Splitting finite antichains in the homomorphism order

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Extended Abstract

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Abstract

A structural condition is given for finite maximal antichains in the homomorphism order of relational structures to have the splitting property. It turns out that non-splitting antichains appear only at the bottom of the order. Moreover, we examine looseness and finite antichain extension property for some subclasses of the homomorphism poset. Finally, we take a look at cut-points in this order.

Keywords: homomorphism order, maximal antichain, splitting property

1 Introduction

A homomorphism from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ that preserves the edges of G, that is if $xy \in E(G)$ then $f(x)f(y) \in$

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$E(H)$. The omission of round or curly brackets indicates that the definition applies to both undirected and directed graphs, as does the discussion in the next few paragraphs. The definition applies equally well to finite and infinite graphs, but all graphs we consider in this paper are finite.

Existence of homomorphisms defines a relation on the class of all graphs, which is reflexive (the identity mapping is a homomorphism) and transitive (the composition of homomorphisms is a homomorphism); thus it is a preorder. Hence we write $G \leq H$ if there exists a homomorphism from G to H.

Furthermore we write $G \sim H$ if $G \leq H$ and at the same time $H \leq G$. Since $\leq$ is a preorder, the relation $\sim$ is an equivalence relation. The preorder $\leq$ induces a partial order on equivalence classes of $\sim$, which is called the *homomorphism order*.

A *core* is a graph which admits a homomorphism to no proper subgraph of itself. It is easy to show that there is exactly one core (up to isomorphism) in each equivalence class of $\sim$ (see [14]). So we have a canonical representative in each class, and sometimes it may be convenient to consider the homomorphism order to be a relation on the set of all cores rather than $\sim$-equivalence classes.

Several properties of the homomorphism order have been examined. Perhaps the earliest result is its universality: Hedrlín [13] proved that the homomorphism order contains every countable partial order as an induced suborder. It came as a surprise that the same is true about the homomorphism order of finite directed paths, which was proved by Hubička and Nešetřil [15, 16]. Another considered property was density. Welzl [20] showed that there is only one gap in the homomorphism order of undirected graphs. There are infinitely many gaps in the order of directed graphs, and they were described by Nešetřil and Tardif [18]; we give an overview in Section 2.3.

We are interested in finite maximal antichains in the homomorphism order.

As one of the first applications of the probabilistic method, P. Erdős [7] showed that there exist undirected graphs with arbitrary large girth and arbitrary large chromatic number. Thus if $\mathcal{A}$ is a set of undirected graphs that are not bipartite, there exists a graph H that is incomparable with all elements of $\mathcal{A}$: it suffices to take H which has both chromatic number and girth larger than any graph in $\mathcal{A}$. Hence each finite maximal antichain contains a bipartite graph or a graph with a loop; but there are only two bipartite cores and only one core with a loop. Therefore there are only three finite maximal antichains: $\{K_1\}$, $\{K_2\}$ and $\{\top\}$, where $\top$ denotes the graph consisting of a single vertex with a loop.

The situation is more intricate in the order of digraphs. There are four one-element maximal antichains: $\{K_1\}$, $\{\vec{P}_1\}$, $\{\vec{P}_2\}$ and $\{\top\}$; where $\top$ is the graph with a single vertex with a loop as before, and $\vec{P}_k$ denotes the directed path with k edges.

Maximal antichains of size 2 were classified by Nešetřil and Tardif [19]. They showed that they are the sets $\{F, D\}$ such that (F, D) is a *homomorphism duality*: a pair of graphs satisfying that $F \leq X$ is for any digraph X equivalent to $X \not\leq D$.

