Some recollections on early work with Jan Pelant

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Abstract

In this note we consider three questions which can be traced to our early collaboration with Jan "Honza" Pelant. We present them from the contemporary perspective, in some cases extending our earlier work. The questions relate to Ramsey Theory, uniform spaces and tournaments.

1 Introduction

Jan's mathematical interactions with the authors date back to early 70's. Jan Pelant was a remarkable man whose influence on his contemporaries transcended Prague's mathematical life. He was an excellent mathematician with a gift for understanding and solving problems. Moreover, Jan Pelant was not just an expert in his own field. His interests and talents were broad and he could have been successful in other areas. His passing is a great loss to all of us.

Here we deal with his work related to 3 problems: Ramsey topological spaces, characters of uniformities and tournament algebras.

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2 Ramsey topological spaces

Ramsey theory was developing very rapidly during the 70's. One of the most significant changes was the fact that the original set theory (and graph theory) setting of Ramsey theory was generalized to other structures. These developments are, for example, nicely described in the first monograph devoted to Ramsey theory [7]. The following is an example of an extension to topology.

Definition 2.1 *A topological space Y is said to be point Ramsey for the space X if for every (set) partition $Y = Y_1 \cup Y_2$ one of the classes Y_i contains a subspace which is homeomorphic to X.*

In the classical Erdős-Rado notation this is denoted by $Y \to (X)^1_2$. If α parts are allowed we write $Y \to (X)^1_\alpha$. We say that a class $\mathcal{T}$ of topological spaces is point Ramsey if for every $X \in \mathcal{T}$ and every cardinal α there exists $Y \in \mathcal{T}$ such that

$$Y \to (X)^1_\alpha.$$

In [26] we proved the following statements:

Theorem 2.2

1. *The class $\mathcal{T}_0$ of all T_0-topological spaces is point Ramsey.*
2. *The class $\mathcal{T}_1$ of all T_1-topological spaces is point Ramsey.*

This is an easy result which is obtained by the lexicographic (nested) product.

It is not known if the class $\mathcal{T}_2$ of Hausdorf topological spaces is point Ramsey. Particularly, the following problem concerning the unit interval I popularized the study of Ramsey topological spaces.

Problem 2.3 *Is it true that for every α there exists β such that $I^\beta \to (I)^1_\alpha$?*

Problem 2.3 is related to the question of whether the class of completely regular spaces is point Ramsey. The above is contained in the conference volume of TOPOSYM'76 [26].

We were pleased to learn that this note was quickly followed by research by W. Weiss, V. I. Malyhin, S. Todorčević and others [14, 39, 40]. A survey article by W. Weiss about this research appeared in [41]. In fact the

TOPOSYM paper [26] contains only a sketch of the proof of Theorem 2.2 and, in hindsight, it proves more. Thus after 30 years we take the liberty to include here the following mild strengthening of [26]:

Theorem 2.4 *For every topological space X, every linear ordering $\leq_X$ of its points and every cardinal α there exists a linearly ordered topological space $(Y, \leq_Y)$ such that $(Y, \leq_Y) \to (X, \leq_X)^1_\alpha$.*

Proof Let $(X, \leq_X)$ and α be given. We define the underlying set of Y as X^α. Let $\leq_Y$ be the lexicographic ordering of sequences $(x_\iota : \iota < \alpha)$. The topology of Y will be defined by the subbasic neighborhoods $U(x^0, W, \gamma)$. For $\gamma < \alpha$, $x^0 = (x^0_\iota : \iota < \alpha)$ and a neighborhood W of x^0_γ (in X) we put $(x_\iota : \iota < \alpha) \in U(x^0, W, \gamma)$ if and only if $x_\iota = x^0_\iota$ for $\iota < \gamma$ and $x_\gamma \in U$.

