{\mathbf{X}}$. Let φ' be the homomorphism $\mathbf{P}_i \to L(\mathbf{B})$ such that $\varphi'(\overrightarrow{R_i}) = \vec{v}$. Again by composing with φ we obtain homomorphism $\mathbf{P}_i \to L(\mathbf{A})$, a contradiction with $\vec{v} \notin X^i_{\mathbf{X}}$. Thus $\mathbf{X}$ is induced by x on $L(\mathbf{B})$.

In the reverse direction if $UW(\mathbf{X})$ was witness, then $\mathbf{X} \in \mathcal{L}$. The conditions listed in the lemma are precisely the conditions on $UW(\mathbf{X})$ being witness. $\square$

Definition 3.2 *For rooted structure* $(\mathbf{X}, \overrightarrow{R})$ *we define* i-rooted homomorphism $(\mathbf{X}, \overrightarrow{R}) \to \mathbf{Y}$ *as a homomorphism* $f : \mathbf{X} \to \mathbf{Y}$ *such that* $f(\overrightarrow{R}) \in X^i_{\mathbf{Y}}$ *iff* $\overrightarrow{R} \in X^i_{\mathbf{X}}$.

For relational structure $\mathbf{A}$ *and* $\mathbf{X}$ *sublift of* $L(\mathbf{A})$*, we say that* $\mathbf{X}$ *is* $\mathbf{A}$-covering *iff there is homomorphism* $f : \mathbf{A} \to UW(\mathbf{X})$.

Similarly for piece $\mathcal{P}_i$ *and* $\mathbf{X}$ *a sublift of* $L(\mathbf{P}_i)$ *such that* $\overrightarrow{R}_i \notin X^i_{\mathbf{X}}$*, we say that* $\mathbf{X}$ *is* $\mathcal{P}_i$-covering *iff* X *contains all roots of* $\overrightarrow{R}_i$ *and there is homomorphism* $\varphi : \mathbf{P}_i \to UW(\mathbf{X})$ *such that* f *is identity on* $\overrightarrow{R}_i$.

Our first characterization of classes $\mathcal{L}$ is in the form of rooted homomorphism and coverings.

Theorem 3.2 *For fixed* $\mathcal{F}$*, class* $\mathcal{L}$ *(defined above before Theorem 2.2) satisfies:*

$\mathbf{X} \in \mathcal{L}$ *if and only if*

(a) there is no homomorphism $\mathbf{Y} \to \mathbf{X}$*, for* $\mathbf{Y}$ *being* $\mathbf{F}$*-covering for some* $\mathbf{F} \in \mathcal{F}$*;*

(b) for every $i = 1, \dots, N$ *and every* $\mathcal{P}_i$*-covering* $\mathbf{Z}$ *there is no* i*-rooted homomorphism* $f : (\mathbf{Z}, \overrightarrow{R}_i) \to \mathbf{X}$.

Lemma 3.3 *Conditions* (a) *and* (b) *holds for every* $\mathbf{X} \in \mathcal{L}$.

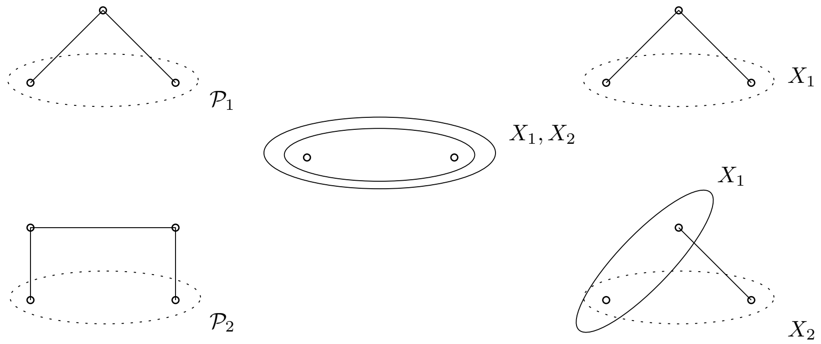


Figure 4: Pieces of 5 cycle C_5 (up to isomorphisms and permutations of roots), inclusion minimal C_5-covering sublifts, and inclusion minimal $\mathcal{P}_1$-covering and $\mathcal{P}_2$-covering sublifts. Roots are denoted by dotted line.

Proof. Fix $\mathbf{X} \in \mathcal{L}$. Assume that (a) does not hold for some $\mathbf{Y}$ that is $\mathbf{F}$-covering for $\mathbf{F} \in \mathcal{F}$. Since there is homomorphism $\mathbf{F} \to UW(\mathbf{Y})$ and a homomorphism $\mathbf{Y} \to \mathbf{X}$ we also have homomorphism $\mathbf{F} \to UW(\mathbf{Y}) \to UW(\mathbf{X})$, a contradiction with Lemma 3.1.

(b) is a rooted analogy of the same proof. $\square$

Proof of Theorem. Take lift $\mathbf{X}$ such that $\mathbf{X} \notin \mathcal{L}$. By Lemma 3.1 we have one of the following cases:

I $\mathbf{X}$ is not induced by $L(UW(\mathbf{X}))$ on $\mathbf{X}$.

In this case we have some homomorphism of $f : \mathbf{P}_i \to UW(\mathbf{X})$ such that $f(R_i) \notin X^i_{\mathbf{X}}$. Assume that i is chosen such that number of vertices of $\mathbf{P}_i$ is minimal.

Denote by $\mathbf{Y}$ maximal (non-induced) sublift of $L(\mathbf{P}_i)$ such that f is also homomorphism from $\mathbf{Y}$ to $\mathbf{X}$. We need to show that $\mathbf{Y}$ is $\mathcal{P}_i$-covering to get contradiction with (b).

Denote by $\mathbf{C}_1 \dots \mathbf{C}_t$ components of $\mathbf{P}_i \setminus Y$. Now denote by $\mathcal{P}_{k_1}, \dots \mathcal{P}_{k_t}$ pieces corresponding to these components and by $f_1, \dots f_t$ homomorphisms $\mathbf{P}_{k_j} \to \mathbf{P}_i$ mapping non-roots of $\mathbf{P}_{k_j}$ to the corresponding component and roots to vertices of $\mathbf{Y}$.