In general, a maximal antichain $\mathcal{A}$ *splits* if it is a disjoint union $\mathcal{A} = \mathcal{F} \cup \mathcal{D}$ such that (using fairly standard notation[1]) the whole poset $\mathcal{P} = \mathcal{F}^{\uparrow} \cup \mathcal{D}^{\downarrow}$. Equivalently, $(\mathcal{F}, \mathcal{D})$ is a splitting of $\mathcal{A}$ if and only if $\mathcal{A}^{\uparrow} = \mathcal{F}^{\uparrow}$ and $\mathcal{A}^{\downarrow} = \mathcal{D}^{\downarrow}$. The result mentioned in the previous paragraph implies that all maximal antichains of size 2 in the homomorphism order of digraphs split, because $\mathcal{A}^{\uparrow} = F^{\uparrow}$ and $\mathcal{A}^{\downarrow} = D^{\downarrow}$. Later, Foniok, Nešetřil and Tardif [12] proved that in the homomorphism order of digraphs, all finite maximal antichains of size greater than one split.

The splitting property of maximal antichains in general posets has been extensively studied, see [1, 2, 4, 5, 6, 8, 9, 10]. Here we are concerned with the splitting property of maximal antichains in the homomorphism order of finite relational structures. Relational structures are generalisations of graphs; they have a set of vertices and multiple sets of edges, each of which is a relation that is not necessarily binary. We introduce relational structures and the homomorphism order in Section 2.

Extension of antichains is the topic of Section 3. There we summarise and extend recent results of [3]. Several properties are defined that imply extensibility of finite antichains to splitting or non-splitting infinite maximal antichains. We also look at the problem of cut-points, elements of the poset that cut an interval into two. Some cut-points are again linked to dualities.

In Section 4, we show that almost all finite maximal antichains in this homomorphism order have the splitting property. The exceptions appear at the very bottom of the order, and are formed by structures with few edges. For the antichains that split we show how to partition them into $\mathcal{F}$ and $\mathcal{D}$. This has been known for digraphs [11] and structures with one relation of arbitrary arity [12].

[1] $\mathcal{F}^{\uparrow} = \{X : \exists F \in \mathcal{F},\ F \leq X\}$ is the upset generated by $\mathcal{F}$; and $\mathcal{D}^{\downarrow} = \{X : \exists D \in \mathcal{D},\ X \leq D\}$ is the downset generated by $\mathcal{D}$

Lastly, we list some open questions and suggestions for future work in Section 5.

2 Homomorphism order of relational structures

2.1 Relational structures

A *type* Δ is a sequence $(\delta_i : i \in I)$ of positive integers; I is a finite set of indices. A *relational structure* A of type Δ is a pair $(X, (R_i : i \in I))$, where X is a finite nonempty set and $R_i \subseteq X^{\delta_i}$; that is, R_i is a δ_i-ary relation on X. We often refer to a relational structure of type Δ as a Δ-*structure*. In this paper, we do not consider unary relations; so we assume that $\delta_i \geq 2$ for all $i \in I$.

If $A = (X, (R_i : i \in I))$, the *base set* X is denoted by $\underline{A}$ and the relation R_i by $R_i(A)$. The elements of the base set are called *vertices* and the elements of the relations R_i are called *edges*; this terminology is motivated by the fact that relational structures of type $\Delta = (2)$ are digraphs. To distinguish between various relations of a Δ-structure we speak about *kinds of edges* (so the elements of $R_i(A)$ are referred to as the *edges of the ith kind*).

Let A and A' be two relational structures of the same type Δ. A mapping $f : \underline{A} \to \underline{A'}$ is a *homomorphism* from A to A' if for every $i \in I$ and for every $u_1, u_2, \dots, u_{\delta_i} \in \underline{A}$ the following implication holds:

$$(u_1, u_2, \dots, u_{\delta_i}) \in R_i(A) \quad \Rightarrow \quad (f(u_1), f(u_2), \dots, f(u_{\delta_i})) \in R_i(A').$$

An *endomorphism* is a homomorphism from a Δ-structure to itself.

If there exists a homomorphism from A to A', we say that A is *homomorphic* to A' and write $A \leq A'$; otherwise we write $A \not\leq A'$. If A is homomorphic to A' and at the same time A' is homomorphic to A, we say that A and A' are *homomorphically equivalent* and write $A \sim A'$. If on the other hand there exists no homomorphism from A to A' and no homomorphism from A' to A, we say that A and A' are *incomparable* and write $A \parallel A'$.