We prove $(Y, \leq) \to (X, \leq)^1_\alpha$. Suppose for contrary that $(Y, \leq) \not\to (X, \leq)^1_\alpha$. Let $c : Y \to \alpha$ be a coloring of points of Y. We construct by transfinite induction points $x_\lambda \in X$ such that $c(u) \neq \lambda$ whenever $u \in Y$ such that $u_\gamma = x_\gamma$, for each $\gamma \leq \lambda$. Suppose that $\lambda < \alpha$ and $x_\gamma \in X(\gamma < \lambda)$ have already been constructed. Suppose on the contrary that there is no x_λ with the required property. This means that for each $v \in X$ there exists $y^v \in Y$ satisfying $y^v_\gamma = x_\gamma$ $(\gamma < \lambda)$, $y^v_\lambda = v$ and $c(y^v) = \lambda$. Then, the set $\{y^v : v \in X\}$ induces an ordered subspace of Y monotone (linearly) homeomorphic to $(X, \leq)$. Clearly the set is homogeneous for the coloring c, a contradiction with the choice of x_λ. Hence, we can construct the elements $x_\lambda(\lambda < \alpha)$ with the required property. Then, the sequence $x = (x_\lambda)_{\lambda<\alpha} \in Y$ satisfies $c(x) \neq \lambda$ for each $\lambda < \alpha$, a contradiction. ■

Remark 2.5 Recall that Theorem 2.4 deals with partitions of points only. Perhaps it makes sense to ask if a similar Ramsey type statement holds when pairs or, more generally, discrete n-tuples are partitioned. Since $\kappa \not\to (\omega)^\omega_2$ for any infinite cardinal κ [4, 5] it is unlikely that there is a Ramsey class of infinite topological spaces. For some related applications see [38]. In [28], we suggested an alternative (graph theoretical) proof of this partition relation. An interesting version of this proof was given in [39].

There are several beautiful Ramsey type results for topological restricted colorings (cf. [3, 6, 13]). For finite topological spaces, the full characterization of Ramsey classes is given in [23, 24]. Ramsey classes of finite structures are related to ultrahomogeneous structures [12, 22, 23], a connection which recently yielded a spectacular application in the context of topological dynamics [12].

Remark 2.6 Ramsey problems depend very much on the underlying category. The more restrictive maps lead to fewer subspaces and thus we can expect a richer spectrum of results. Examples of this phenomenon are Euclidean and geometric Ramsey theorems [16] and also metric Ramsey theorems [2, 17] (which should be distinguished from Ramsey theorem for finite metric spaces [25]). However, these questions were studied much later.

3 The point character of $\ell_p(\kappa)$

Let (X, ρ) be a metric space. An open covering $\mathcal{U}$ of (X, ρ) is a family of open subsets of X with $X = \bigcup \mathcal{U}$. We say that $\mathcal{U}$ is bounded if there exists $b > 0$ with the property that $diam\, U < b$ for all $U \in \mathcal{U}$. We also say that $\mathcal{U}$ is b-bounded if $diam\, U < b$ for all $U \in \mathcal{U}$. The covering $\mathcal{U}$ is called uniform if there exists $\varepsilon > 0$ such that for every $x \in X$ there is a $U \in \mathcal{U}$ which contains the ε-ball $B(x, \varepsilon) = \{y : \rho(x, y) < \varepsilon\}$. By a well-known theorem of A.H. Stone [36], every metric space is paracompact and hence every open covering $\mathcal{U}$ of (X, ρ) has a locally finite open refinement $\mathcal{V}$, i.e., there exists an open covering $\mathcal{V}$ with the following two properties:

1. for each $x \in X$ there is a neighborhood of x which meets only finitely many members of $\mathcal{V}$,

2. for every $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $V \subset U$.

A.H. Stone [37] asked whether the theorem remains valid when replacing the open covering and its refinement by uniform ones (see also [9]). In other words, is it true that in any metric space every uniform covering has a locally finite refinement? A space satisfying this property is said to have the Stone uniform property. It is clear that Euclidean spaces and more generally separable spaces have the Stone uniform property. However, it was shown independently by Pelant [29] and Schepin [35] that the space $\ell_\infty(\kappa)$ for κ sufficiently large does not have the Stone uniform property. Subsequently in [33] and in [32] we proved that the space $\ell_p(\kappa)$, $1 \le p < \infty$ and κ sufficiently large, does not have the Stone uniform property either. Here we present the result from [32] which is related to a paper from this volume [1].

