Facial colorings using Hall's Theorem[*]

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Abstract

A vertex coloring of a plane graph is ℓ-facial if every two distinct vertices joined by a facial walk of length at most ℓ receive distinct colors. It has been conjectured that every plane graph has an ℓ-facial coloring with at most $3\ell + 1$ colors. We improve the currently best known bound and show that every plane graph has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. Our proof uses the standard discharging technique, however, in the reduction part we have successfully applied Hall's Theorem, which seems to be quite an innovative approach in this area.

1   Introduction

The Cyclic Coloring Conjecture of Ore and Plummer [17] is a well-studied problem in graph theory which also appears as Problem 2.5 in the monograph of Jensen and Toft [14]. The conjecture asserts that every plane

graph has a cyclic coloring with $\left\lfloor \frac{3}{2}\Delta^* \right\rfloor$ colors such that no face is incident with two vertices with the same color, where Δ^* is the maximum number of distinct vertices incident with a single face.

A generalization of the Cyclic Coloring Conjecture is provided through the notion of *facial colorings*. Two vertices of a plane graph G are ℓ-*facially adjacent* if they are joined by a walk with at most ℓ edges that is a subwalk of a facial walk of G. A coloring of a plane graph is ℓ-*facial* if no two distinct ℓ-facially adjacent vertices receive the same color. Observe that cyclic and ℓ-facial colorings coincide if $\Delta^* \leq 2\ell + 1$.

The Facial Coloring Conjecture [15] asserts that every plane graph has an ℓ-facial coloring with at most $3\ell + 1$ colors. If true, the Facial Coloring Conjecture implies the Cyclic Coloring Conjecture for odd Δ^* and yields the conjectured bound increased by 1 for even Δ^*. Observe that the bound offered by this conjecture is tight: for every $\ell \geq 1$, there exists a plane graph that is not ℓ-facially 3ℓ-colorable. Indeed, consider a plane embedding of the complete graph on 4 vertices and subdivide each of the three edges incident with one of the vertices $\ell - 1$ times. In the obtained plane graph, any two vertices are ℓ-facially adjacent and hence any ℓ-facial coloring must use a dedicated color for each of the $3\ell + 1$ vertices.

It has been proven that every plane graph has an ℓ-facial coloring with at most $\lfloor 18\ell/5 \rfloor + 2$ colors [15, 16]. In the case of 3-facial colorings, it is known that every plane graph has a 3-facial coloring with at most 11 colors [11]. In the present paper, we improve the general bound by showing that every plane graph has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. Our proof uses the standard discharging technique, which implies proving the reducibility of some configurations. We do so by applying Hall's Theorem.

We now briefly survey known upper bounds on cyclic colorings of plane graphs. The first upper bound of $2\Delta^*$ was proven by Ore and Plummer [17]. Borodin [5] slightly improved the bound to $2\Delta^* - 3$ for $\Delta^* \geq 8$. Progress has been made at the end of the nineties: Borodin, Sanders and Zhao [7] proved the bound of $\left\lfloor \frac{9}{5}\Delta^* \right\rfloor$, and the currently best known general bound $\left\lceil \frac{5}{3}\Delta^* \right\rceil$ is due to Sanders and Zhao [20]. Recently, Amini, Esperet and van den Heuvel [1] proved that for every $\varepsilon > 0$, there exists Δ_ε such that every plane graph of maximum face size $\Delta^* \geq \Delta_\varepsilon$ admits a cyclic coloring with at most $\left(\frac{3}{2} + \varepsilon \right)\Delta^*$ colors. They cleverly used a result by Havet, van den Heuvel, McDiarmid and Reed [9, 10] that the chromatic number of the square of a planar graph of maximum degree Δ is at most $\frac{3}{2}\Delta (1 + o(1))$.

There are also numerous results on plane graphs with small maximum face sizes Δ^*. The case of cyclic colorings of plane triangulations, i.e.,

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$\Delta^* = 3$, is equivalent to the famous Four Color Theorem [2, 3, 19]. The case of $\Delta^* = 4$ is Ringel's problem that was solved by Borodin [4, 6]. The conjecture is open for $\Delta^* \geq 5$. A related conjecture by Plummer and Toft [18] on cyclic colorings of 3-connected plane graphs is proven for graphs with large maximum face sizes [8, 12, 13].

2   Notations

A plane graph G is said to be ℓ-*minimal* if G has no ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors and every plane graph with less edges than G has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. Note that since every plane graph has an ℓ-facial coloring with at most $\lfloor 18\ell/5 \rfloor + 2$ colors, there are no ℓ-minimal graphs for $\ell \leq 40$. However, we will not use this assumption in Sections 3, 4 and 5, as the lemmas will be stated in full generality.

Given a graph and one of its edges $e = uw$, the *contraction* of e consists of replacing u and w by a new vertex adjacent to all the former neighbors of u and v. In doing so, we keep parallel edges if they arise. *Suppressing* a vertex means contracting one of its incident edges. The *skeleton* G^+ of a plane graph G is the graph obtained by recursively suppressing each vertex of degree two. There is a natural one-to-one correspondence between the faces of G and G^+, therefore we understand the faces of G and G^+ to be the same. An edge of G^+ which is also an edge of G is called *real*.

A vertex v of degree d is referred to as a *d-vertex*. A face f of G is said to be a *d-face* if it is incident with d edges in G^+ (since we show that every ℓ-minimal graph is 2-connected in Section 3 and we will use this notion only for ℓ-minimal graphs, we can afford being imprecise on whether bridges incident with f are counted once or twice). A vertex of degree at most d is referred to as a *$(\leq d)$-vertex*. We use a $(\geq d)$-vertex, a $(\leq d)$-face and a $(\geq d)$-face in analogous meanings.

Now, we state the well-known Hall Theorem.

Theorem 1 (Hall, 1935). *A bipartite graph with parts A and B admits a matching that covers every vertex of A if and only if for every set $S \subseteq A$ the number of vertices of B with a neighbor in S is at least $|S|$.*

We apply it in two different situations, which we briefly describe now. In the first one, we consider two graphs G_1 and G_2 that we want to glue, say, on a vertex v to form a new graph G. We have an ℓ-facial coloring of each of them, and we may assume that they agree on v. We aim at finding

a permutation of the colors for the coloring of, say G_2, such that the ℓ-facial coloring of G given by the coloring of G_1 and the new coloring of G_2 is ℓ-facial. We define an auxiliary bipartite graph H as follows. The vertex-set of H is composed of two sets A and B, each being a copy of the set of all colors, but the one of v. Next, for any pair of nodes $(a, b) \in A \times B$, we add an edge between a and b unless there is a vertex of G_1 colored a which is ℓ-facially adjacent in G to a vertex of G_2 colored b. Thus, the sought permutation is precisely a perfect matching of H.

The second application is the following. We consider a set of vertices, each of them having a list of prescribed colors. We want to color each vertex with a color from its list, so that no two vertices are assigned the same color. We construct a bipartite graph H with parts A and B. The part A is composed of a copy of each vertex, and the part B of a copy of each available color. There is an edge between a node $a \in A$ and a node $b \in B$ if the color corresponding to b belongs to the list of the vertex corresponding to a. Thus, the desired coloring is precisely a matching of H that covers A.

3   Connectivity

In this section, we establish that every ℓ-minimal graph G is 2-connected and its skeleton is 3-connected. We start with 2-connectivity.

Lemma 2. *Every ℓ-minimal graph G is 2-connected.*

Proof. Suppose the opposite. Let v be a cut-vertex of G such that one of the components of $G - v$ is as small as possible. Let C be this component. Let G_1 be the subgraph of G induced by the vertex v and the set of vertices $V(C)$ of C. Let G_2 be the graph $G - V(C)$. Note that we can assume that the subgraphs G_1 and G_2 of G share the outer face of G. Also observe that G_1 is either an edge or its outer face is bounded by a cycle as G_1 is 2-connected by the choice of v and C. Since G is an ℓ-minimal graph, there exist an ℓ-facial coloring c_1 of G_1 and an ℓ-facial coloring c_2 of G_2 using at most $\lfloor 7\ell/2 \rfloor + 6$ colors. We can assume without loss of generality that $c_1(v) = c_2(v)$.

Let $\mathcal{C}$ be the set of all $\lfloor 7\ell/2 \rfloor + 5$ colors different from $c_1(v)$. Our next aim is to find a permutation σ of $\mathcal{C}$ such that the coloring c defined on G by $c(w) = \sigma(c_1(w))$ if the vertex w belongs to C and $c(w) = c_2(w)$ otherwise is an ℓ-facial coloring of G. Note that there are at most 2ℓ vertices of $G_2 - v$ that are ℓ-facially adjacent with a vertex of C in G. Let $\mathcal{C}_2$ be the set of

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colors assigned by c_2 to such vertices. If the size of the outerface of G_1 is at most $\ell + 1$, then let $\mathcal{C}_1$ be the set of at most ℓ colors assigned by c_1 to the vertices of the outerface of G_1 distinct from v. We choose a permutation of $\mathcal{C}$ such that no color of $\mathcal{C}_1$ is mapped to a color of $\mathcal{C}_2$. This is possible since $|\mathcal{C}| \geq 3\ell$.

We next assume that the size of the outerface of G_1 is greater than ℓ. The existence of a permutation σ will then be obtained by applying Hall's Theorem. To this end, an auxiliary bipartite graph H is constructed. Its vertex-set is composed of two copies $\mathcal{C}^1$ and $\mathcal{C}^2$ of the set of all $\lfloor 7\ell/2 \rfloor + 5$ colors contained in $\mathcal{C}$. We call those vertices *nodes* to avoid confusion with the vertices of the graph G. We add an edge between two nodes $x \in \mathcal{C}^1$ and $y \in \mathcal{C}^2$ if there is no pair of two ℓ-facially adjacent vertices in G such that one of the vertices is a vertex of G_1 with the color x and the other is a vertex of G_2 with the color y. Observe that any perfect matching of H corresponds to a suitable permutation σ.