A Δ-structure C is a *core* if it is not homomorphic to any proper substructure of itself. Equivalently, C is a core if every endomorphism of C is an automorphism. It is well-known (consult [14]) that every Δ-structure A is homomorphically equivalent up to isomorphism to exactly one core C;

then C is called *the core* of A. The class of all Δ-structures which are cores is denoted by $\mathcal{C}(\Delta)$.

2.2 Homomorphism order

Let Δ be a fixed type. The relation $\leq$ of being homomorphic is reflexive, as the identity mapping is a homomorphism from a Δ-structure to itself, and it is transitive, since the composition of two homomorphisms is a homomorphism. Thus $\leq$ is a preorder on the class of all Δ-structures.

This preorder induces a partial order on $\sim$-equivalence classes, which is naturally equivalent to the partial order $\leq$ on the class $\mathcal{C}(\Delta)$ of all core Δ-structures (taken up to isomorphism). This order is called the *homomorphism order*.

Note that $\mathcal{C}(\Delta)$ is a distributive lattice: The supremum of two structures A, B is their disjoint union $A + B$ and the infimum is the categorical product $A \times B$ (a precise definition is found e.g. in [12, 14]).[2]

2.3 Dualities and gaps

In this section we sum up the results of [12, 18]. We give the definition and the characterisation of homomorphism dualities and describe their connection to gaps in the homomorphism order.

Let $\mathcal{F}$ and $\mathcal{D}$ be two finite sets of core Δ-structures such that no homomorphisms exist among the structures in $\mathcal{F}$ and among the structures in $\mathcal{D}$. We say that $(\mathcal{F}, \mathcal{D})$ is a *finite homomorphism duality* (often just a *finite duality*) if for every Δ-structure A there exists $F \in \mathcal{F}$ such that $F \leq A$ if and only if for all $D \in \mathcal{D}$ we have $A \nleq D$. In the special case that $|\mathcal{F}| = |\mathcal{D}| = 1$, if $(\{F\}, \{D\})$ is a duality pair, that is if $\{A : F \leq A\} = \{A : A \nleq D\}$, the pair (F, D) is a *duality pair*.

For the full description of finite dualities we need some more notions. The *incidence graph* $\mathrm{Inc}(A)$ of a Δ-structure A is the bipartite multigraph $(V_1 \cup V_2, E)$ with parts $V_1 = \underline{A}$ and

$$V_2 = \mathrm{Block}(A) := \big\{(i, (a_1, \dots, a_{\delta_i})) : i \in I,\ (a_1, \dots, a_{\delta_i}) \in R_i(A)\big\},$$

and one edge between a and $(i, (a_1, \dots, a_{\delta_i}))$ for each occurrence of a as some a_j in an edge $(a_1, \dots, a_{\delta_i}) \in R_i(A)$.

[2] Strictly we should say that the supremum is *the core of* the disjoint union; similarly for infimum. We allow ourselves the concession to be a little imprecise, which we expect to make the exposition clearer rather than more confused.

A Δ-structure A is *connected* if its incidence graph $\mathrm{Inc}(A)$ is connected; it is a Δ-tree if $\mathrm{Inc}(A)$ is a tree; and it is a Δ-forest if $\mathrm{Inc}(A)$ is a Δ-forest. A *component* of a Δ-structure is its maximal connected substructure. Note that a Δ-structure is a Δ-forest if and only if each of its components is a Δ-tree.

Let us now give the characterisation of finite dualities.