For a family $\mathcal{E}$ of sets, we define $ord(\mathcal{E}) = sup\,\{|\mathcal{D}|^+ : \mathcal{D} \subset \mathcal{E},\ \bigcap \mathcal{D} \neq \emptyset\}$.

Definition 3.1 *Let (X, ρ) be a metric space. The point character $pc(X, \rho)$*

of (X,ρ) is the least cardinal β such that every uniform cover $\mathcal{U}$ of X has a uniform refinement $\mathcal{V}$ with $ord(\mathcal{V}) \leq \beta$.

A space with $pc(X,\rho) \leq \aleph_0$ is also called point finite. Point finite spaces are those satisfying the Stone uniform property. For any Euclidean space E_n we have that $pc(E_n) = n+2$. So the point character provides a generalization of the notion of dimension for the "infinite dimensional case".

For an infinite cardinal κ and $p \geq 1$ recall that $\ell_p(\kappa)$ is the Banach space whose elements are the real functions on κ such that $\sum_{i<\kappa} |f(i)|^p$ converges. The operations are pointwise and the norm is defined by

$$\|f\| = \left(\sum_{i<\kappa} |f(i)|^p \right)^{1/p}.$$

The main objective of this paragraph is to prove the following.

Theorem 3.2 *For any limit ordinal α we have*

$$pc\left(\ell_1(\omega_\alpha)\right) \geq \omega_\alpha.$$

For the proof we shall need the following lemma. Let X be a set. We denote the system of all n-element subsets of X by $[X]^n$.

Lemma 3.3 *Let $n \geq 2$ be an integer and let γ be any ordinal. For every mapping $f : [\omega_{\gamma+n-1}]^n \to \omega_{\gamma+n-1}$ with the property that for any $x, y \in [\omega_{\gamma+n-1}]^n$, $x \cap y = \emptyset$ implies $f(x) \neq f(y)$, there exists $C \subset [\omega_{\gamma+n-1}]^n$ with the following properties:*

1. *$|C| = \omega_\gamma$,*
2. *for any $x_1, x_2 \in C$, $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$,*
3. *$|\bigcap_{c \in C} c| = n-1$.*

For the proof see [1].

Proof of Theorem 3.2 Let $\mathcal{U} = \{B(x, \frac{1}{2}), x \in X\}$ be a cover consisting of all balls of diameter 1. We will show that any refinement $\mathcal{V}$ of $\mathcal{U}$ satisfies $ord(\mathcal{V}) \geq \omega_\alpha$. In fact, we will show that any 1-bounded covering $\mathcal{V}$ has this property.

Let us consider the topological subspace of $\ell_1(\omega_\alpha)$ on the set

$$\{f \mid f : \omega_\alpha \to [0,1],\ |cozf| < \omega_0 \text{ and } f(x) = 1/|cozf|, \text{ for } x \in cozf\}$$

where $cozf = \{m | \ f(m) \neq 0\}$. We denote this subspace by $F(\omega_\alpha)$.

As $\mathcal{V}$ is a uniform covering, there exists $\varepsilon > 0$ such that for every $x \in F(\omega_\alpha)$ there is a $V \in \mathcal{V}$ with $B(x, \varepsilon) \subset V$. Let us take n sufficiently large so that $1/n < \varepsilon/2$. Consider

$$F^n(\omega_\alpha) = \{f \mid f \in F(\omega_\alpha) \text{ and } |cozf| = n\}.$$

For any $M \in [\omega_\alpha]^n$, we denote by f_M the unique map in $F^n(\omega_\alpha)$ satisfying $coz(f_M) = M$. Let us define the mapping $g : [\omega_\alpha]^n \to \mathcal{V}$ so that for every $M \in [\omega_\alpha]^n$, $B(f_M, \varepsilon) \in g(M)$. In other words, the map g "chooses" for each $M \in [\omega_\alpha]^n$ a set of $\mathcal{V}$ containing $B(f_M, \varepsilon)$.