Because $f(f_j(x))$ is homomorphism $\mathbf{P}_{k_j} \to UW(\mathbf{X})$ we have from minimality of counterexample $f(f_j(\overrightarrow{R}_{k_j})) \in X^{k_j}_{\mathbf{X}}$ and thus also $f_j(\overrightarrow{R}_{k_j}) \in$

$X_{\mathbf{Y}}^{k_j}$. This holds for every $j = 1, \ldots, t$ and thus we have also $\mathbf{P_n} \to UW(\mathbf{Y})$ such that it is identity on Y. This prove that $\mathbf{Y}$ is $\mathcal{P}_i$-covering.

II There is a homomorphism f of some $\mathbf{F} \in \mathcal{F}$ to $UW(\mathbf{X})$.

Assume that $f(F) \cup X$ is empty. In this case there is i such that $f(F)$ is contained in vertices of copy of $\mathcal{P}_i$ in $UW(X)$. In this case lift $\mathbf{X}$ is covering, because it contains tuple in $X_{\mathbf{X}}^{i}$. A contradiction.

Denote by $\mathbf{Y}$ maximal (non-induced) sublift of $L(\mathbf{F})$ so f is also a homomorphism $\mathbf{Y} \to \mathbf{X}$. Because there is non-empty intersection of $f(F)$ and X, Y is nonempty. We can show that $\mathbf{Y}$ is covering by same argument as in I getting contradiction with (a) too.

□

Observe that properties (a) and (b) directly translate to the family $\mathcal{F}'$ with the desired property that the shadow of $Forb_e(\mathcal{F}')$ is $Forb_h(\mathcal{F})$. This leads to the desired explicit characterization of the class $\mathcal{L}$.

Theorem 3.4 *Let $\mathcal{F}'$ be a class of $\mathcal{F}$-lifts satisfying the following:*

1. *$\mathbf{X} \in \mathcal{F}'$ for every lift $\mathbf{X}$ such that there is $\mathbf{F}$-covering lift $\mathbf{Y}$ for some $\mathbf{F} \in \mathcal{F}$ together with a surjective homomorphism $\mathbf{Y} \to \mathbf{X}$,*
2. *$\mathbf{X} \in \mathcal{F}'$ for every structure $\mathbf{X}$ such that there is $1 \le i \le N$ and $\mathcal{P}_i$-covering rooted structure $(\mathbf{Y}, \overrightarrow{R})$ together with a surjective i-rooted homomorphism $\mathbf{Y} \to \mathbf{X}$,*
3. *$\mathcal{F}'$ contains no other structures.*

Then we have:

1. *$\mathcal{F}'$ is a finite family;*
2. *$Forb_e(\mathcal{F}') = \mathcal{L}$ and thus $Forb_e(\mathcal{F}')$ is amalgamation class. The shadows $\mathbf{U}$ of the generic $\mathbf{U}' = \lim Forb_e(\mathcal{F}')$ is universal structure for $Forb_h(\mathcal{F})$.*

Proof. $\mathcal{F}'$ is necessarily finite, because number of vertices of lifts $\mathbf{X} \in \mathcal{F}'$ is bounded by number of vertices of structures $\mathbf{A} \in \mathcal{F}$. From construction above it follows that $Forb_e(\mathcal{F}')$ is precisely class of structures satisfying conditions (a) and (b). □

We used notion of rooted homomorphisms (and thus classes $Forb_e(\mathcal{F}')$) to define our lifted classes. It is easy to see that classes $Forb_h(\mathcal{F}')$ are not powerful enough to extend the expressive power.

Lemma 3.5 *Assume that there is a class $\mathcal{F}$ and a lifted class $\mathcal{F}'$ such that $Forb_h(\mathcal{F}')$ contains a generic object whose shadow is universal graph of class $Forb_h(\mathcal{F})$. Then the class $Forb_h(\mathcal{F})$ itself contains a generic object.*

Proof. Observe that all classes $Forb_h(\mathcal{F}')$ are monotonous. That is for any $\mathbf{X} \in Forb_h(\mathcal{F}')$ a relational structures $\mathbf{Y}$ created from $\mathbf{X}$ by removing some of its edges also belongs to $Forb_h(\mathcal{F}')$.

In particular $Forb_h(\mathcal{F}')$ is closed for constructing shadows and thus $Forb_h(\mathcal{F})$ may be thought as subclass of $Forb_h(\mathcal{F}')$ (modulo the signature of relational structures).

Now take any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in Forb_h(\mathcal{F})$ and their lifts $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ such that they contain no new edges. These lifts are in class $Forb_h(\mathcal{F}')$. Now consider $\mathbf{W}$ amalgam of $\mathbf{X}$ and $\mathbf{Y}$ over $\mathbf{Z}$ and its shadow $\mathbf{D}$. $\mathbf{D}$ is amalgam of $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$. $\square$

4 Bounding arities

The expressive power of lifts can be limited in several ways. For example, it is natural to restrict arities of the newly added relations. It follows from the above proof that the arities of new relations in our lifted amalgamation class $\mathcal{L}$ depend on the size of maximal inclusion minimal cut of the Gaifman graph of the forbidden structure.

In this section we completely characterize the minimal arity of ultrahomogeneous lifts of classes $Forb_h(\mathcal{F})$. This involves a non-trivial Ramsey type statement stated bellow as Lemma 4.1. As a warm up, we first show that generic universal graph for the class $Forb_h(C_5)$ can not be constructed by finite monadic lifts:

Consider, for contradiction, a monadic lift $\mathbf{U}'$ which is both a generic relational structure and whose shadow $\mathbf{U}$ is universal for the class $Forb_h(C_5)$. Since all extended relations are monadic, we can view them as finite coloring of vertices. For $v \in \mathbf{U}$ we will denote by $c(v)$ the color of v or, equivalently, set of all extended relations $X^i_{\mathbf{U}}$ such that $(v) \in X^i_{\mathbf{U}}$.