We now analyze the degrees of the nodes in H. Let $v_{-\ell}, \ldots, v_0, \ldots, v_\ell$ be a part of the facial walk of the outer face of G_1 such that $v_0 = v$. Note that if the size of the outer face of G_1 is smaller than $2\ell + 1$, some of these vertices coincide. The number of times a color is assigned is counted with multiplicity—i.e., a color assigned to a vertex appearing t times is considered to be assigned to t vertices of the walk. Each node of $\mathcal{C}^1$ has degree at least $\lfloor 5\ell/2 \rfloor + 5$: indeed, if a color of $\mathcal{C}$ is assigned to at most one of the vertices $v_{-\ell}, \ldots, v_{-1}, v_1, \ldots, v_\ell$, then the corresponding node of $\mathcal{C}^1$ is not adjacent in H to at most ℓ nodes of $\mathcal{C}^2$. If a color of $\mathcal{C}$ is assigned to two vertices, say v_i and v_j with $i < 0 < j$, then $j - i \geq \ell + 1$: otherwise, the vertices v_i and v_j must coincide (two ℓ-facially adjacent vertices that are distinct cannot have the same color), and hence $j - i \leq \ell$ would imply that the size of the outer face of G_1 is at most ℓ (as G_1 is 2-connected), which is the case that was already dealt with. Consequently, a node of $\mathcal{C}^1$ corresponding to such a color is not adjacent to at most $2\ell - (j - i) < \ell$ nodes of $\mathcal{C}^2$ in H.

On the other hand, each node y of $\mathcal{C}^2$ has degree at least $\lfloor 3\ell/2 \rfloor + 5$ in H since y can be non-adjacent only to the nodes corresponding to the colors assigned to the vertices $v_{-\ell}, \ldots, v_{-1}, v_1, \ldots, v_\ell$.

It remains to verify Hall's condition for H. Let $X \subseteq \mathcal{C}^1$. If $|X| \leq \lfloor 5\ell/2 \rfloor + 5$, then the set of neighbors of X in H has size at least $\lfloor 5\ell/2 \rfloor + 5$ since the minimum degree of a node of $\mathcal{C}^1$ is at least $\lfloor 5\ell/2 \rfloor + 5$. On the other hand, if $|X| > \lfloor 5\ell/2 \rfloor + 5$, then each node y of $\mathcal{C}^2$ is adjacent to at least one node of X as the degree of y is at least $\lfloor 3\ell/2 \rfloor + 5 > |\mathcal{C}^1| - |X|$. Hence, the neighbors of the nodes of X are all the nodes of $\mathcal{C}^2$. By Hall's
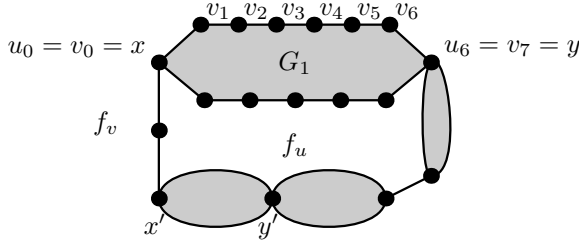
Figure 1: Notation used in the proof of Lemma 3.

Theorem, we conclude that H has a perfect matching, which completes the proof of the lemma. □

In the next lemma, we address the structure of 2-cuts in ℓ-minimal graphs.

Lemma 3. *Let G be an ℓ-minimal graph. If x and y are two (≥ 3)-vertices forming a 2-cut of G, then $G - \{x, y\}$ contains two components and one of the components is a path of 2-vertices between x and y.*

Proof. By Lemma 2, the graph G is 2-connected and we will use this fact without explicit reference in the proof. Let $\{x, y\}$ be a 2-cut of G composed of (≥ 3)-vertices such that $G - \{x, y\}$ has either three components, or two components neither being a path of 2-vertices. Unless G is formed by three paths of 2-vertices with the same end-vertices (in which case G has an ℓ-facial coloring with $3\ell + 1$ colors and thus it cannot be ℓ-minimal), the components of $G - \{x, y\}$ can be grouped to form subgraphs G_1 and G_2 whose intersection is precisely $\{x, y\}$, and such that G_1 is 2-connected and G_2 is not a path.

Let f_u and f_v be the two faces of G that contain both x and y, are not in G_1 but are adjacent to faces of G_1. Let $u_0 \ldots u_{k_u-1}$ and $v_0 \ldots v_{k_v-1}$ be the facial walks bounding the faces f_u and f_v such that $x = u_0 = v_0$ and u_1 and v_1 belongs to G_1. Set $u_{k_u} = u_0 = v_{k_v} = v_0 = x$. Finally, set d_u to be the index such that $u_{d_u} = y$ and d_v such that $v_{d_v} = y$; see Figure 1. For $i \in \{1, 2\}$, construct the graph G'_i from G_i by adding the edge xy. Since G is ℓ-minimal, all the graphs G_1, G'_1, G_2 and G'_2 have ℓ-facial colorings with at most $\lfloor 7\ell/2 \rfloor + 6$ colors.

We use an approach similar to that of Lemma 2. We fix a coloring of G_1 or G'_1, and of G_2 or G'_2, according to three different cases considered

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below. Let us say, for instance, that we have colorings c_1 and c_2 of G'_1 and G'_2, respectively. Note that x and y have different colors in those colorings, and we may assume that $c_1(x) = c_2(x)$ and $c_1(y) = c_2(y)$. We aim to find a permutation σ of the remaining $\lfloor 7\ell/2 \rfloor + 4$ colors such that the coloring of vertices of G'_1 with their original colors and recoloring vertices of G'_2 with the colors assigned by the permutation σ is an ℓ-facial coloring of G. To this end, we construct an auxiliary bipartite graph H with each part of size $\lfloor 7\ell/2 \rfloor + 4$. More precisely, let $\mathcal{C}^1$ and $\mathcal{C}^2$ be the two parts of H, where $\mathcal{C}^i$ corresponds to the colors of the vertices of G'_i. Two nodes $\alpha \in \mathcal{C}^1$ and $\beta \in \mathcal{C}^2$ are joined by an edge in H if and only if no vertex of G'_1 with the color α is ℓ-facially adjacent in G to a vertex of G'_2 colored β. A perfect matching of H then defines a suitable permutation σ of the colors as in the proof of Lemma 2.

In the rest of the proof, we consider several cases based on the values of d_u and d_v. These cases will also determine whether an ℓ-facial coloring of G_i or G'_i for $i \in \{1, 2\}$ should be used in the construction of the coloring of the whole graph G. In all the considered cases, we establish that the minimum degree of H is at least $\ell + 5$, and we later proceed jointly for all the cases.

- **The sum of d_u and d_v is at most $2\ell + 1$.** Note that d_u or d_v is at most ℓ and thus the vertices x and y are ℓ-facially adjacent in G_1. Hence, we can consider the ℓ-facial colorings of G_1 and G'_2. Let us estimate the minimum degree of H. A node α of $\mathcal{C}^1$ is not adjacent to at most 2ℓ nodes of $\mathcal{C}^2$ since there is a unique vertex of G_1 with the color α incident with f_u or f_v. The uniqueness follows from the assumption that $d_u + d_v \leq 2\ell + 1$. On the other hand, a node of $\mathcal{C}^2$ is not adjacent to at most $2\ell - 1$ nodes of $\mathcal{C}^1$ since it can be non-adjacent only to the nodes corresponding to the colors of (at most) $2\ell - 1$ vertices of G_1 incident with f_u or f_v. We conclude that the minimum degree of H is at least $\lfloor 3\ell/2 \rfloor + 4$.

- **The sum of d_u and d_v is bigger than $2\ell + 1$ and d_u or d_v is at most $\lfloor \ell/2 \rfloor$.** By symmetry, let us suppose that $d_u \leq \lfloor \ell/2 \rfloor$, and thus $d_v > \ell$. We again consider the ℓ-facial colorings of G_1 and G'_2. The colors of x and y are distinct in both the considered colorings. Let us proceed with estimating the minimum degree of H. If a color $\alpha \in \mathcal{C}^1$ is not assigned to a vertex u_i with $0 < i < d_u$, then there are at most 2ℓ edges from α missing in H. Similarly, there are at most 2ℓ missing edges if α is assigned to no vertex v_i with $0 < i < d_v$. Hence, assume

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that there are vertices u_i with $0 < i < d_u$ and v_j with $0 < j < d_v$ that are colored with α, and we choose the smallest i and j among all such vertices. Since the considered coloring is an ℓ-facial coloring of G_1, it must hold that $i + j > \ell$.

The vertex u_i is ℓ-facially adjacent in G to at most $\ell - i$ vertices of $G_2 - x$ through a facial walk including the vertex $x = u_0$ and the vertex v_j is ℓ-facially adjacent in G to at most $\ell - j$ vertices of $G_2 - y$ through a facial walk including the vertex $x = v_0$. Thus, there are at most $2\ell - i - j \le \ell - 2$ vertices of G_2 that are ℓ-facially adjacent in G through a facial walk including x to a vertex of G_1 colored with α. Similarly, there are at most $\ell - 2$ such vertices of G_2 that are ℓ-facially adjacent in G to vertices of G_1 with the color α through a facial walk including y. We conclude that there are at most $2\ell - 4$ edges missing at α in H and thus the degree of α is at least $\lfloor 3\ell/2 \rfloor + 8$.

A vertex of G_2 can only be ℓ-facially adjacent to at most 2ℓ vertices of $\{v_1, \ldots, v_{d_v-1}\}$. Since there are only $\lfloor \ell/2 \rfloor - 1$ vertices u_i with $i \in \{1, 2, \ldots, d_u - 1\}$, there are at most $\lfloor 5\ell/2 \rfloor - 1$ edges missing at every node β of $\mathcal{C}^2$ and thus its degree is at least $\ell + 5$.

- **The sum of d_u and d_v is bigger than $2\ell + 1$, both d_u and d_v are bigger than $\lfloor \ell/2 \rfloor$.** Since $d_u + d_v > 2\ell + 1$, we can also assume by symmetry that $d_v > \ell$. Let us next realize that we can assume that $k_u - d_u > \lfloor \ell/2 \rfloor$ and $k_v - d_v > \lfloor \ell/2 \rfloor$. Indeed, if $k_u - d_u \le \lfloor \ell/2 \rfloor$, we can choose a 2-cut $\{x', y'\}$ among the vertices $u_{d_u}, \ldots, u_{k_u}$ such that the 2-cut has the properties stated at the beginning of the proof and the role of G_1 will now be played by a subgraph of G_2 (see Figure 1). This will bring us to the first or second case (that was already analyzed) since $k_u - d_u < \lfloor \ell/2 \rfloor$. Similarly, we can assume that $k_u - d_u$ or $k_v - d_v$ is bigger than ℓ.