For any two disjoint M,$N \in [\omega_\alpha]^n$ we have $\mathrm{dist}(f_M, f_N) = 2$. Since $\mathcal{V}$ is 1-bounded, $g(M)$ and $g(N)$ must be different. Hence, the mapping g satisfies the assumption of Lemma 3.3.

Let now $\gamma < \alpha$. As α is a limit ordinal we have also $\omega_{\gamma+n-1} < \omega_\alpha$ and thus, by Lemma 3.3, there is a family $C \subset [\omega_{\gamma+n-1}]$ satisfying the following properties:

1. $|C| = \omega_\gamma$,
2. for any $c_1, c_2 \in C$, if $c_1 \neq c_2$, then $g(c_1) \neq g(c_2)$,
3. $|\bigcap_{c \in C} c| = n - 1$.

Fix $c \in C$. For each $c' \in C$ we have $\rho(f_c, f_{c'}) = \frac{2}{n} < \varepsilon$, and so $f_c \in B(f_{c'}, \varepsilon) \subset g(c')$. Hence c is contained in ω_γ elements of $\mathcal{V}$. Since $\gamma < \alpha$ was arbitrary, we infer that $pc(\ell_1(\omega_\alpha)) \geq \omega_\alpha$. ■

Finally, let us note that the proof for $p > 1$ is analogous. For more details see [29, 30, 34].

4 Tournaments and algebras

The first two papers [19, 31] of Jan Pelant deal with relations: [31] can be traced to a dimension question of M. Katětov while [19] is an abstract of the main activity of the combinatorial seminars in 1970 – 1971. It deals with the following notion:

Definition 4.1 *A tournament (X, R) is a reflexive relation which is complete and antisymmetric. Explicitly, R satisfies*

$$R \cup R^{-1} = X^2, \quad R \cap R^{-1} = \{(x,x) : x \in X\}.$$

Thus for $x, y \in X$, $x \neq y$ we have $(x,y) \in R \Longleftrightarrow (y,x) \notin R$.

In [19, 20, 21] we studied tournaments from the algebraic point of view: every tournament $T = (X, R)$ corresponds uniquely to the binary tournament algebra $(X, \cdot_T)$ defined by

$$x \cdot_T y = \begin{cases} x & \text{if } (x,y) \in R, \\ y & \text{if } (y,x) \in R. \end{cases}$$

In [19, 20, 21] we studied tournaments from the algebraic point of view: Every tournament $T = (X, R)$ corresponds uniquely to the binary tournament algebra $(X, \cdot_T)$ defined by $x \cdot_T y = z$ if $(x, y) \in R$ and $x = z$.

Clearly tournament algebras are just quasitrivial ($x \cdot y \in \{x, y\}$), commutative and idempotent algebras. Note also that $f : (X, R) \to (X', R')$ is a (relational) homomorphism if and only if $f : (X, \cdot_T) \to (X', \cdot_{T'})$ is an (algebraic) homomorphism.

This connection led us to investigate the tournament algebras thoroughly. This resulted in papers [20, 21] where we (among other things) characterized the congruence lattices of tournaments algebras. It also led to new notions such as the simple tournament.

Definition 4.2 *A tournament $T = (X, R)$ is simple if every non-constant homomorphism $f : T \to T$ is an automorphism. (These are now called core tournaments [8].)*

Inspired by the characterization of the groups of automorphisms of tournaments we proved that every such group can be represented by a simple tournament. We also characterized scores of simple tournaments and scores for which every tournament is simple (these are just scores $(1, 1, 1)$, $(2, 2, 2, 2, 2)$,$(3, 3, 3, 3, 3, 3, 3)$). It came then as a surprise that the this notion was studied independently at the same time by P. Erdős, A. Hajnal, E. Milner and Moon [5, 18]. We found this very encouraging.