Since graphs in $Forb_h(C_5)$ have unbounded chromatic number, we know that chromatic number of $\mathbf{U}$ is infinite. Consider decomposition of $\mathbf{U}$ implied by c. Since the range of c is finite, one of the graphs in this decomposition has infinite chromatic number. Denote this subgraph by $\mathbf{S}$.

In fact is suffices that $\mathbf{S}$ is not bipartite. Thus $\mathbf{S}$ contains an odd circle. The shortest odd cycle has length ≥ 7 and thus $\mathbf{S}$ contains induced path of length 3 formed by vertices p_1, p_2, p_3, p_4. Additionally there is a vertex v of degree at least 2. Because the graph is triangle free, the vertices v_1 and v_2 connected to v are not connected by an edge.

From genericity of $\mathbf{U}$ we know that the partial isomorphism mapping $v_1 \to p_1$ and $v_2 \to p_4$ can be extended to an automorphism φ of $\mathbf{U}'$. The vertex $\varphi(v)$ is connected to p_1 and p_4 and thus together with p_1, p_2, p_4 contains either triangle or 5-cycle. It follows that generic $\mathbf{U}'$ cannot be defined by monadic lifts of $Forb_h(\mathcal{F})$.

In this section we prove that there is nothing special here about arity 2 and about the pentagon. One can determine the minimal arity of the ultrahomogeneous lifts for general classes $Forb_h(\mathcal{F})$. Towards this end we shall need a Ramsey type statement which we formulate after introducing the following:

Let S be finite set with partition $S_1 \cup S_2 \cup \ldots \cup S_n$. For $v \in S$ we denote by $i(v)$ the index i such that $v \in S_i$. Similarly, for a tuple $\vec{x} = (x_1, x_2, \ldots, x_t)$ of elements of S we denote by $i(A)$ the tuple $(i(x_1), i(x_2), \ldots, i(x_t))$. We make use of the following:

Lemma 4.1 *For every $n \geq 2$, $r < n$ and K integers there is relational structure $\mathbf{S} = (S, R_{\mathbf{S}})$, with vertices $S = S_1 \cup S_2 \cup \ldots S_n$ (sets S_i are mutually disjoint) and single relation $R_{\mathbf{S}}$ of arity $2n$ with the following properties:*

1. *Every $(v_1, u_1, v_2, u_2, \ldots v_n, u_n) \in R_{\mathbf{S}}$, $v_1, u_1 \in S_1$, $v_2, u_2 \in S_2$ satisfies $v_i \neq u_i \in S_i, i = 1 \ldots n$*

2. *For every $\vec{v}, \vec{u} \in \mathbf{S}$, $\vec{v} \neq \vec{u}$, $\vec{v}$ and $\vec{u}$ has at most r common vertices.*

3. *For every coloring of tuples on S of size r using K colors there is tuple $\vec{v} \in R_{\mathbf{S}}$ such that every two tuples $\vec{x}, \vec{x}'$ consisting of vertices of $\vec{v}$ such that $i(\vec{x}) = i(\vec{x}')$ have same color.*

Proof. This statement follows from results obtained by Nešetřil and Rödl [30]. Although not stated explicitely, this is "partite version" of the main result of [30]. It can be also obtained directly by means of amalgamation method, see [26, 31]. In this paper this result plays auxiliary role only and we omit the proof. $\square$

Given a relational structure $\mathbf{S} = (S, R_{\mathbf{S}})$ with relation $R_{\mathbf{S}}$ of arity $2n$ and rooted relational structure $(\mathbf{A}, \overrightarrow{R})$ of type Δ with $\overrightarrow{R} = (r_1, r'_1, \dots r_n, r'_n)$, we denote by $\mathbf{S} * (\mathbf{A}, \overrightarrow{R})$ the following relational structure $\mathbf{B}$ of type Δ:

The vertices of B are equivalence classes of equivalence $\sim$ on $R_{\mathbf{S}} \times A$ generated by the following pairs:

$$(\vec{v}, r_i) \sim (\vec{u}, r_i) \text{ iff } \vec{v}_{2i} = \vec{u}_{2i}$$

$$(\vec{v}, r'_i) \sim (\vec{u}, r'_i) \text{ iff } \vec{v}_{2i+1} = \vec{u}_{2i+1}$$

$$(\vec{v}, r_i) \sim (\vec{u}, r'_i) \text{ iff } \vec{v}_{2i} = \vec{u}_{2i+1}$$

Denote by $[\vec{v}, r_i]$ the equivalence class of $\sim$ containing $(\vec{v}, r_i)$.

We put $\vec{v} \in R^j_{\mathbf{B}}$ iff $\vec{v} = ([\vec{u}, v_1], [\vec{u}, v_2], \dots [\vec{u}, v_t])$ for some $\vec{u} \in R_{\mathbf{S}}$ and $(v_1, v_2, \dots v_t) \in R^j_{\mathbf{A}}$.

This construction is commonly used in the graph homomorphism context as *indicator construction*, see [15]. It essentially means replacing every tuple of $R_{\mathbf{S}}$ by disjoint copy of $\mathbf{A}$ with roots $\overrightarrow{R}$ identified with vertices of the tuple.

For given vertex v of $\mathbf{S} * (\mathbf{A}, \overrightarrow{R})$ such that $v = [\vec{u}, r_i]$ (or $v = [\vec{u}, r'_i]$) we will call vertex $v' = \vec{u}_{2i}$ (or $v' = \vec{u}_{2i+1}$ respectively) *the vertex corresponding to v in* $\mathbf{S}$. Note that this gives correspondence between vertices of $\mathbf{S}$ and $\mathbf{S} * (\mathbf{A}, \overrightarrow{R})$ restricted to vertices $[\vec{v}, r_i]$ and $[\vec{v}, r'_i]$.