2.1 Theorem ([12, 18]). *If (F, D) is a duality pair, then F is a Δ-tree. Conversely, if F is a Δ-tree, there exists a unique Δ-structure D (*the dual of F*) such that (F, D) is a duality pair.*

If $(\mathcal{F}, \{D\})$ is a finite duality, then all elements of $\mathcal{F}$ are Δ-trees and D is the product of their duals. If $(\mathcal{F}, \mathcal{D})$ is a finite duality, then all elements of $\mathcal{F}$ are Δ-forests and each element of $\mathcal{D}$ is the product of duals of some components of elements of $\mathcal{F}$. In this case, $\mathcal{D}$ is determined uniquely by $\mathcal{F}$. $\square$

In a poset $\mathcal{P}$, a *gap* is a pair (B, T) of elements of $\mathcal{P}$ such that $B < T$ and whenever $B \leq X \leq T$, then $X = B$ or $X = T$. In the homomorphism order of undirected graphs, there is exactly one gap: (K_1, K_2). For digraphs and general Δ-structures, the gaps are as follows:

2.2 Theorem ([18]). *A pair (B, T) is a gap in $\mathcal{C}(\Delta)$ if and only if there exists a Δ-tree F with dual D such that $T \times D \leq B \leq D$ and $T = B + F$. If T is connected, then there exists a Δ-structure B such that (B, T) is a gap if and only if T is a Δ-tree. Then $B = T \times D$.* $\square$

3 Antichains, looseness and cut-points

In [3], a number of properties of the homomorphism order of graphs and digraphs were investigated.

First we introduce some of the notions defined there. Let $(\mathcal{P}, \leq)$ be a poset. An element $Y \in \mathcal{P}$ is a *cut-point* if there exist $X, Z \in \mathcal{P}$ such that $X < Y < Z$ and the interval $[X, Z] = [X, Y] \cup [Y, Z]$. A subset $\mathcal{A} \subseteq \mathcal{P}$ is *cut-free* if there are no $Y \in \mathcal{A}$ and no $X, Z \in \mathcal{P}$ such that $X < Y < Z$ and $\mathcal{A} \cap [X, Z] = \mathcal{A} \cap ([X, Y] \cup [Y, Z])$.

We write $\mathcal{A} \parallel X$ to mean that the element X is incomparable with each element of $\mathcal{A}$.

A subset $\mathcal{A} \subseteq \mathcal{P}$ is an *upward loose kernel* in $\mathcal{P}$ if for every finite $\mathcal{S} \subseteq \mathcal{A}$ and every $X \in \mathcal{P} \setminus \mathcal{S}^{\uparrow}$ there exists $Y \in \mathcal{A}$ such that $X < Y$ and $\mathcal{A} \parallel Y$.

Analogously $\mathcal{A}$ is an *downward loose kernel* in $\mathcal{P}$ if for every finite $\mathcal{S} \subseteq \mathcal{A}$ and every $X \in \mathcal{P} \setminus \mathcal{S}^{\downarrow}$ there exists $Y \in \mathcal{A}$ such that $X > Y$ and $\mathcal{S} \parallel Y$.

A subset $\mathcal{A} \subseteq \mathcal{P}$ *has no finite maximal antichains* in $\mathcal{P}$ if there is no finite subset $\mathcal{S} \subseteq \mathcal{A}$ that is a maximal antichain in $\mathcal{P}$. Furthermore, $\mathcal{A}$ has the *finite antichain extension property* in $\mathcal{P}$ if for every finite antichain $\mathcal{S} \subseteq \mathcal{A}$ and every $X \in \mathcal{P} \setminus \mathcal{S}$ there exists $Y \in \mathcal{A}$ such that $\mathcal{S} \parallel Y$ but Y is comparable with X. Observe that if $\mathcal{A}$ is both an upward and a downward loose kernel in $\mathcal{P}$, then $\mathcal{A}$ has the finite antichain extension property in $\mathcal{P}$.

In short, the importance of these notions is this: Finite antichains in subsets with the finite antichain extension property extend to (infinite) splitting maximal antichains. And non-maximal antichains in loose kernels extend to (infinite) non-splitting maximal antichains. For details see [3].

Now let $\mathbb{G}$ be the homomorphism poset of undirected graphs and let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$ be the set of non-bipartite cores. Then we have:

3.1 Theorem ([3]). *The subset $\mathbb{G}'$ is both an upward loose kernel and a downward loose kernel in $\mathbb{G}$. Hence $\mathbb{G}'$ has the finite antichain extension property in $\mathbb{G}$ and it has no finite maximal antichains in $\mathbb{G}$. Moreover, $\mathbb{G}'$ is cut-free in $\mathbb{G}$.* $\square$

In addition, $\mathbb{G}'$ is dense, that is it contains no gaps.