Consider now ℓ-facial colorings of G_1' and G_2'. If a color $\alpha \in \mathcal{C}^1$ is assigned to a single vertex u_i with $0 < i < d_u$, then at most $2\ell - \lfloor \ell/2 \rfloor - 1 \le \lfloor 3\ell/2 \rfloor$ vertices of $G_2 - \{x, y\}$ are ℓ-facially adjacent in G to u_i. If there are more such vertices u_i, let i and i' be the smallest and the largest index of such vertices. The vertex u_i is ℓ-facially adjacent in G to at most $\ell - i$ vertices of $G_2 - \{x, y\}$ and $u_{i'}$ to at most $\ell - (d_u - i')$ vertices. Since the vertices u_i and $u_{i'}$ are not ℓ-facially adjacent in G_1', it holds that $i + (d_u - i') \ge \ell + 1$. Hence, the vertices u_i and $u_{i'}$ are ℓ-facially adjacent in G to at most $\ell - 1$ vertices

of $G_2 - \{x, y\}$. Consequently, each of the vertices $u_1, \ldots, u_{d_u - 1}$ is ℓ-facially adjacent in G to at most $\lfloor 3\ell/2 \rfloor$ vertices of $G_2 - \{x, y\}$.

We can argue analogously for the vertices $v_1, \ldots, v_{d_v - 1}$. If there is a single vertex v_i with the color α, then it is ℓ-facially adjacent in G to at most $\ell - 1$ vertices of G_2 since $d_v > \ell$. If there are more such vertices v_i, then they all are ℓ-facially adjacent in G to at most $\ell - 1$ vertices of G_2. We conclude that at most $\lfloor 5\ell/2 \rfloor$ edges are missing at α and the degree of α in H is at least $\ell + 6$.

A completely symmetric argument applies for colors $\beta \in \mathcal{C}^2$ as both $k_u - d_u$ and $k_v - d_v$ are bigger than $\lfloor \ell/2 \rfloor$ and one of them is bigger than ℓ.

We now proceed jointly for all the three cases above. Let us count the number of edges between $\mathcal{C}^1$ and $\mathcal{C}^2$ that are missing in H. We consider first the vertices u_i with $0 < i < d_u$. If $i \leq \ell - 1$ then u_i can be ℓ-facially adjacent to at most $\ell - i$ vertices of G_2 because of a facial walk going through u_0. Similarly, if $d_u - \ell < i < d_u$, the vertex u_i is ℓ-facially adjacent to at most $\ell - (d_u - i)$ vertices of G_2 because of a facial walk going through u_{d_u}. Therefore, the number of edges missing in H between $\mathcal{C}^1$ and $\mathcal{C}^2$ due to the colors of the vertices u_i for $i \in \{1, 2, \ldots, d_u - 1\}$ is at most

$$\sum_{i=1}^{\ell-1} \ell - i + \sum_{i=d_u-\ell+1}^{d_u-1} \ell - (d_u - i) = 2 \cdot \sum_{i=1}^{\ell-1} \ell - i.$$

The same holds for the vertices v_j with $j \in \{1, 2, \ldots, d_v - 1\}$. Hence, the total number of edges missing in H between $\mathcal{C}^1$ and $\mathcal{C}^2$ is at most

$$m = 4 \sum_{i=1}^{\ell-1} (\ell - i) = 2\ell^2 - 2\ell.$$

We are now ready to verify the condition of Hall's Theorem for H. Let $X \subseteq \mathcal{C}^1$. If $|X| \leq \ell + 5$, then the condition holds since each node of X has at least $\ell + 5$ neighbors in $\mathcal{C}^2$. Similarly, if $|X| \geq \lfloor 5\ell/2 \rfloor$ then each node of $\mathcal{C}^2$ is adjacent to a node of X and the condition of Hall's Theorem is also fulfilled. Suppose that $\ell + 6 \leq |X| \leq \lfloor 5\ell/2 \rfloor - 1$. If the nodes of X have less than $|X|$ neighbors in $\mathcal{C}^2$, then the number of edges missing in H between $\mathcal{C}^1$ and $\mathcal{C}^2$ is at least

$$|X| \left(\left\lfloor \frac{7\ell}{2} \right\rfloor + 4 - |X| \right) \geq \left(\left\lfloor \frac{5\ell}{2} \right\rfloor - 1 \right)(\ell + 5) > m,$$

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a contradiction. $\qquad\square$

Lemma 3 immediately implies the following.

Lemma 4. *The skeleton G^+ of an ℓ-minimal graph G is 3-connected with no parallel edges.*

4 Small Faces

In this section, we analyze the structure of small faces of the skeleton of an ℓ-minimal graph. We start with showing that the edges of the skeleton cannot correspond to long paths.

Lemma 5. *Let G^+ be the skeleton of an ℓ-minimal graph G and e an edge of G^+. Let $v_0 \cdots v_{k+1}$ be the path of G corresponding to e, i.e., the vertices $v_1, \ldots, v_k$ are 2-vertices. Then $k \leq \max\{0, \lfloor \ell/2 \rfloor - 6\}$.*

Proof. Suppose on the contrary that $k > \lfloor \ell/2 \rfloor - 6$ and $k \geq 1$. Let G' be the graph obtained from G by suppressing the 2-vertex v_1. Since G is ℓ-minimal, G' has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. Based on this coloring, we construct an ℓ-facial coloring of G. The vertices distinct from v_1 preserve their colors. Each of the two faces incident with v_1 forbids assigning at most 2ℓ colors to v_1 but $k+1$ of these colors are counted twice (the colors assigned to $v_0, v_2, v_3, \ldots, v_{k+1}$). Hence, there are at most $4\ell - k - 1 \leq \lfloor 7\ell/2 \rfloor + 5$ colors that cannot be assigned to v_1. Consequently, there is a color that can be assigned to v_1 since there are $\lfloor 7\ell/2 \rfloor + 6$ colors in total. $\qquad\square$

A straightforward consequence of Lemma 5 is that edges incident with (≤ 4)-faces are real.

Lemma 6. *Let G^+ be the skeleton of an ℓ-minimal graph G. Every edge incident with a (≤ 4)-face in G^+ is real.*

Proof. If $\ell/2 - 6 < 1$, there is nothing to prove since Lemma 5 implies that every edge is real. In the rest, we assume that $\ell/2 - 6 \geq 1$ and establish that all edges incident with a d-face f of G^+ are real for $d \leq 4$.

Let $\alpha_1, \ldots, \alpha_d$ be the number of 2-vertices on the paths in G which are contracted to the d edges incident with f. Assume on the contrary that $\alpha_1 > 0$ and let v be one of the 2-vertices on the corresponding path.

The graph G' obtained from G by suppressing the vertex v has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors since G is ℓ-minimal. We aim to extend the coloring to v: there are at most 2ℓ colors that cannot be assigned to v because of the vertices of the face incident with v distinct from f. There are also at most $\sigma = \sum_{i=2}^{d} \alpha_i + 2$ additional colors that cannot be assigned to v since they appear on the vertices of f. By Lemma 5, we know that $\sigma \leq \lfloor 3\ell/2 \rfloor - 16$. Thus, there are at most $2\ell + \sigma \leq \lfloor 7\ell/2 \rfloor - 16$ colors that cannot be assigned to v. So there exists a color that can be assigned to v, which contradicts our assumption that G is ℓ-minimal. □

Since the edges incident with (≤ 4)-faces in the skeleton of an ℓ-minimal graphs are real, no two such faces can be adjacent, as stated in the next lemma.

Lemma 7. *The skeleton G^+ of an ℓ-minimal graph G contains no two adjacent (≤ 4)-faces if $\ell \geq 3$.*

Proof. By Lemma 6, all the edges incident with the two adjacent (≤ 4)-faces in G^+ are real. Let G' be the graph obtained from G by removing the edge shared by the two faces. Observe that every two vertices that are ℓ-facially adjacent in G are also ℓ-facially adjacent in G'. Since G is ℓ-minimal, G' has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. Consequently, G has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors which contradicts our assumption that G is ℓ-minimal. □

We use the following definitions in the sequel (see Figure 2 for examples). A face f' of G^+ is *strongly adjacent* to a face f if f' is adjacent to f and f' is not a (≤ 4)-face sharing a 3-vertex with f. Two adjacent faces f_1 and f_2 of G^+ *touch* if the faces f_1 and f_2 share in G a 2-vertex or a 3-vertex incident with a (≤ 4)-face distinct from f_1 and f_2. Such 2-vertices and 3-vertices are *strongly shared* by the faces f_1 and f_2.

We classify the faces f of the skeleton G^+ of an ℓ-minimal graph G as follows. Let k be the number of faces strongly adjacent to f. If $k \leq 2$ then f is a *circular face*. If $k = 3$ then f is a *triangular face*, and if $k = 4$ then f is a *quadrangular face*. If $k = 5$, the face f is *pentagonal*, if $k = 6$, the face f is *hexagonal*, and otherwise f is *polygonal*.

In the next lemma, we establish that G^+ has no circular faces, and moreover its triangular and quadrangular faces are precisely the 3-faces and 4-faces of G, respectively.
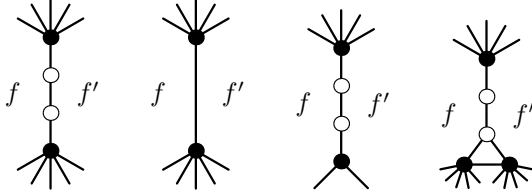
Figure 2: Examples of pairs of faces f and f' that are strongly adjacent; the vertices strongly shared by f and f' are represented by empty circles. The faces f and f' also touch in the first, third and fourth example.

Lemma 8. *Let G^+ be the skeleton of an ℓ-minimal graph G. A face of G^+ is triangular if and only if it is a 3-face of G^+, and it is quadrangular if and only if it is a 4-face of G^+. Moreover, G^+ has no circular face.*

Proof. Let k be the number of faces strongly adjacent to f. If f is a (≤ 4)-face, then it is strongly adjacent to each of its adjacent faces by Lemma 7. In particular $k \geq 3$ since G^+ is a simple graph by Lemma 4.

For the converse, we suppose on the contrary that $k \in \{2, 3, 4\}$ and yet f is a (≥ 5)-face of G^+. Let d be the number of faces adjacent to f in G^+, let $f_1, \ldots, f_d$ be these faces in the cyclic order around f, and for $i \in \{1, 2, \ldots, d\}$, let e_i be the edge shared by f and f_i in G^+. Further, let $i_1, \ldots, i_k$ be the indices of the faces strongly adjacent to f.

For $j \in \{1, 2, \ldots, k\}$, we define α_j to be the number of vertices strongly shared by f and f_{i_j}. By Lemma 5, it holds that $\alpha_j \leq \lfloor \ell/2 \rfloor - 4$.

We assert that the face f is incident with at most $k + \alpha_1 + \alpha_2 + \ldots + \alpha_k$ vertices in G. To see this, first note that f is incident to $d + x$ vertices in G, where x is the number of 2-vertices incident with f. Each face that is adjacent but not strongly adjacent to f is not incident to a 2-vertex of G. By Lemmas 6 and 7, each such face is incident with exactly one 3-vertex that is strongly shared by f and one of the faces f_{i_j}. As there are $d - k$ such faces, we infer that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \geq d - k + x.$$

Consequently, the face f is incident with at most

$$d + x = k + (d - k) + x \leq k + \alpha_1 + \alpha_2 + \ldots + \alpha_k$$

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vertices in G, as asserted.