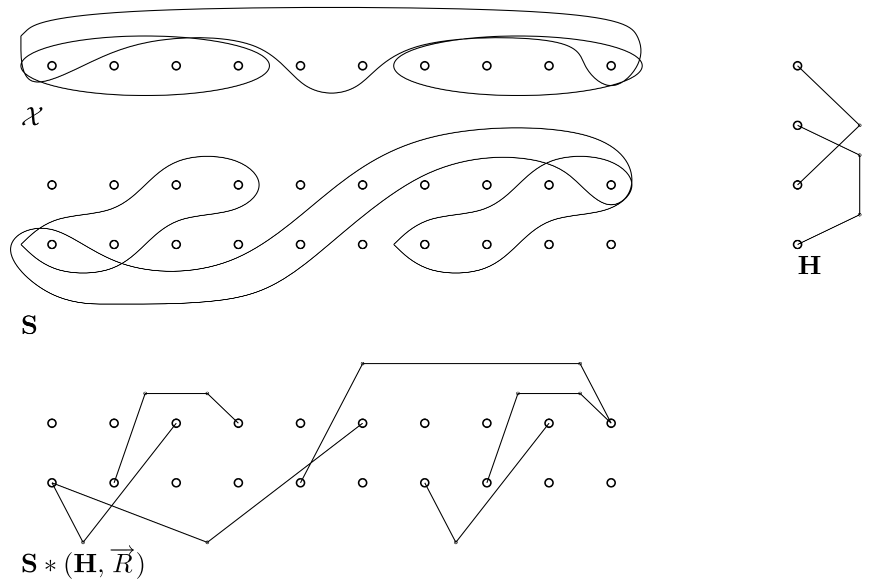
Finite family finite relational structures is called *minimal* iff there all structures in $\mathcal{F}$ are cores and there is no homomorphism in between two structures in $\mathcal{F}$.

The following is the main result of this section.

Theorem 4.2 *Denote by $\mathcal{F}$ minimal family of finite connected relational structures. There is lift of class $Forb_h(\mathcal{F})$ that contains new relations of arity at most r that is amalgamation class iff all minimal cuts of $\mathbf{F} \in \mathcal{F}$ consist of at most r vertices.*

Proof. Construction of lifted class $\mathcal{L}$ in proof of Theorem 2.2 adds relations of arities corresponding to the sizes of minimal cuts of $\mathbf{F} \in \mathcal{F}$ so one direction of Theorem 4.2 follows directly from the proof of Theorem 2.2.

In the opposite direction fix class $\mathcal{F}$, $r \geq 1$ and a relational structure $\mathbf{F} \in \mathcal{F}$ containing minimal cut $C = \{r_1, r_2, \dots r_n\}$ of size $n > r$. Assume, for contradiction, that there exists lift $\mathcal{K}$ of class $Forb_h(\mathcal{F})$ such that $\mathcal{K}$ is an amalgamation class and contains new relations of arities at most r. Denote



by K the number of different relational structures on r vertices appearing in $\mathcal{K}$.

For the brevity, assume that $\mathbf{F} \setminus C$ has only two connected components. Denote by $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ and $\mathcal{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ the pieces generated by C such that $\overrightarrow{R}_1 = \overrightarrow{R}_2 = (r_1, r_2, \ldots r_n)$. (For 3 and more pieces we can proceed analogously.)

Now we construct relational structure $\mathbf{H}$ as follows:

$$H = (P_1 \times \{1\}) \cup (P_2 \times \{2\})$$

and put

$$((v_1, 1), \ldots (v_t, 1)) \in R^i_{\mathbf{H}} \text{ iff } (v_1, \ldots v_t) \in R^i_{\mathbf{P_1}}$$
$$((v_1, 2), \ldots (v_t, 2)) \in R^i_{\mathbf{H}} \text{ iff } (v_1, \ldots v_t) \in R^i_{\mathbf{P_2}}$$

with no other tuples. In other words, $\mathbf{H}$ is disjoint union of $\mathbf{P}_1$ and $\mathbf{P}_2$. We will consider $\mathbf{H}$ rooted by tuple

$$\overrightarrow{R} = ((r_1, 1), (r_1, 2), (r_2, 1), (r_2, 2), \ldots, (r_n, 1), (r_n, 2)).$$

Take relational structure $\mathbf{S}$ from Lemma 4.1 and put $\mathbf{D} = \mathbf{S} * (\mathbf{H}, \overrightarrow{R})$. This construction for the two pieces of 5-cycle is shown at Figure 4. For vertex $v \in S$ denote by $m(v)$ the vertex of $\mathbf{D}$ corresponding to v (if it exists) or arbitrary vertex of $\mathbf{D}$ otherwise.

Denote by f mapping $[(a,1),\vec{x})] \mapsto a$ for $a \in P_1$ and $[(a,2),\vec{x}] \mapsto a$ for $a \in P_2$. It is easy to check that f is homomorphism $\mathbf{D} \to \mathbf{F}$. Additionally for $v \in \mathbf{D}$ put

$$M(v) = \{\vec{x}; ((a,t),\vec{x}) \text{ is in equivalence class } v\}.$$

Observe that for vertex $v \in \mathbf{D}$ such that $f(v) \in C$, $M(v)$ may contain multiple tuples, while for all other vertices $M(v)$ contains precisely one tuple.

Assume, to the contrary, that there is homomorphism $\varphi : \mathbf{F} \to \mathbf{D}$. By composition we have that $\varphi \circ f$ is homomorphism $\mathbf{F} \to \mathbf{F}$. Because $\mathbf{F}$ is core, we also know that $\varphi \circ f$ is automorphism of $\mathbf{F}$. It follows that φ is injective $\mathbf{F} \to \mathbf{D}$. For $v \in \mathbf{F}$ denote by $M(v)$ the set $M(\varphi(v''))$ where v'' is uniquely defined by $\varphi \circ f(v'') = v$.