Tournament algebras proved to be useful. Denote by $\mathcal{V}_T$ the variety (in the sense of Birkhoff) generated by the finite tournament algebras. In [20] we isolated infinitely many irreducible equations valid in $\mathcal{V}_T$ and posed as a problem whether $\mathcal{V}_T$ is finitely axiomatizable. This problem was solved by J. Ježek, M. Mároti and R. McKenzie [10] (there is no finite axiomatization). It appeared that tournament algebras form an important class (see, e.g., [15]). They played a role in Ramsey theory as well. We finish this paper by stating explicitly this connection.

Let $\mathcal{K}$ be a class of idempotent algebras (by this we mean that every single element subset induces a subalgebra). The notation $B \to (A)^1_k$ has the analogous meaning as above in Section 2 (for topological spaces). More generally given algebras A, B we also write $C \to (B)^A_k$ if the following statement holds:

For every partition of the set $\binom{C}{A}$ of all subalgebras of C which are isomorphic to A into k classes there exists a subalgebra B' of C, $B' \simeq B$, such that $\binom{B'}{A}$ is a subset of one of the classes of the partition. We say that $\mathcal{K}$ has the A-Ramsey property if for every positive k and every $A, B \in \mathcal{K}$ there exists C such that $C \to (B)^A_k$.

In [11] we proved:

Theorem 4.3

1. *Every variety $\mathcal{V}$ of idempotent algebras has the point Ramsey property.*
2. *The variety $\mathcal{V}_T$ generated by the tournament algebras has the A-Ramsey property if and only if A is the singleton.*

In [27] we investigated varieties of partially ordered sets and lattices. Particularly we characterized those lattices A for which the class of all finite distributive lattices has the A-Ramsey property and for which the class of all lattices have the A-Ramsey property. However, for the class $\mathcal{M}$ of all finite modular lattices the situation is not clear and still presents an open problem:

Problem 4.4 *Characterize those modular lattices A for which the class $\mathcal{M}$ has the A Ramsey property.*

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References

[1] C. Avart, P. Komjáth, T. Łuczak, and V. Rödl. Colorful flowers. *submitted*, 2007.

[2] Y. Bartal, N. Linial, M. Mendel, and A. Naor. On metric Ramsey-type phenomena. *Ann. of Math.*, 162:643–709, 2005.

[3] T. J. Carlson and S. G. Simpson. Topological Ramsey theory. In Nešetřil J. and Rödl V., editors, *Mathematics of Ramsey Theory*, pages 172–183. Springer Verlag, 1990.

[4] P. Erdős, A. Hajnal, A. Maté, and R. Rado. *Combinatorial Set Theory.* North-Holland, 1964.

[5] P. Erdős, A. Hajnal, and E. C. Milner. Simple one-point extensions of tournaments. *Mathematika*, 19:57–62, 1972.

[6] F. Galvin and K. Prikry. Borel sets and Ramsey's theorem. *J. Symbolic Logic*, 38:193–198, 1973.

[7] R. Graham, B. L. Rothschild, and J. Spencer. *Ramsey Theory.* Wiley, 1980.

[8] P. Hell and J. Nešetřil. *Graphs and Homomorphisms.* Oxford University Press, 2005.

[9] J. R. Isbell. *Uniform Spaces, Mathematical Surveys.* AMS, Providence, 1964.

[10] J. Ježek, M. Mároti, and R. McKenzie. Equations of tournaments are not finitely based. *Discrete Math.*, 211:243–248, 2000.

[11] J. Ježek and J. Nešetřil. Ramsey varieties. *European J. Combin.*, 4:143–147, 1983.

[12] A. S. Kechris, V. G. Pestov, and S. Todorcevic. Fraissé limits, Ramsey theory and topological dynamics of automorphism group. *GAFA*, 15:106–189, 2005.

[13] A. S. Kechris, S. Solecki, and S. Todorčević. Borel chromatic numbers. *Adv. Math.*, 141:1–44, 1999.

[14] V. I. Malyhin. On Ramsey spaces. *Sov. Math.*, 20:894–898, 1979.

[15] M. Mároti. *The variety generated by tournaments*. PhD thesis, Vanderbilt University, 2002.