If $\alpha_1+\alpha_2+\ldots+\alpha_k = 0$ then $d \leq k \leq 4$, a contradiction. (Again, k cannot be 2 in this case since $d \geq 5$.) Let us assume that $\alpha_1 + \alpha_2 + \ldots + \alpha_k > 0$. By symmetry, we can assume $\alpha_1 > 0$ and there is a vertex v strongly shared by f and f_{i_1}. Contract an edge incident with v and the face f in G. Since G is ℓ-minimal, the obtained graph has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. The vertices of G distinct from v keep their colors and we aim to extend the coloring to the vertex v. The vertex v cannot be assigned at most 2ℓ colors of ℓ-facially adjacent vertices on f_{i_1}, at most additional $k + \alpha_2 + \ldots + \alpha_k$ colors of vertices on f, and at most one additional color of the vertex of a possible quadrangular face incident with v. Hence, there are at most

$$2\ell + k + \alpha_2 + \ldots + \alpha_k + 1 \leq 2\ell + k + 3 \cdot \left(\left\lfloor \frac{\ell}{2} \right\rfloor - 4 \right) + 1 \leq \left\lfloor \frac{7\ell}{2} \right\rfloor - 7$$

colors that cannot be assigned to v. Hence, the coloring can be extended to v. $\qquad\square$

The next lemma bounds the size of a non-polygonal face in terms of ℓ.

Lemma 9. *Let G^+ be the skeleton of an ℓ-minimal graph G with $\ell \geq 6$. Every face of G^+ that is not polygonal has size at most $2\ell + 1$ in G.*

Proof. Let f be a non-polygonal face of G^+, and let k be the number of faces strongly adjacent to f. If $k \in \{3, 4\}$, then by Lemmas 6 and 8 the face f is a k-face of G. So we assume that $k \in \{5, 6\}$. Let d be the size of f in G, and d^+ the size of f in G^+. Set $\delta = \lfloor d/2 \rfloor$. Assume for the sake of contradiction that $d \geq 2\ell + 2$, and so $\delta \geq \ell + 1$. Note also that $d \geq 14$ since $\ell \geq 6$. Let $v_1, \ldots, v_d$ be the vertices incident with f in the cyclic order around f in G, let $f_1, \ldots, f_{d^+}$ be the faces incident with f in the cyclic order around it in G^+, and e_i the edge shared in G^+ by f and f_i for $i \in \{1, \ldots, d\}$. Further, let $i_1, \ldots, i_k$ be the indices of the strongly adjacent faces. Recall that $k \in \{5, 6\}$.

For $j \in \{1, 2, \ldots, k\}$, let A_j be the set of vertices strongly shared by f and f_{i_j}, and set $\alpha_j = |A_j|$. By Lemma 5, it holds that $\alpha_j \leq \lfloor \ell/2 \rfloor - 4$ for $j \in \{1, \ldots, k\}$. Since f is pentagonal or hexagonal, there are at most 6 vertices not included in $\cup_{j=1}^k A_j$. Therefore, the size d of the face f is at most $3\ell - 18$.

Let P_0 be the set of δ pairs formed by the vertices v_i and $v_{i+\delta}$ for $i \in \{1, \ldots, \delta\}$. Since $\delta \geq \ell + 1$, the pairs of vertices in P_0 are not ℓ-facially

adjacent: they are at facial distance $\delta \geq \ell + 1$ in f and they cannot be ℓ-facially adjacent through a different face by Lemma 4. Remove from P_0 the pairs such that at least one of the two vertices in the pair is not contained in $\cup_{j=1}^{k} A_j$. Let P be the resulting set of pairs and W the vertices contained in pairs in P. Since we have removed at most six pairs of vertices from P_0 and at most one vertex (in case that d is odd) is not included in a pair in P_0, it holds that $d - |W| \leq 13$.

Recall that $d > 13$ and choose an arbitrary vertex $v \in W$. Observe that v is either a 2-vertex or a 3-vertex incident to a (≤ 4)-face. The graph G' obtained from G by suppressing the vertex v has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors since G is ℓ-minimal. Uncolor now all the vertices of W; the other vertices of G keep their colors.

Let $L(v)$ for $v \in W$ be the list of all colors that can be assigned to the vertex v. If $v \in A_j$, there are at most $2\ell - (\alpha_j - 1)$ colors that cannot be assigned to v because of the face f_{i_j}, at most one additional color because of a possibly quadrangular face containing v, and at most $d - |W|$ colors because of the vertices incident with f that are not contained in W. Hence, $L(v)$ contains all but at most $2\ell - \alpha_j + 2 + d - |W|$ colors if $v \in A_j$. We conclude that

$$|L(v)| \geq \left\lfloor \frac{3\ell}{2} \right\rfloor + \alpha_j + 4 + |W| - d \geq \left\lfloor \frac{3\ell}{2} \right\rfloor + 4 + |W| - d \ .$$

If the vertices v and v' form a pair contained in P_0 and $L(v) \cap L(v') \neq \emptyset$, then color the vertices v and v' with a color $c \in L(v) \cap L(v')$ and remove c from the lists of all uncolored vertices. (Recall that v and v' are not ℓ-facially adjacent, so they may be assigned the same color.) Let ρ be the number of pairs of vertices colored in this way. Let W_0 be the subset of W of vertices not colored during this phase. Note that $2\rho = |W| - |W_0|$. If $v \in W_0$, then the size of the list of colors still available for v is at least

$$|L(v)| \geq \left\lfloor \frac{3\ell}{2} \right\rfloor + 4 + |W| - d - \rho.$$

We now show that the remaining vertices can be colored using Hall's Theorem. We consider an arbitrary subset $W' \subseteq W_0$ and aim to establish that

$$\left| \bigcup_{v \in W'} L(v) \right| \geq |W'| \ .$$

If W' does not include two vertices contained in the same pair in P, then $|W'| \le |W_0|/2$. Moreover, for an arbitrary vertex $v \in W'$ (recall that $d \le 3\ell - 18$),

$$
\begin{aligned}
|L(v)| &\ge \left\lfloor \frac{3\ell}{2} \right\rfloor + 4 + |W| - d - \rho \\
&\ge \left\lfloor \frac{3\ell}{2} \right\rfloor + 4 - \frac{d}{2} + \frac{|W| - d}{2} + \frac{|W| - 2\rho}{2} \\
&\ge \left\lfloor \frac{3\ell}{2} \right\rfloor + 4 - \frac{d}{2} - \frac{13}{2} + \frac{|W_0|}{2} \\
&\ge \frac{|W_0|}{2} \ge |W'| .
\end{aligned}
$$

Thus, the condition of Hall's Theorem is satisfied for W'.

If W' contains two vertices v and v' in the same pair in P, the lists $L(v)$ and $L(v')$ are disjoint. Thus,

$$
\begin{aligned}
|L(v) \cup L(v')| &\ge 3\ell + 7 + 2|W| - 2d - 2\rho \\
&\ge 3\ell + 7 + 2|W| - 2d - (|W| - |W_0|) \\
&\ge 3\ell + 7 - (d - |W|) - d + |W_0| \\
&\ge 3\ell - 6 - d + |W_0| \\
&\ge |W_0| \ge |W'| .
\end{aligned}
$$

Hence, the condition of Hall's Theorem is satisfied for all $W' \subseteq W_0$ and the coloring can be extended to all the vertices W. This contradicts our assumption that G is ℓ-minimal. $\qquad\square$

We finish this section with an auxiliary lemma on pentagonal faces.

Lemma 10. *Let G^+ be the skeleton of an ℓ-minimal graph G, f a pentagonal face of G^+, and f' a face adjacent to f. Suppose that $\ell \ge 5$. If f' is a triangular or quadrangular face that shares no 3-vertex with f, or f' is a pentagonal face, then the edge shared by f and f' in G^+ is not real.*

Proof. We proceed similarly to the proof of Lemma 8. Let d be the number of faces adjacent to f in G^+, let $f_1, \ldots, f_d$ be these faces in the cyclic order around f, and let e_i be the edge shared by f and f_i for $i \in \{1, \ldots, d\}$. Further, let $i_1, \ldots, i_5$ be the indices of the faces strongly adjacent to f. As in the proof of Lemma 8, we define α_j to be the number of vertices strongly

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shared by f and f_{i_j}. By symmetry, we can assume that $f_{i_5} = f'$ and thus $\alpha_5 = 0$. As in the proof of Lemma 8, we can argue that the face f is incident with at most $5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ vertices and that $\alpha_j \leq \lfloor \ell/2 \rfloor - 4$ for $j \in \{1, 2, 3, 4\}$.

If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > 0$, we consider a vertex v strongly shared by f and another face. If any, we choose v to be a 2-vertex, otherwise v is a 3-vertex incident to a (≤ 4)-face. Contract an edge incident with v and the face f in G. Since G is ℓ-minimal, the obtained graph has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. The vertices of G distinct from v keep their colors and we count the number of colors that cannot be assigned to v: there are at most 2ℓ colors of ℓ-facially adjacent vertices on the face distinct from f, at most additional $5 + 3 \cdot \alpha_i$ colors of vertices on f, and if v is a 3-vertex, at most one additional color of the vertex of a possible 4-face incident with v. Recall that $\alpha_i \leq \lfloor \ell/2 \rfloor - 4$. Hence, if v is a 2-vertex then there are at most

$$2\ell + 5 + 3 \cdot \left(\left\lfloor \frac{\ell}{2} \right\rfloor - 4 \right) = \left\lfloor \frac{7\ell}{2} \right\rfloor - 7$$

colors that cannot be assigned to v and the coloring can be extended to v. If v is a 3-vertex, then f is not incident to a 2-vertex. Consequently, each α_i is at most 2. Therefore, the number of colors that cannot be assigned to v is at most

$$2\ell + 5 + 3 \cdot 2 + 1 \leq \lfloor 7\ell/2 \rfloor + 5,$$

since $\ell \geq 5$, so the coloring can be extended to v. We conclude that all α_i are equal to 0.