It follows that for $v \in F \setminus C$, $M(v)$ consists of single tuple. For tuple $\vec{x} \in R^i_{\mathbf{F}}$, there is tuple $\varphi(\vec{x}) \in R^i_{\mathbf{D}}$ iff sets $M(v), v \in \varphi(\vec{x})$ have nonempty intersection (i.e. all belongs to single copy of some piece $\mathbf{P}_i$) and thus also sets $M(v), v \in \varphi \circ f(\vec{x})$ have nonempty intersection.

As relational systems $\mathbf{P}_i \setminus C$ are connected, it follows that all $M(v), v \in \mathbf{P}_i \setminus C$ are equivalent singleton sets. Denote by $\vec{x}_1$ tuple such that $M(v) = \{\vec{x}_1\}$ for $x \in P_1 \setminus C$ and by $\vec{x}_2$ tuple such that $M(v) = \{\vec{x}_2\}$ for $v \in P_2 \setminus C$. Because copies of pieces in $\mathbf{D}$ corresponding to single tuple $\vec{x} \in \mathcal{X}$ are not connected, we have $\vec{x}_1 \neq \vec{x}_2$. Finally because every vertex in C is connected to $P_1 \setminus C$ by some tuple (from minimality of cut C), we have $\vec{x}_1 \in M(v)$ for every $v \in C$ and analogously $\vec{x}_2 \in M(v)$ for every $v \in C$. It follows that sets $\vec{x}_1$ and $\vec{x}_2$ overlaps on whole C. This $\vec{x}_1$ and $\vec{x}$ overlaps on r or more vertices that is a contradiction with construction of relational system $\mathbf{S}$. It follows that there is no homomorphism $\mathbf{F} \to \mathbf{D}$.

There is also no homomorphism $\mathbf{F}' \to \mathbf{D}$ for any $\mathbf{F}' \in \mathcal{F}, \mathbf{F}' \neq \mathbf{F}$ because composing such homomorphism with f would lead to homomorphism $\mathbf{F}' \to \mathbf{F}$ that does not exist. It follows that $\mathbf{D} \in Forb_h(\mathcal{F})$.

Take generic lift $\mathbf{U} \in \mathcal{K}$. Every embedding $\Phi : \mathbf{D} \to \psi(\mathbf{U})$ ($\psi(\mathbf{U})$ is shadow of $\mathbf{U}$) imply K coloring of r-tuples from elements of D (colors are defined by the additional relations of $\mathbf{U}$) and thus also K coloring of r-tuples of $\mathbf{S}$. Subsequently, using Lemma 4.1, there is tuple $\vec{v} \in \mathbf{S}$, such that $\vec{v} = (u_1, v_1, u_2, v_2, \ldots u_n, v_n)$ and the relations added by lift $\mathcal{K}$ are equivalent on $\Phi(u_i)$ and $\Phi(v_i), i = 1, \ldots, n$. Thus $\mathbf{U}$ induce on both sets

$\{\Phi(u_1), \Phi(u_2), \dots \Phi(u_n)\}$ and $\{\Phi(v_1), \Phi(v_2), \dots \Phi(v_n)\}$ same lift $\mathbf{X}$. ($\mathbf{X}$ is lift of relational structure induced by $\mathbf{F}$ on C.) Subsequently there is partial isomorphism of $\mathbf{U}$ mapping $\Phi(u_i) \to \Phi(v_i)$. From genericity of the $\mathbf{U}$ this partial isomorphism extends to automorphism Ψ of $\mathbf{U}$. From the construction of relational system $\mathbf{D}$ this mapping Ψ sends a root of image of piece $\mathcal{P}_1$ to corresponding roots of image of piece $\mathcal{P}_2$ and thus shadow of $\mathbf{U}$ contains copy of $\mathbf{F} \in \mathcal{F}$, a contradiction. $\square$

5 Special cases of small arities

By Theorem 4.2 it follows that only minimal classes of finite relational structures $\mathcal{F}$ such that class $Forb_h(\mathcal{F})$ has monadic lift that forms an amalgamation class are precisely classes $\mathcal{F}$ such that all minimal vertex cuts of their Gaifman graph have size 1. Examples forming an amalgamation class include graphs with all its blocks being complete graphs.

Consider even more restricted classes $\mathcal{F}$ of structures consisting from (relational) trees only (relational trees are defined bellow). In this case we can claim a much stronger result: there exists a finite universal object $\mathbf{D}$ which is a retract of an universal structure $\mathbf{U}$.

5.1 Finite Dualities and Constraint Satisfaction Problems

A Constraint Satisfaction Problem (CSP) is the following decision problem:

Instance: A finite structure $\mathbf{A}$
Question: Does there exists a homomorphism $\mathbf{A} \to \mathbf{H}$?

We denote by $CSP(\mathbf{H})$ the class of all finite structures $\mathbf{A}$ with $\mathbf{A} \to \mathbf{H}$. It is easy to see that class $CSP(\mathbf{H})$ coincides with a particular instance of lifts and shadow.

A *finite duality* (for structures of given type) is any equation

$$Forb_h(\mathcal{F}) = CSP(\mathbf{D})$$

where $\mathcal{F}$ is s finite set [28, 32, 15]. $\mathbf{D}$ is called *dual of* $\mathcal{F}$. We also write $\mathbf{D}_{\mathcal{F}}$ for dual of $\mathcal{F}$ (it is easy to see that $\mathbf{D}_{\mathcal{F}}$ is up to homomorphism equivalence uniquely determined). The pair $(\mathcal{F}, \mathbf{D})$ is called *dual pair.* In a sense duality is a simple constraint satisfaction problem: the existence of homomorphism into $\mathbf{D}$ (i.e. a $\mathbf{D}$-coloring) is equivalently characterized by a finite set of forbidden substructures. Dualities play a role not only in the complexity problems but also in logic, model theory, partial orders and categories. Particularly it follows from [1] and [35] that dualities coincide with those first order definable classes which are homomorphism closed.