[16] J. Matoušek. *Lectures on Discrete Geometry*. Springer Verlag, 2002.

[17] J. Matoušek and V. Rödl. On Ramsey sets in spheres. *J. Combin. Theory Ser. A*, 70:30–44, 1995.

[18] J. W. Moon. Embedding tournaments in simple tournaments. *Discrete Math.*, 2:389–395, 1972.

[19] V. Müller, J. Nešetřil, and J. Pelant. Either tournaments or algebras? *Comment. Math. Univ. Carol.*, 13:801–807, 1972.

[20] V. Müller, J. Nešetřil, and J. Pelant. Either tournaments or algebras? *Discrete Math.*, 11:37–66, 1975.

[21] V. Müller and J. Pelant. On strongly homogeneous tournaments. *Czechoslovak Math. J.*, 24(99):378–391, 1974.

[22] J. Nešetřil. For graphs there are only four types of hereditary Ramsey classes. *J. Combin. Theory Ser. B*, 46:127–132, 1989.

[23] J. Nešetřil. Ramsey classes and homogeneous structures. *Combin. Probab. Comput.*, 14(1):171–189, 2005.

[24] J. Nešetřil. Ramsey classes of topological and metric spaces. *Ann. Pure Appl. Logic*, 4(143):147–154, 2006.

[25] J. Nešetřil. Metric spaces are Ramsey. *European J. Combin.*, 28:457–468, 2007.

[26] J. Nešetřil and V. Rödl. Ramsey topological spaces. In Novák J., editor, *General topology and its relations to modern analysis and algebra IV, part B*, pages 333–337. Society of Czechoslovak Math. and Phys., 1977.

[27] J. Nešetřil and V. Rödl. Combinatorial partitions of finite posets and lattices - Ramsey lattices. *Algebra Univ.*, 19:106–119, 1984.

[28] J. Nešetřil and V. Rödl. Two remarks on Ramsey's theorem. *Discrete Math.*, 54:339–341, 1985.

[29] J. Pelant. Cardinal reflections and point character of uniformities. In Frolik Z., editor, *Seminar Uniform Spaces 1973-74*, pages 149–158. Math. Institute of Czechoslovak Academy of Science, 1975.

[30] J. Pelant, P. Holický, and O. Kalenda. $c(k)$ spaces which cannot be uniformly embedded into $c_0(\gamma)$. *Fund.Math.*, 192:245–254, 2006.

[31] J. Pelant and V. Rödl. On generating of relations. *Comment. Math. Univ. Carol.*, 14:95–105, 1973.

[32] J. Pelant and V. Rödl. On coverings of infinite-dimensional metric spaces. *Discrete Math.*, 108:75–81, 1992.

[33] V. Rödl. Canonical partition relations and point-character of ℓ_p-spaces. *Seminar Uniform Spaces (1976-77), Math. Institute of Czechoslovak Academy of Science*, pages 79–82, 1978.

[34] V. Rödl. Small spaces with large point character. *European J. Combin.*, 8:55–58, 1987.

[35] E. V. Schepin. On a problem of Isbell. *Dokl. Akad. Nauk, USSR*, 222:541–543, 1975.

[36] A. H. Stone. Paracompactness and product spaces. *AMS*, 54:977–982, 1948.

[37] A. H. Stone. Universal space for some metrizable uniformities quart. *J. Math.*, 11:105–115, 1960.

[38] R. Thomas. A combinatorial construction of a nonmeasurable set. *Amer. Math. Monthly*, 92:421–422, 1985.

[39] S. Todorčević. Partitions problems in topology. *Amer Math. Soc., Contemporary Math.*, 84, 1989.

[40] W. Weiss. Partitioning topological spaces. In Császár A., editor, *Császár Topology II*, pages 1249–1255. North-Holland, Amsterdam, 1980.

[41] W. Weiss. Partitioning topological spaces. In Nešetřil J. and Rödl V., editors, *Mathematics of Ramsey Theory*, pages 154–171. Springer Verlag, 1990.