Since $\alpha_i = 0$ for every $i \in \{1, \ldots, 5\}$, the face f is a 5-face in G. Note that if f' is a pentagonal face, we can symmetrically argue that f' is a 5-face in G^+. Let G' be the graph obtained from G by removing the edge shared by f and f'. As both f and f' are (≤ 5)-faces in G, the new face of G' is a (≤ 8)-face. It follows from the ℓ-minimality of G that G' has an ℓ-facial coloring of G with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. Since the new face of G' is a (≤ 8)-face and $\ell \geq 4$, the ℓ-facial coloring of G' is also an ℓ-facial coloring of G, a contradiction. $\qquad\square$

5   Adjacent Faces

In this section, we finish our analysis of configurations in the skeleton of ℓ-minimal graphs. We start with showing that no two pentagonal faces can share an edge.

Lemma 11. *The skeleton G^+ of an ℓ-minimal graph G contains no two adjacent pentagonal faces. In particular, no two pentagonal faces of G^+ touch.*

Proof. Assume for the sake of contradiction that G contains two pentagonal faces f^a and f^b that share an edge e_{ab} in G^+. By Lemma 10, the edge e_{ab} is not real, i.e., the faces f^a and f^b touch. Let $f_1^a, \ldots, f_4^a$ be the other faces strongly adjacent to f^a. For $j \in \{1, 2, 3, 4\}$, we let A_j be the set of vertices strongly shared by f^a and f_j^a, and $\alpha_j = |A_j|$. Let C be the vertices strongly shared by f^a and f^b and let $\gamma = |C|$. Observe that the size of f^a in G is $k^a + \alpha_1 + \cdots + \alpha_4 + \gamma$ where k^a is the number of vertices incident with f^a that are not strongly shared with another face, so $k^a \leq 5$. Analogously, we use k^b, $f_1^b, \ldots, f_4^b$, $B_1, \ldots, B_4$ and $\beta_1, \ldots, \beta_4$.

Let G' be the graph obtained from G by suppressing a 2-vertex lying on the path corresponding to the edge e_{ab}. Since G is ℓ-minimal, G' has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. The vertices not contained in $A_1, \ldots, A_4, B_1, \ldots, B_4$ and C keep their colors and we extend the coloring to the vertices contained in $A_1, \ldots, A_4, B_1, \ldots, B_4$ and C. Observe that the set $L(v)$ of colors that can be assigned to a vertex $v \in A_j$ contains at least

$$\lfloor 7\ell/2 \rfloor + 6 - (2\ell - \alpha_j + 1) - 1 - k^a = \lfloor 3\ell/2 \rfloor + \alpha_j + 4 - k^a$$

colors since there are at most k^a colored vertices incident with f^a and at most $2\ell - \alpha_j + 1$ vertices of $f_{i_j}^a$ that are ℓ-facially adjacent to v. The "-1" in the formula is needed in case v is incident with a 4-face.

The face f_1^a can coincide with at most one of the faces $f_1^b, \ldots, f_4^b$ since G^+ is 3-connected by Lemma 4. An analogous statement is true for f_2^a, f_3^a and f_4^a. Hence, we can form four disjoint pairs each containing one of the faces f_1^a, f_2^a, f_3^a and f_4^a, and one of the faces f_1^b, f_2^b, f_3^b and f_4^b such that each pair is formed by distinct faces. Among these pairs of faces, choose the pair f_j^a and $f_{j'}^b$ such that $\alpha_j + \beta_{j'}$ is maximum. For each $v \in A_j$, let $L_a(v)$ be the list $L(v)$ enhanced by the k^a colors of the vertices not contained in $A_1 \cup \cdots \cup A_4$, and for each $v' \in B_{j'}$, let $L_b(v')$ be the list $L(v')$ enhanced by the k^b colors not contained in $B_1 \cup B_2 \cup B_3 \cup B_4$. Color as many pairs of vertices $v \in A_j$ and $v' \in B_{j'}$ with the same color from $L_a(v)$ and $L_b(v')$ as possible, assigning distinct pairs distinct colors. Since $|L_a(v)| \geq \lfloor 3\ell/2 \rfloor + \alpha_j + 4$ for every $v \in A_j$, $|L_b(v')| \geq \lfloor 3\ell/2 \rfloor + \beta_{j'} + 4$ for every $v' \in B_{j'}$, and there are $\lfloor 7\ell/2 \rfloor + 6$ available colors, at least (and we

may assume, exactly)

$$\Delta = \left\lfloor \frac{3\ell}{2} \right\rfloor + \alpha_j + 4 + \left\lfloor \frac{3\ell}{2} \right\rfloor + \beta_{j'} + 4 - \left\lfloor \frac{7\ell}{2} \right\rfloor - 6 = \alpha_j + \beta_{j'} - \left\lceil \frac{\ell}{2} \right\rceil + 2$$

pairs of vertices are colored during this step. Note that $\Delta \le \min\{\alpha_j, \beta_{j'}\}$ since $\alpha_j \le \lfloor \ell/2 \rfloor - 4$ and $\beta_{j'} \le \lfloor \ell/2 \rfloor - 4$. Uncolor now the vertices $v \in A_j$ with the color conflicting with one of the k^a colors and $v' \in B_{j'}$ with the color conflicting with one of the k^b colors. Observe that there are still at least Δ pairs of vertices incident with f^a and f^b with the same color and there are no ℓ-facially adjacent vertices with the same color. By the choice of f_j^a and $f_{j'}^b$, it holds that

$$\Delta \ge \alpha_j + \beta_{j'} - \left\lceil \frac{\ell}{2} \right\rceil + 2 \ge \frac{1}{4} \left(\sum_{i=1}^{4} \alpha_i + \sum_{i=1}^{4} \beta_i \right) - \left\lceil \frac{\ell}{2} \right\rceil + 2 \ .$$

Next, we color the non-colored vertices of $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$ greedily by colors that can be assigned to such vertices. Let us verify that there is always at least one color available for every vertex $v \in A_1 \cup \cdots \cup A_4$; the analysis is analogous for an arbitrary vertex of $B_1 \cup \cdots \cup B_4$. When a vertex $v \in A_j$ is supposed to be colored, there are at most $\alpha_1 + \cdots + \alpha_4 - 1$ vertices of $A_1 \cup \cdots \cup A_4$ colored. Hence, the number of colors remaining in the list $L(v)$ is at least

$$\lfloor 3\ell/2 \rfloor + \alpha_j + 4 - k^a - \sum_{i=1}^{4} \alpha_i + 1 \ge \lfloor 3\ell/2 \rfloor - 3 \cdot (\lfloor \ell/2 \rfloor - 4) \ge 12,$$

and thus there is at least one color that can be assigned to v.

It remains to color the vertices of C. Since there are at least Δ colors assigned to both a vertex incident with f^a and a vertex incident with f^b, the number of colors that cannot be assigned to a vertex $v \in C$ is at most

$$\sum_{i=1}^{4} \alpha_i + \sum_{i=1}^{4} \beta_i + k^a + k^b + 1 - \Delta$$

$$\le \frac{3}{4} \left(\sum_{i=1}^{4} \alpha_i + \sum_{i=1}^{4} \beta_i \right) + \left\lceil \frac{\ell}{2} \right\rceil - 1 + k^a + k^b$$

$$\le \frac{3}{4} \left(\sum_{i=1}^{4} \alpha_i + \sum_{i=1}^{4} \beta_i \right) + \left\lceil \frac{\ell}{2} \right\rceil + k^a + k^b \ ,$$

where the additional "+1" in the first line corresponds to a possible additional vertex of a 4-face in case v has degree 3. Since there are $\lfloor 7\ell/2 \rfloor + 6$ available colors in total, the number of colors that can be assigned to a vertex $v \in C$ is at least

$$3\ell + 6 - k^a - k^b - \frac{3}{4}\left(\sum_{i=1}^{4}\alpha_i + \sum_{i=1}^{4}\beta_i \right) . \qquad (1)$$

The size of the face f^a is $k^a + \sum_{i=1}^{4}\alpha_i + \gamma$, and the size of f^b is $k^b + \sum_{i=1}^{4}\beta_i + \gamma$. By Lemma 9, each of these sizes is at most $2\ell + 1$. Thus,

$$\frac{3}{4}\left(\sum_{i=1}^{4}\alpha_i + \sum_{i=1}^{4}\beta_i \right) \leq \frac{3}{4}\left(4\ell + 2 - 2\gamma - k^a - k^b \right).$$

Plugging this inequality into (1), the number of colors yet available for a vertex $v \in C$ is at least

$$\begin{aligned}
& 3\ell + 6 - \left(k^a + k^b\right) - 3\ell - \frac{3}{2} + \frac{3}{2}\cdot\gamma + \frac{3}{4}\left(k^a + k^b\right) \\
\geq{} & \frac{3}{2}\cdot\gamma + \frac{9}{2} - \frac{1}{4}\left(k^a + k^b\right) \\
\geq{} & \frac{3}{2}\cdot\gamma + 2 \\
\geq{} & \gamma = |C| .
\end{aligned}$$

Hence, the vertices of C can be assigned mutually distinct colors and the coloring can be completed to an ℓ-facial coloring of G with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. $\qquad \square$

The last structural result we need asserts that the skeleton of an ℓ-minimal graph does not contain two adjacent hexagonal faces adjacent to the same pentagonal face.

Lemma 12. *Let G^+ be the skeleton of an ℓ-minimal graph G. If $\ell \geq 6$, then G^+ does not contain hexagonal faces f^a and f^b and a pentagonal face f^c such that the following three conditions hold simultaneously:*

1. *the faces f^a, f^b and f^c share a 3-vertex, or each of the faces f^a, f^b and f^c share an edge with a triangular face f' incident with 3-vertices only;*
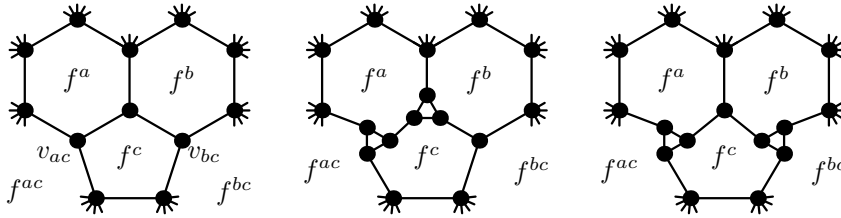
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Figure 3: Some of the configurations that cannot appear in the skeleton of an ℓ-minimal graph by Lemma 12.

2. the faces f^a and f^c share a 3-vertex v_{ac} that is not on a face adjacent with f^b, and if the third face incident with v_{ac} is triangular, then all its vertices have degree 3; and

3. the faces f^b and f^c share a 3-vertex v_{bc} that is not on a face adjacent with f^a, and if the third face incident with v_{bc} is triangular, then all its vertices have degree 3.

See Figure 3 for possible configurations in G^+ excluded by Lemma 12.