Finite dualities for monadic lifts include all classes $CSP(\mathbf{H})$. We formulate this as follows:

Proposition 5.1 *For a class $\mathcal{K}$ of structures the following two statements are equivalent:*

1. *$\mathcal{K} = CSP(\mathbf{H})$ for finite $\mathbf{H}$.*
2. *There exists a class $\mathcal{K}'$ of monadic lifts such that*
   - *(a) Shadow of $\mathcal{K}'$ is $\mathcal{K}$.*
   - *(b) $\mathcal{K}' = Forb_h(\mathcal{F}') \cap Forb_e(\mathbf{K}_1)$ where $\mathcal{F}'$ is a finite set of monadic covering lifts of edges (i.e. every $\mathbf{F} \in \mathcal{F}'$ contains at most one non-unary tuple.) while every vertex belongs to a unary lifted tuple.*

Proof (sketch). 1. obviously imply 2.

In the opposite direction construct $\mathbf{H}$ as follows: Let $\mathbf{H}_0$ be lift with vertex for every consistent combination of new relations $X^i_{\mathbf{H_0}}$ and relations $R^i_{\mathbf{H_0}}$ empty. Now construct lift $\mathbf{H}$ on same vertex set as $\mathbf{H}_0$ with $X^i_{\mathbf{H}} = X^i_{\mathbf{H_0}}$. Put tuple $\vec{x} \in R^i_{\mathbf{H}}$ iff the structure induced by $\vec{x}$ is on $\mathbf{H}_0$ with $\vec{x}$ added to $R^i_{\mathbf{H}}$ is in $Forb_h(\mathcal{F}')$. □

In the language of dualities this amounts to saying that the classes $CSP(\mathbf{H})$ are just classes described by shadow dualities of the simplest kind: forbidden lifts are just vertex colored edges.

A (relation) tree can be defined as follows: The *incidence graph* $ig(\mathbf{A})$ of relational structure $\mathbf{A}$ is the bipartite graph with parts A and

$$Block(A) = \{(i, (a_1, \ldots, a_{\delta_i})) : i \in I, (a_1, \ldots, a_{\delta_i}) \in R^i_{\mathbf{A}}\},$$

and edges $[a, (i, (a_1, \ldots, a_{\delta_i}))]$ such that $a \in (a_1, \ldots, a_{\delta_i})$. (Here we write $x \in (x_1, \ldots, x_n)$ when there exists an index k such that $x = x_k$; $Block(A)$ is a multigraph.) $\mathbf{A}$ is called a *tree* when $ig(\mathbf{A})$ is a graph tree (see e.g. [24]). The definition of relational trees by the incidence graph $ig(\mathbf{A})$ allows us to use graph terminology for relation trees.

Finite dualities are characterized (see [32])

Theorem 5.1 *For every type Δ and for every set $\mathcal{F}$ of relational trees there exists a dual Δ-structure $\mathbf{D}_{\mathcal{F}}$. Up to homomorphism equivalence there are no other dual pairs.*

Given $\mathcal{F}$ various constructions of structure duals are known [33]. It follows from this Section that we have a yet another approach to this problem:

Corollary 5.1 *Let $\mathcal{F}$ be a set of finite tree-structures, then there exists finite set of lifted structures $\mathcal{F}'$ with the following properties:*

(i) $Forb_e(\mathcal{F}')$ *is amalgamation class (and thus there is universal* $\mathbf{U}' \in Forb_e(\mathcal{F}')$*),*

(ii) all lifts in $Forb_e(\mathcal{F}')$ *are monadic,*

(iii) $\psi(\mathbf{U}') = \mathbf{U}$ *is universal for* $\mathcal{K}$*,*

(iv) $\mathbf{U}'$ *has a finite retract* $\mathbf{D}'_{\mathcal{F}}$ *and consequently* $\psi(\mathbf{D}'_{\mathcal{F}}) = \mathbf{D}_{\mathcal{F}}$ *is a dual of* $\mathcal{F}$*.*

Proof. Observe that the inclusion minimal cuts of a relational tree are all having arity 1. Thus for fixed family $\mathcal{F}$ of finite relational trees our Theorem 2.2 establishes the existence monadic lift that give an generic structure $\mathbf{U}'$ whose shadow is (homomorphism) universal for $Forb_h(\mathcal{F})$.

This structure $\mathbf{U}'$ is countable. To get a dual, we find finite $\mathbf{X}' \in \mathcal{L}$ which is retract of $\mathbf{U}'$ and still there is homomorphism $\mathbf{Y} \to \mathbf{U}'$ iff there is homomorphism $\mathbf{Y} \to \mathbf{X}'$.

The set $\mathcal{F}'$ is given by construction in Theorem 3.2. Observe that every inclusion minimal covering set of every piece of tree is induced by single tuple and thus the class $\mathcal{L}$ is defined by forbidden (rooted) homomorphism of structures covered by single tuple. This means that the generic graph $\mathbf{U}'$ has finite retract defined by all consistent combinations of new relations of its vertices. $\square$

Note that it is also possible to construct $\mathbf{D}_{\mathcal{F}}$ in a finite way without using the Fraïssé limit: for every possible combination of new relations on single vertex create single vertex of $\mathbf{D}_{\mathcal{F}}$ and then keep adding tuples as long as possible so $\mathbf{D}_{\mathcal{F}}$ is still in $\mathcal{L}$.

Finally, let us remark that one can prove that $\mathbf{U}'$ has a finite presentation in the sense of [17].

The situation of course changes if we consider more complicated (non-monadic and non-trees) lifted classes $Forb_e(\mathcal{F}')$ and in fact it has been proved in [21] that any NP language is polynomally equivalent to a class $\psi(Forb_e(\mathcal{F}'))$ for a finite set $\mathcal{F}'$ set of lifted structures. As these structures may be supposed to be connected (by adding dummy edges) we have the following corollary of Theorem 1.3:

Corollary 5.2 *Every NP problem is polynomally equivalent to membership problem for a class $Age(\mathbf{U})$ where $\mathbf{U}$ is shadow of an ultrahomogeneous structure $\mathbf{U}'$.*

Constraint Satisfaction Problems $CSP(\mathbf{H})$ for countable templates $\mathbf{H}$ were investigated in [3] and it has been shown there that the basic results of so called "algebraic method" [4] hold also for ω_0-categorical templates [3]. Recently, [2] proved that $CSP(\mathbf{H})$ for general countable template $\mathbf{H}$ encodes any problem in NP. It is not known whether $\mathbf{H}$ may be chosen ω_0-categorical. Corollary 5.2 complements this: lifts and shadows enable to code any NP language by membership problem for ages of shadows of ultrahomogeneous structures.

5.2 Forbidden cycles and Urysohn spaces (binary lifts)

We turn briefly our attention to binary lifts. This relates some of the earliest results on universal graphs with recently intensively studied Urysohn spaces:

We will consider a finite family $\mathcal{F}$ consisting of graphs of odd cycles of lengths $3, 5, 7, \ldots l$. As shown by [19] (see also [6]) those families have universal graphs in $Forb_e(\mathcal{F})$ and as shown by [7] those are the only classes defined by forbidding a finite set of cycles. These classes also form especially easy families of pieces. In fact each piece is an undirected path of length at most l, where l is the length of longest cycle in $\mathcal{F}$ with both ends of the patch being roots. This allows particularly easy description of the lifted structure.

We use the following definition which is motivated by metric spaces. When specialized to graphs, this definition is analogous to (corrected form) of an s-structure [19]. However this approach also gives new easy description (i.e. finite presentation) of the Urysohn space [27].

Definition 5.1 *Pair (a, b) is considered to be* even-odd pair *if a is even non-negative integer or ω and b is odd non-negative integer or ω.*

For even-odd pairs (a, b) and (c, d) we say that $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$. Consider $a + \omega = \omega$ and $\omega + b = \omega$. Put:

$$(a, b) + (c, d) = (\min(a + c, b + d), \min(a + d, b + c)).$$

For a set S, function d from S to even-odd pairs is called even-odd distance function on S *if following conditions are satisfied:*

1. $d(x, y) = (0, b)$ *if and only if* $x = y$*;*
2. $d(x, y) = d(y, x)$*;*
3. $d(x, z) \leq d(x, y) + d(y, z)$*.*

Finally pair (S, d) where d is even-odd distance function for S is called even-odd metric space*.*

Note that the even-odd metric spaces differ from usual notion of metric space primarily by the fact that the ordering of values of distance function is not linear, but forms a 2-dimensional partial order. Some basic results about metric spaces are valid even in this setting.

Even-odd metric space can form a stronger version of the distance metric on the graph. For graph $\mathbf{G}$ we can put $d(x, y) = (a, b)$ where a is length of the shortest walk of even length connecting x and y, while b is the length of shortest walk of odd length.

The even-odd distance metric specify length of all possible walks: for a graph $\mathbf{G}$ and a even-odd distance metric d we now have a walk connecting x and y of length a iff $d(x, y) = (b, c)$ such that $b \leq a$ for a even or $c \leq a$ for a odd.

It is well known that universal and homogeneous metric space exists for several classes of metric spaces [11, 34]. Analogously we have:

Lemma 5.2 *There exists a generic even-odd metric space* $\mathbf{U}$*.*

Proof. We prove that class $\mathcal{M}$ of all even-odd metric spaces is an amalgamation class.

To show that $\mathcal{M}$ has the amalgamation property, take a free amalgam $\mathbf{D}$ of even-odd metric spaces $\mathbf{A}$, $\mathbf{B}$ over $\mathbf{C}$. This amalgam is not even-odd metric space, since some distances are not defined.

We can however define a walk from v_1 to v_t of length l in $\mathbf{D}$ as a sequence of vertices $v_1, v_2, v_3, \ldots v_t$ and distances $d_1, d_2, d_3, \ldots d_{t-1}$ such that $\sum_{i=1}^{t-1} d_i = l$ and d_i is present in the even-odd pair $d(v_i, v_i + 1)$ for $i = 1, \ldots, t - 1$.

We produce even-odd metric space $\mathbf{U}$ on same vertex set as $\mathbf{D}$ where distance of some vertices $a, b \in E$ is even-odd pair (l, l') such that l is the smallest even value such that there exists walk joining a and b of length l

in $\mathbf{D}$. l' is the smallest odd value such that there exists walk from a to b of length l'.

It is easy to see that $\mathbf{U}$ is even-odd metric space (every triangular inequality is supported by the existence of a walk) and other properties of amalgamation class follows from definition. $\square$

The graphs omitting odd cycles up to length l can be axiomatized by simple condition on their even-odd distance metric: there are no vertices x, y such that $d(x, y) = (a, b)$ where $a + b \leq l$. Denote by $\mathcal{K}_l$ the class of all countable even-odd metric spaces such that there are no vertices x, y such that $d(x, y) = (a, b)$ with $a + b \leq l$. The existence of universal and homogeneous even-odd metric space $\mathbf{U}_l = (U_l, d_l)$ for class $\mathcal{K}_l$ is simple consequence of Lemma 5.2. In fact $\mathbf{U}_l$ is a subspace of $\mathbf{U}$ induced by all those vertices v of $\mathbf{U}$ satisfying $d(v, v) = (0, b)$ and $b > l$.

Theorem 5.3 *For metric space $\mathbf{U}_l = (U_l, d_l)$ denote by $\mathbf{G}_l = (U_l, E_l)$ a graph on vertex set U_l where $\{x, y\} \in E_l$ iff $d(x, y) = (a, 1)$.*

For every choice of odd integer $l \geq 3$, $\mathbf{G}_l$ is universal graph for class of all graphs omitting odd cycles of length at most l.

Proof. Graph $\mathbf{G}_l$ is omitting odd cycles up to length l from the fact that any two vertices x, y on a odd cycle of length k have distance $d(x, y) = (a, b)$ where $a + b$ is at most l.