Proof. Let f^{ac} be the face incident with v_{ac} different from f^a and f^c, if it is not triangular. Otherwise, let f^{ac} be the face different from f^a and f^c and incident with the triangular face containing v_{ac}. By Lemma 4, f^{ac} is different from f^b, not triangular, and strongly adjacent to both f^a and f^c. The face f^{bc} is defined analogously. In particular, f^{bc} is a non-triangular face different from f^a and strongly adjacent to both f^b and f^c.

Let $f_1^a, \ldots, f_4^a$ be the faces strongly adjacent to f^a distinct from f^b and f^c, let $f_1^b, \ldots, f_4^b$ be the faces strongly adjacent to f^b distinct from f^a and f^c, and let f_1^c, f_2^c, f_3^c be the faces strongly adjacent to f^c distinct from f^a and f^b. By symmetry, we can assume that $f_1^a = f_1^c = f^{ac}$ and $f_1^b = f_3^c = f^{bc}$. For $j \in \{1, 2, 3, 4\}$, let A_j be the set of vertices strongly shared by f_j^a and f^a. The sets B_j for $j \in \{1, \ldots, 4\}$, and C_j for $j \in \{1, 2, 3\}$, are defined analogously. Further, for two distinct elements x and y of $\{a, b, c\}$, let D_{xy} be the set of the vertices strongly shared by the faces f^x and f^y. Let X be the union of all the sets A_j, B_j, C_j, D_{ab}, D_{ac} and D_{bc}, and let k^a, k^b and k^c be the number of vertices of f^a, f^b and f^c not contained in X, respectively. Since f^a and f^b are hexagonal and f^c is pentagonal, $k^a \leq 6$, $k^b \leq 6$ and $k^c \leq 5$. Finally, let $\alpha_j = |A_j|$, $\beta_j = |B_j|$, $\gamma_j = |C_j|$ and

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$\delta_{xy} = |D_{xy}|$. Without loss of generality, we can assume that $\alpha_2 \geq \alpha_3 \geq \alpha_4$ and $\beta_2 \geq \beta_3 \geq \beta_4$.

If $X = \emptyset$, then the faces f^a and f^b are 6-faces of G and the face f^c is a 5-face of G. Removing an edge shared by the faces f^a and f^c yields a graph with an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. As $\ell \geq 4$, this coloring is also an ℓ-facial coloring of G, which cannot exist since G is an ℓ-minimal graph. Hence, $X \neq \emptyset$.

Let G' be the graph obtained by suppressing a vertex of X with an edge contained in f^a, f^b or f^c. Since G is ℓ-minimal, G' has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors. The vertices not contained in the set X preserve their colors and we extend the obtained coloring to an ℓ-facial coloring of G. Let $L(v)$ be the set of colors available for a vertex $v \in X$. As in the proof of Lemma 11, we can argue that $|L(v)| \geq \lfloor 3\ell/2 \rfloor + 4 + \alpha_j - k^a$ for $v \in A_j$, $|L(v)| \geq \lfloor 3\ell/2 \rfloor + 4 + \beta_j - k^b$ for $v \in B_j$ and $|L(v)| \geq \lfloor 3\ell/2 \rfloor + 4 + \gamma_j - k^c$ for $v \in C_j$.

Since $f^{ac} = f_1^a = f_1^c$ and G^+ is 3-connected (by Lemma 4), it follows that f_1^a and f_2^c are distinct, and so are f_1^c and f_2^a. Similarly as in the proof of Lemma 11, for each $v \in A_1$ we let $L_a(v)$ be the list $L(v)$ enhanced by the k^a colors of the vertices not in $A_1 \cup \cdots \cup A_4$. (Note that this k^a colors are pairwise distinct by Lemma 9.) For $v \in C_2$, the list $L_c(v)$ is defined analogously. So $|L_a(v)| \geq \lfloor 3\ell/2 \rfloor + 4 + \alpha_1$ if $v \in A_1$ and $|L_c(v)| \geq \lfloor 3\ell/2 \rfloor + 4 + \gamma_2$ if $v \in C_2$. We color as many pairs of vertices from the sets A_1 and C_2 with the same color as possible, using the colors in the lists L_a and L_c. As there are $\lfloor 7\ell/2 \rfloor + 6$ colors in total, we deduce that at least

$$\alpha_1 + \gamma_2 + 2 - \left\lceil \frac{\ell}{2} \right\rceil$$

pairs of vertices are colored. Note that this number is smaller than α_1 and smaller than γ_2 by Lemma 5. We uncolor the vertices of A_1 that have been assigned one of the k^a colors already appearing on the vertices of f^a. Similarly, we uncolor those vertices of C_2 that received one of the k^c colors of $L_c(v) \setminus L(v)$. Observe that, at the end of this phase, there are at least $\alpha_1 + \gamma_2 + 2 - \lceil \ell/2 \rceil$ vertices of f^a that have the same color as a vertex of f^c.

We now color as many pairs of vertices from the sets A_2 and C_1 with the same color as possible. The list $L(v)$ of colors that can be assigned to a vertex $v \in C_1$ has size at least $\lfloor 3\ell/2 \rfloor + 4 + \gamma_1 - k^c$. Note that the fact that we colored some vertices of A_1 does not decrease this bound, since when computing it we implicitly assumed that all the vertices of f^{ac} were already colored. The list $L(u)$ of colors that can be assigned to a vertex $u \in A_2$ has

size at least $\lfloor 3\ell/2 \rfloor + 4 + \alpha_2 - k^a - |C|$ where C is the set of colors assigned to the vertices of A_1 in the previous step. As we just noted, no color of C is in $L(v)$. So, the size of $L(v) \cap L(u)$ for $v \in C_1$ and $u \in A_2$ is at least

$$\alpha_2 + \gamma_1 - k^a - k^c + 2 - \left\lceil \frac{\ell}{2} \right\rceil,$$

and hence we can color at least that number of pairs of vertices during this phase. By our previous arguments, the following estimate on the number Δ_{ac} vertices with the same color incident with f^a and f^c holds.

$$
\begin{aligned}
\Delta_{ac} \;\geq\;\; & \alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 - k^a - k^c + 4 - 2 \cdot \left\lceil \frac{\ell}{2} \right\rceil \\
\geq\;\; & \alpha_1 + \frac{\alpha_2 + \alpha_3 + \alpha_4}{3} + \gamma_1 + \gamma_2 - k^a - k^c + 3 - \ell .
\end{aligned}
$$

The face f_3^a can coincide with at most one of the faces f_3^b and f_4^b since G^+ is 3-connected by Lemma 3. Similarly, the face f_4^a coincides with at most one of these faces. Hence, we can form two pairs of distinct faces out of the faces f_3^a, f_4^a, f_3^b and f_4^b and choose the pair $(f_j^a, f_{j'}^b)$ such that $\alpha_j + \beta_{j'}$ is the biggest possible. Without loss of generality, we may assume that $j = 3$ and $j' = 4$, i.e., $\alpha_3 + \beta_4 \geq \alpha_4 + \beta_3$. We color as many pairs of vertices of A_3 and B_4 with the same color as possible. In doing so, we use the original list of available colors for the vertices of A_3, enhanced by the k^a colors initially assigned to the vertices of f^a. So some vertices of A_3 may get a color already assigned to a vertex of f^a. We uncolor each such vertex of A_3 at the end of this procedure. Similarly, we use for the vertices of B_4 their original list, enhanced by the k^b colors already assigned to vertices of f^b. Any vertex that is assigned one of the already used colors is uncolored at the end of the procedure. Consequently, the number of pairs of vertices (u, v) with the same color, and such that u is incident to f^a and v is incident to f^b is at least

$$\Delta_{ab} \geq \alpha_3 + \beta_4 + 2 - \left\lceil \frac{\ell}{2} \right\rceil \geq \frac{\alpha_3 + \alpha_4 + \beta_3 + \beta_4}{2} + 2 - \left\lceil \frac{\ell}{2} \right\rceil .$$

Finally, we do a similar coloring with pairs of vertices of B_2 and C_3, i.e., we do not remove the colors of the vertices of $B_3 \cup B_4$ from the lists of available colors for the vertices of B_2, and we add the k^b colors initially assigned to vertices of f^b; we do not remove the colors of the vertices of

$C_1 \cup C_2$ from the lists of available colors for the vertices C_3, but enhance them with the k^c colors initially assigned to vertices of f^c. Using those lists, we color as many pairs as possible with the same color. Then, we eventually uncolor those vertices whose color conflicts with a color we previously assigned to a vertex incident with the same face. Similarly, as in the previous two cases, there are at least

$$\Delta_{bc} \geq \beta_2 + \gamma_3 + 2 - \left\lceil \frac{\ell}{2} \right\rceil \geq \frac{\beta_2 + \beta_3 + \beta_4}{3} + \gamma_3 + 2 - \left\lceil \frac{\ell}{2} \right\rceil$$

pairs of vertices with the same color incident with f^b and f^c.

We now greedily color all the vertices of $A_1 \cup \cdots \cup A_4$, afterwards of $B_1 \cup \cdots \cup B_4$ and finally of $C_1 \cup \cdots \cup C_3$. Let us verify that this is indeed possible by examining one case in more detail (the others being similar). Assume that the last vertex of $A_1 \cup \cdots \cup A_4$ that is colored is a vertex $v \in A_4$. The number of colors still available for this vertex is at least

$$\left(\left\lfloor \frac{3\ell}{2} \right\rfloor + \alpha_4 + 4 - k^a \right) - \alpha_1 - \alpha_2 - \alpha_3 - (\alpha_4 - 1)$$
$$\geq \quad \left\lfloor \frac{3\ell}{2} \right\rfloor - 3 \cdot \left(\left\lceil \frac{\ell}{2} \right\rceil - 4 \right) - 1$$
$$\geq \quad 11 \ .$$

Next, we color greedily the vertices of D_{ab}. The number of colors that can be assigned to any vertex of D_{ab} before we start coloring the vertices of D_{ab} is at least (recall that $\alpha_1 + \cdots + \alpha_4 + \delta_{ab} + \delta_{ac} + k^a \leq 2\ell + 1$ and $\beta_1 + \cdots + \beta_4 + \delta_{ab} + \delta_{bc} + k^b \leq 2\ell + 1$)

$$\left\lfloor \frac{7\ell}{2} \right\rfloor + 6 - \sum_{i=1}^{4} (\alpha_i + \beta_i) - k^a - k^b + \Delta_{ab}$$
$$\geq \quad 3\ell + 7 - \frac{\sum_{i=1}^{4}(\alpha_i + \beta_i) + k^a + k^b}{2} - \frac{k^a + k^b}{2} - \frac{\alpha_1 + \alpha_2 + \beta_1 + \beta_2}{2}$$
$$\geq \quad \delta_{ab} + \ell + 6 - \frac{k^a + k^b}{2} - (\ell - 8)$$
$$\geq \quad \delta_{ab} + 14 - \frac{12}{2} \geq \delta_{ab} \ .$$

Hence, all the vertices except for those of $D_{ac} \cup D_{bc}$ are now colored.