Now consider any countable graph $\mathbf{G} = (V, E)$ omitting odd cycles of length at most l. Construct corresponding even-odd distance metric space $(V, d_{\mathbf{G}})$. By universality argument $(V, d_{\mathbf{G}})$ is subspace of $\mathbf{U}_l$ and thus also $\mathbf{G}$ subgraph of $\mathbf{G}_l$. $\square$

The explicit construction of the Urysohn space, without using Fraïssé limit argument, is described in [27] and same technique can be carried to even-odd metric. This is captured by the following definition.

Definition 5.2 *The vertices of $\mathcal{U}$ are functions f such that:*

(1) The domain D_f of f is finite (possibly empty) set of functions and $\emptyset$.

(2) The range of f are even-odd pairs.

(3) For every $g \in D_f$ and $h \in D_g$, we have $h \in D_f$.

(4) D_f using metric $d_{\mathcal{U}}$ defined bellow forms a even-odd metric space.

(5) f defines an extension of even-odd metric space on vertices D_f by adding a new vertex. This means that $f(\emptyset) = (0, x)$ and for every $g, h \in D_f$ we have $f(g) + f(h) \leq d_{\mathcal{U}}(g, h)$ and $f(g) \geq f(h) + d_{\mathcal{U}}(g, h)$.

The metric $d_{\mathcal{U}}(f, g)$ is defined by:

1. *if $f = g$ then $d_{\mathcal{U}}(f, g) = f(\emptyset)$,*
2. *if $f \in D_g$ then $d_{\mathcal{U}}(f, g) = g(f)$,*
3. *if $g \in D_f$ then $d_{\mathcal{U}}(f, g) = f(g)$,*
4. *if none of above holds then $d_{\mathcal{U}}(f, g) = min_{h \in D_f \cap D_g} f(h) + g(h)$. Minimum is taken elementwise on the pairs.*

Theorem 5.4 *$(\mathcal{U}, d_{\mathcal{U}})$ is the generic even-odd metric space.*

This may be seen as yet another incarnation of Katětov functions [37], see also [34].

6 Indivisibility results

We call pair (A, B) a *partition of structure* $\mathbf{R}$ if A and B are disjoint sets of vertices of $\mathbf{R}$ and $A \cup B = R$. We denote by $\mathbf{A}$ the structure induced on A by $\mathbf{R}$ and by $\mathbf{B}$ the structure induced on B by $\mathbf{R}$.

The structure $\mathbf{R}$ is *weakly indivisible* if for every partion (A, B) of R for which some finite induced substructure of $\mathbf{R}$ does not have copy in $\mathbf{A}$, there exists copy of $\mathbf{R}$ in $\mathbf{B}$.

For finite minimal family of finite structures $\mathcal{F}$, we call structure $\mathbf{A}$ a *minimal homomorphic image* of $\mathbf{F} \in \mathcal{F}$ iff $\mathbf{A}$ is homorphic image of $\mathbf{F}$ and every proper substructure of $\mathbf{A}$ is in $Forb_h(\mathcal{F})$.

The weak indivisibility of homogeneous structures was studied by [36]. In this section we briefly discuss basic (in)divisibility results on universal structures of classes $Forb_h(\mathcal{F})$.

We say that age of structure $\mathbf{A}$ has *free vertex amalgamation* if any two finite substructures of $\mathbf{A}$ whose intersection is a singleton have free amalgamation. We make use of the following:

Theorem 6.1 ([36]) *Let $\mathbf{H}$ be a countable homogeneous structure with free vertex amalgamation. Then $\mathbf{H}$ is weakly indivisible.*

The construction of universal structures as shadows of ultrahomogeneous makes this result particularly easy to apply to obtain indivisibility results for universal structures for classes $Forb_h(\mathcal{F})$. This leads to the following partial classification of classes $\mathcal{F}$ that do admit weakly indivisible structure universal for $Forb_h(\mathcal{F})$.

Theorem 6.2 *Fix minimal finite family of finite structures $\mathcal{F}$.*

1. *The class $Forb_h(\mathcal{F})$ contains universal structure that is weakly indivisible if every vertex minimal cut C of every homomorphic image $\mathbf{A}$ of $\mathbf{F} \in \mathcal{F}$ is of size at least 2 and additionally the structure induced by $\mathbf{A}$ on C is connected and has no cuts of size 1.*

2. *All universal structures $\mathbf{U}$ are divisible if there is a structure $\mathbf{A}$ which is a minimal homomorphic image of $\mathbf{F} \in \mathcal{F}$ such that $\mathbf{A}$ contains cut C of size 1.*

Proof. To prove 2., fix $\mathbf{A}$, a minimal homomorphic image of $\mathbf{F} \in \mathcal{F}$ with vertex cut C of size 1. Denote by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ all pieces of $\mathbf{A}$ generated by cut C. Fix $\mathbf{U}$, universal structure for $Forb_h(\mathcal{F})$. Denote by U_i, $i = 1\ldots\ldots n$ set of all vertices v of $\mathbf{U}$ such that there is rooted homomorphism of $\mathcal{P}_i$ to $\mathbf{U}$ mapping root of $\mathcal{P}_i$ to u.

Structure induced on U_i on $\mathbf{U}$ is not universal as it does not contains a homomorphic image of $\mathbf{P}_i$. Similarly structure induced on $U \setminus \bigcup_{i=1}^{n} U_i$ is not universal since there is no homomorphic image of $\mathbf{P}_1$. As a result $\mathbf{U}$ is divided to finitely many substructures such that none is universal resulting in divisibility of $\mathbf{U}$.

1. follows from weak indivisibility of class $\mathcal{L}$. To apply Theorem 6.1 we only need to show that class $\mathcal{L}$ admits free vertex amalgamation. This follows directly from the construction of amalgam in proof of Theorem 1.3: the amalgam constructed is not free in general, every new tuple added has the property that there is homomorphism from the structure induced by cut generating the corresponding piece into the vertices of tuple. But since we have free amalgam of the shadow and since all cuts of all homomorphic images do not have cuts of size 1, we have free vertex amalgamation property. □

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