For $x \in \{a, b\}$, we define N_{xc} to be the number of colors available for each vertex of D_{xc}. It is straightforward to check that $N_{xc} \geq \delta_{xc}$. Let us verify this statement for a vertex $v \in D_{bc}$, the other case being similar.

$$
\begin{aligned}
N_{bc} &\geq \left\lfloor \frac{7\ell}{2} \right\rfloor + 6 - \sum_{i=1}^{4} \beta_i - \delta_{ab} - k^b - k^c - \sum_{i=1}^{3} \gamma_i \\
&\geq \left\lfloor \frac{3\ell}{2} \right\rfloor + 5 + \delta_{bc} - k^c - 3 \cdot (\lfloor \ell/2 \rfloor - 4) \\
&\geq \delta_{bc} + 12 \ .
\end{aligned}
$$

Let us further estimate the number N_{ac}.

$$
\begin{aligned}
N_{ac} &\geq \left\lfloor \frac{7\ell}{2} \right\rfloor + 6 - \sum_{i=1}^{4} \alpha_i - \delta_{ab} - \sum_{i=1}^{3} \gamma_i - k^a - k^c + \Delta_{ac} \\
&\geq \left\lfloor \frac{5\ell}{2} \right\rfloor + 9 - \frac{2}{3} \sum_{i=2}^{4} \alpha_i - \delta_{ab} - \gamma_3 - 2k^a - 2k^c \ .
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
N_{bc} &\geq \left\lfloor \frac{7\ell}{2} \right\rfloor + 6 - \sum_{i=1}^{4} \beta_i - \delta_{ab} - \sum_{i=1}^{3} \gamma_i - k^b - k^c + \Delta_{bc} \\
&\geq 3\ell + 7 - \frac{2}{3} \sum_{i=1}^{4} \beta_i - \frac{\beta_1}{3} - \delta_{ab} - \gamma_1 - \gamma_2 - k^b - k^c \ .
\end{aligned}
$$

Next, we show that $N_{ac} + N_{bc} \geq 2(\delta_{ac} + \delta_{bc})$. Hence, at least one of the numbers N_{ac} and N_{bc} is $\delta_{ac} + \delta_{bc}$ or more. Therefore the vertices of D_{ac} and D_{bc} can be colored greedily. Indeed, if $N_{xc} \geq N_{yc}$ then we first color the vertices of D_{yc}, which is possible since $N_{xc} \geq \delta_{xc}$ as we noted earlier, and then those of D_{xc}. This yields the desired conclusion.

It only remains to verify that $\sigma = N_{ac} + N_{bc} \geq 2(\delta_{ac} + \delta_{bc})$.

$$
\begin{aligned}
\sigma &\geq \left\lfloor \frac{11\ell}{2} \right\rfloor + 16 - \frac{2}{3}\left(\sum_{i=1}^{4} \alpha_i + \beta_i \right) - 2\delta_{ab} - \frac{\beta_1}{3} - \sum_{i=1}^{3} \gamma_i - 2k^a - k^b - 3k^c \\
&\geq \frac{\ell}{3} + \left\lfloor \frac{\ell}{2} \right\rfloor + 13 + \frac{2}{3} + \frac{5}{3}(\delta_{ac} + \delta_{bc}) - \frac{2}{3}\delta_{ab} - \frac{\beta_1}{3} - \frac{4k^a}{3} - \frac{k^b}{3} - 2k^c \\
&\geq \frac{\ell}{3} + \left\lfloor \frac{\ell}{2} \right\rfloor - 7 + \frac{2}{3} + \frac{5}{3}(\delta_{ac} + \delta_{bc}) - \frac{\beta_1}{3} - \frac{2\delta_{ab}}{3} \\
&\geq \frac{1}{3} + 2(\delta_{ac} + \delta_{bc}) \geq 2(\delta_{ac} + \delta_{bc}) \, .
\end{aligned}
$$

The proof of the lemma is now finished. $\qquad\square$

6 The Discharging Phase

We will be discharging in the skeleton of an ℓ-minimal graph G. Each d-vertex of G^+ receives a charge of $2d - 6$ units and each d-face receives a charge of $d - 6$ units. Euler's formula implies that the sum of the initial amounts of charge assigned to all vertices and faces of G^+ is negative. We then apply the following rules to redistribute charge between vertices and faces of G^+:

Rule V1 Each (≥ 4)-vertex v incident with a 3-face $f = vv'v''$ sends 1 unit of charge to the face f unless one of the other faces incident with the edges vv' and vv'' is pentagonal.

Rule V2 Each (≥ 4)-vertex v incident with a 4-face $f = vv'v''v'''$ sends $1/2$ unit of charge to the face f unless one of the other faces incident with the edges vv' and vv''' is pentagonal.

Rule V3 Each (≥ 4)-vertex v sends 1 unit of charge to each incident pentagonal face.

Rule F1 Each face f that shares an edge vv' with a 3-face $f' = vv'v''$ sends 1 unit of charge to f' if the degree of v or v' is three unless both v and v' are 3-vertices and Rule V1 applies to v'' with respect to f'.

Rule F2 Each face f that shares an edge vv' with a 4-face f' sends $1/2$ unit of charge to f' if the degree of v or v' is three.

Rule F2$^+$ Each pentagonal face f that shares an edge vv' with a 4-face f' sends $1/2$ unit of charge to f' in addition to the charge sent by Rule F2 if one of the vertices v, v' is a 3-vertex and the other one is a (≥ 4)-vertex.

Rule F3 Each polygonal face f adjacent to a pentagonal face f' sends $1/3$ unit of charge to f' with the following two exceptions:

1. f' is incident with a (≥ 4)-vertex; or
2. there is a 3-face $v_1 v_2 v_3$ such that $v_1 v_2$ is an edge of f, $v_1 v_3$ is an edge of f', both v_1 and v_3 are 3-vertices and v_2 is a (≥ 4)-vertex.

In a series of lemmas, we show that the final charge of every vertex and every face in G^+ is non-negative. We start with analyzing the amount of the final charge of G^+.

Lemma 13. *The final charge of every vertex v of the skeleton G^+ of an ℓ-minimal graph is non-negative.*

Proof. If the degree d of v is three, the vertex v neither receives nor sends out any charge, and so its final charge is equal to zero. Hence, we can assume that v is a (≥ 4)-vertex. Let $f_1, \ldots, f_d$ be the faces incident with v in the cyclic order around v. We show that v sends to any pair of consecutive faces f_i and f_{i+1} at most 1 unit of charge in total, for $i \in \{1, 2, \ldots, d\}$. Fix i and let v' be the neighbor of v shared by the faces f_i and f_{i+1}.

By Lemma 7, at most one of the face f_i and f_{i+1} is a 3- or 4-face. If neither f_i nor f_{i+1} is a 3- or 4-face, then v can send charge to both f_i and f_{i+1} only if both f_i and f_{i+1} are pentagonal faces. This is excluded by Lemma 11. Consequently, at most one of the faces f_i and f_{i+1} is pentagonal and Rule V3 applies to at most one of the faces.

It remains to analyze the case where f_i or f_{i+1} is a 3- or 4-face. By symmetry, we can assume f_i to be such a face. Unless f_{i+1} is a pentagonal face, v sends at most 1 unit of charge to f_i (by Rule V1 or V2) and no charge to f_{i+1}. If f_{i+1} is a pentagonal face, v sends no charge to f_i and sends 1 unit of charge to f_{i+1} (by Rule V3).

We have shown that v sends to any two faces f_i and f_{i+1} at most 1 unit of charge. An averaging argument readily yields that v sends out at most $d/2$ units of charge. Since $d \geq 4$ and the initial charge of v is $2d - 6$, the statement of the lemma follows. $\square$

We now continue with analyzing the final charge of faces, starting with 3-faces. Recall that G has no circular face.

Lemma 14. *The final charge of every* 3*-face* $f = v_1 v_2 v_3$ *of the skeleton* G^+ *of an* ℓ*-minimal graph is non-negative.*

Proof. Let f_{ij} be the other face incident with the edge $v_i v_j$. Since neither of the faces f_{ij} can be a 3- or 4-face by Lemma 7, f does not send out any charge by Rules F1 or F2. We next distinguish four cases based on the number of (≥ 4)-vertices incident with f.

First, suppose that f is incident with 3-vertices only. Hence, Rule V1 applies to none of these vertices and each face sharing an edge with f sends 1 unit of charge to f by Rule F1. Since the initial charge of f is -3, the final charge of f is equal to zero.

Suppose now that f is incident with a single (≥ 4)-vertex. By symmetry, let v_1 be a (≥ 4)-vertex, and let v_2 and v_3 be 3-vertices. If Rule V1 does not apply to v_1 with respect to f, the face f receives 1 unit of charge from each of the faces f_{12}, f_{13} and f_{23} by Rule F1. If Rule V1 applies, then f receives 1 unit of charge from v_1 and 1 unit of charge from each of f_{12} and f_{13}. In both cases, the face f receives 3 units of charge in total, so its final charge equals zero.

If f is incident with exactly two (≥ 4)-vertices, say v_1 and v_2, then f receives 1 unit of charge from each of the faces f_{13} and f_{23} by Rule F1. Since the edge $v_1 v_2$ is real by Lemma 6, the face f_{12} cannot be pentagonal by Lemma 10. By Lemma 11, at most one of the faces f_{13} and f_{23} is pentagonal. Hence, Rule V1 applies to v_1 or v_2 with respect to f. In particular, the face f receives at least 1 unit of charge from v_1 or v_2. Since the face f receives at least 3 units of charge in total, its final charge is non-negative.

If all the vertices v_1, v_2 and v_3 are (≥ 4)-vertices, then none of the faces f_{12}, f_{13} and f_{23} is pentagonal by Lemmas 6 and 10. Hence, Rule V1 applies to all the three incident vertices with respect to f, and so f receives 3 units of charge in total, as desired. $\square$

Let us now analyze the final charge of 4-faces.

Lemma 15. *The final charge of every* 4*-face* $f = v_1 v_2 v_3 v_4$ *of the skeleton* G^+ *of an* ℓ*-minimal graph is non-negative.*

Proof. Let $f_{i\,i+1}$ be the other face incident with the edge $v_i v_{i+1}$ (indices modulo four). Since none of the faces $f_{i\,i+1}$ can be a 3- or 4-face by Lemma 7,

f does not send out any charge by Rules F1 or F2. We next distinguish several cases based on the number of (≥ 4)-vertices incident with f.

If f is incident with at most one (≥ 4)-vertex, it receives $1/2$ unit of charge from each adjacent face by Rule F2. Since, the initial charge of f is -2, the final charge of f is at least zero. A similar argument applies if f is incident with exactly two (≥ 4)-vertices which are not consecutive on f.

Suppose now that the face f is incident with exactly two (≥ 4)-vertices, which are consecutive on f. Let v_1 and v_2 be these two vertices. If f_{23} is a pentagonal face, then the face f receives 1 unit of charge from f_{23} by Rules F2 and F2$^+$ and $1/2$ unit of charge from each of the faces f_{34} and f_{41} by Rule F2. If f_{23} is not a pentagonal face, then f receives $1/2$ unit of charge from each of the faces f_{23}, f_{34} and f_{41} by Rule F2 and $1/2$ unit from the vertex v_2 by Rule V2 since the face f_{12} is not pentagonal by Lemma 10.

Suppose next that f is incident with three (≥ 4)-vertices, say v_1, v_2 and v_3. By Lemma 10, neither the face f_{12} nor the face f_{23} is pentagonal, and by Lemma 11, at most one of the faces f_{34} and f_{41} is pentagonal. Hence, Rule V2 applies to the vertex v_2 and at least one of the vertices v_1 and v_3 with respect to f. This yields that f receives at least 1 unit of charge from the incident (≥ 4)-vertices by Rule V2. Since Rule F2 applies to both f_{34} and f_{41}, the face f receives in total at least 2 units of charge, as desired.

Finally, we consider the case where the face f is incident with (≥ 4)-vertices only. As none of the adjacent faces can be pentagonal by Lemma 10, the face f receives $1/2$ unit of charge from each incident vertex by Rule V2, and hence its final charge is equal to zero. $\qquad\square$

The analysis of the final charge of hexagonal faces is quite straightforward.

Lemma 16. *The final charge of every hexagonal face f of the skeleton G^+ of an ℓ-minimal graph is non-negative.*

Proof. Let k be the number of faces adjacent to f and k' the number of 3- or 4-faces sharing a 3-vertex with f. Hence, $k - k' = 6$. The face f receives no charge by any of the rules, and it can send out charge only by Rules F1 and F2. Note that the amount of charge sent out by Rules F1 and F2 is at most k' units. Since the initial charge of f is $k - 6 = k'$ units, the final amount of charge of f is non-negative. $\qquad\square$

We next analyze the final charge of pentagonal faces.

Lemma 17. *The final charge of a pentagonal face f of the skeleton G^+ of an ℓ-minimal graph is non-negative.*

Proof. Let k be the number of faces adjacent to f and k' the number of 3- or 4-faces sharing a 3-vertex with f. Then, $k - k' = 5$. We distinguish two main cases based on whether f is incident with a (≥ 4)-vertex.

Suppose first that f is incident with a (≥ 4)-vertex. The face f can send out charge only by Rules F1, F2 and F2$^+$. By these rules, it can send at most 1 unit of charge to each 3- or 4-face that shares a 3-vertex with f. Hence, the amount of charge sent out by f is at most k' units. On the other hand, f receives at least 1 unit of charge from the incident (≥ 4)-vertex by Rule V3. Therefore, the final charge of f is at least

$$k - 6 - k' + 1 = 0.$$

In the rest of the proof, we assume that all the vertices incident with f are 3-vertices. Let $f_1, \ldots, f_k$ be the faces adjacent to f in the cyclic order around f, and let $l_1, \ldots, l_5$ be the indices of the strongly adjacent faces. By Lemma 11, each face f_{l_i} is hexagonal or polygonal.

Observe that $l_{i+1} - l_i \in \{1, 2\}$ for every $i \in \{1, \ldots, 5\}$ (indices modulo five). Indeed, if $l_{i+1} - l_i > 2$, then f_{l_i+1} and f_{l_i+2} are 3- and 4-faces. Since no two 3- or 4-faces can be adjacent by Lemma 7, the vertex shared by the faces f, f_{l_i+1} and f_{l_i+2} must be a (≥ 4)-vertex, which contradicts our assumption.

We next show that any 3-face f' adjacent to f is incident to 3-vertices only. If it were not the case, there would exist an index i such that $l_{i+1} - l_i = 2$, the vertex w incident with f_{l_i+1} and not incident with f is a (≥ 4)-vertex, and f_{l_i+1} is a 3-face. Since the faces f_{l_i} and f_{l_i+1} are hexagonal or polygonal, Rule V1 applies to w with respect to f_{l_i+1}. However, Rule F1 does not apply to f with respect to f_{l_i+1} and thus the amount of charge sent out by f totals to at most $k' - 1$ units. Consequently, the final amount of charge of f is non-negative. We conclude that all the vertices incident with f are 3-vertices.

In the rest of the proof, we call a pair of faces f_{l_i} and $f_{l_{i+1}}$ a *direct pair* if either $l_{i+1} - l_i = 1$ or f_{l_i+1} is a 3-face. In the latter case, all vertices incident with f_{l_i+1} must be 3-vertices. Lemma 12 implies that at least one of the faces forming a direct pair is polygonal.

Let k'' denote the number of direct pairs. Since at least one of the faces of a direct pair is polygonal, the face f receives $1/3$ from at least $\lceil k''/2 \rceil$ adjacent polygonal faces by Rule F3. Note that the exceptional case

described in Rule F3 cannot appear since all vertices incident with 3-faces sharing an edge with f are 3-vertices. On the other hand, if the faces f_{l_i} and $f_{l_{i+1}}$ do not form a direct pair, then $l_{i+1} = l_i + 2$ and $f_{l_{i+1}}$ is a 4-face. The face f sends to such a face $f_{l_{i+1}}$ only $1/2$ by Rule F2 and Rule F2$^+$ does not apply. We conclude that the face f sends out $(5 - k'') \cdot (1/2)$ units of charge to adjacent 4-faces and at most $(k' - (5 - k'')) \cdot 1$ units of charge to adjacent 3-faces. Hence, the total charge sent out by f is at most

$$k' - (5 - k'') + \frac{5 - k''}{2} = k' - \frac{5 - k''}{2} \ .$$

Since the initial charge of f is equal to $k - 6$ and f receives at least $(\lceil k''/2 \rceil) \cdot (1/3)$ units of charge, the final charge of f is at least

$$
\begin{aligned}
k - 6 + \left\lceil \frac{k''}{2} \right\rceil \cdot \frac{1}{3} - \left(k' - \frac{5 - k''}{2} \right) &= (k - k') - 6 + \frac{5}{2} + \left\lceil \frac{k''}{2} \right\rceil \cdot \frac{1}{3} - \frac{k''}{2} \\
&= \frac{3}{2} + \left\lceil \frac{k''}{2} \right\rceil \cdot \frac{1}{3} - \frac{k''}{2} \geq 0 \ .
\end{aligned}
$$

Note that we have used the fact that $k - k' = 5$ as f is pentagonal. Since $k'' \in \{0, \ldots, 5\}$, the estimate on the charge of f is always non-negative. This finishes the proof of the lemma. □

It remains to analyze the final charge of polygonal faces.

Lemma 18. *The final charge of a polygonal face f of the skeleton G^+ of an ℓ-minimal graph is non-negative.*

Proof. Let k be the number of faces adjacent to f and k' the number of 3- or 4-faces sharing a 3-vertex with f. Then, $k - k' \geq 7$. Further, let k'_4 be the number of 4-faces sharing a 3-vertex with f. Finally, let $f_1, \ldots, f_k$ be the faces adjacent to f in the cyclic order around f, and let $l_1, \ldots, l_{k-k'}$ be the indices of the strongly adjacent faces. Note that $l_{i+1} - l_i \in \{1, 2, 3\}$ for every $i \in \{1, \ldots, k - k'\}$ (indices modulo $k - k'$).

The face f does not receive any charge from neighboring vertices or faces. We now estimate the amount of charge sent out by f. By Rule F1, f sends out at most $k' - k'_4$ units of charge and by Rule F2, f sends out $k'_4/2$ units of charge. Rule F2$^+$ cannot apply to f. Altogether, f sends out at most $k' - k'_4/2$ units of charge to faces that are not strongly adjacent.

Let k'' be the number of indices i such that f sends $1/3$ unit of charge both to f_{l_i} and $f_{l_{i+1}}$ by Rule F3. Let us fix one such index i. Observe that

both the faces f_{l_i} and $f_{l_{i+1}}$ are incident with 3-vertices only. By Lemma 11, $l_{i+1} - l_i \geq 2$. If $l_{i+1} - l_i = 2$, the face $f_{l_{i+1}}$ cannot be a 3-face by Lemma 11. Hence, $f_{l_{i+1}}$ is a 4-face. Finally, if $l_{i+1} - l_i = 3$, then both $f_{l_{i+1}}$ and $f_{l_{i+2}}$ are not 3-faces, for otherwise the vertex shared by f, $f_{l_{i+1}}$ and $f_{l_{i+2}}$ would be a (≥ 4)-vertex by Lemma 7 and Rule F3 would not apply. Hence, at least one of $f_{l_{i+1}}$ and $f_{l_{i+2}}$ is a 4-face. We conclude that it is possible to associate to each index i such that f sends $1/3$ unit of charge both to f_{l_i} and $f_{l_{i+1}}$ by Rule F3, a 4-face adjacent to f, which is $f_{l_{i+1}}$ or $f_{l_{i+2}}$. Hence, $k'' \leq k'_4$.

As $k'' \leq k'_4 \leq k - k'$, we deduce that f sends out $1/3$ unit of charge by Rule F3 at most $\lfloor (k - k' + k'')/2 \rfloor$ times. Since the initial amount of charge of f is $k - 6$ units, the final amount of charge of f is at least

$$
\begin{aligned}
(k - 6) - \left(k' - \frac{k'_4}{2} \right) - \frac{1}{3} \left\lfloor \frac{k - k' + k''}{2} \right\rfloor \;&=\; \frac{1}{3} \left\lceil \frac{5(k - k') - 36 - k''}{2} \right\rceil + \frac{k'_4}{2} \\
&\geq\; \frac{k'_4}{2} + \frac{1}{3} \left\lceil \frac{-1 - k''}{2} \right\rceil \\
&\geq\; \frac{k'_4}{2} - \frac{k''}{3} \geq 0 \,.
\end{aligned}
$$

The lemma now follows. □

Lemmas 13–18 yield the main result of this paper.

Theorem 19. *Every plane graph has an ℓ-facial coloring with at most $\lfloor 7\ell/2 \rfloor + 6$ colors.*

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