As usual, the situation is more complex for digraphs or Δ-structures. To begin with, it is not entirely clear what subset should play the role of $\mathbb{G}'$. Let us have a look at some conditions, which are equivalent for undirected graphs.

3.2 Observation. *Consider $\mathbb{G}$, the homomorphism order of undirected graphs. The subset $\mathbb{G}'$ of $\mathbb{G}$ is each of the following:*

(1) the set of all cores that are not homomorphic to any tree;

(2) the set of all cores that have no component homomorphic to a tree;

(3) the set of all cores that contain a cycle;

(4) the set of all cores that contain a cycle in each connected component;

(5) the set of all cores that contain an odd cycle;

(6) the set of all cores that contain an odd cycle in each connected component. $\square$

In the homomorphism order of digraphs $\mathbb{D}$, though, the situation is contrasting: no two of the sets defined by the conditions (1)–(6) coincide. This fact motivates separate study of these subsets.

Hence we define subsets $\mathbb{D}_1, \mathbb{D}_2, \dots, \mathbb{D}_6$ of $\mathbb{D}$ so that we set $\mathbb{D}_k$ to be the set satisfying condition (k) of Observation 3.2. For instance, $\mathbb{D}_1$ is the set of all core digraphs that are not homomorphic to an orientation of a tree.

Analogously, in the homomorphism order of Δ-structures, let $\mathcal{C}(\Delta)_k$, for $k = 1, 2, 3, 4$, be the subset of $\mathcal{C}(\Delta)$ satisfying condition (k) of Observation 3.2.

Some properties of subsets of $\mathbb{D}$ defined in this way were examined in [3].

3.3 Theorem ([3]). *The subset $\mathbb{D}_5$ is a downward loose kernel in $\mathbb{D}$. The subset $\mathbb{D}_6$ is an upward loose kernel in $\mathbb{D}$; $\mathbb{D}_6$ has no finite maximal antichains; but $\mathbb{D}_6$ does not have the finite antichain extension property.* $\square$

The main tool for proving this theorem is *sparse incomparability*. This essentially guarantees the existence of digraphs (or Δ-structures) that are locally trees (they have large girth) but do not admit homomorphisms to and from some prescribed graphs. Using similar methods as in [3], and exploiting the characterisation of finite maximal antichains for digraphs, given in [11], we can show the following.

3.4 Theorem. *Neither $\mathbb{D}_3$ nor $\mathbb{D}_4$ has finite maximal antichains in $\mathbb{D}$. The subsets $\mathcal{C}(\Delta)_1$ and $\mathcal{C}(\Delta)_2$ have no finite maximal antichains. The subset $\mathcal{C}(\Delta)_1$ is a downward loose kernel in $\mathcal{C}(\Delta)$, and $\mathcal{C}(\Delta)_2$ is an upward loose kernel in $\mathcal{C}(\Delta)$; but neither $\mathcal{C}(\Delta)_1$ nor $\mathcal{C}(\Delta)_2$ has the finite antichain extension property.* $\square$

Cut-points are also related to the splitting property of antichains. It follows from [1, Theorem 2.1] as well as [10, Theorem 2.10] that a finite maximal antichain splits if it contains no cut-point. Let us look at cut-points a little closer.

3.5 Proposition. *Let T be a Δ-tree and let D be its dual. Then the Δ-structures $T+D$ and $T\times D$ are cut-points in the homomorphism order $\mathcal{C}(\Delta)$.*

Proof. Consider the interval $[K_1, T]$, which is equal to the downset generated by T. Suppose that X is a Δ-structure such that $X < T$. Then $X \le D$, because $T \not\le X$. Thus $X \le T \times D$. Hence the interval $[T \times D, T]$ contains only its end-points, that is $[T \times D, T] = \{T \times D, T\}$. Moreover, $[K_1, T \times D] \cup [T \times D, T] = [K_1, T]$, so $T \times D$ is a cut-point.

Similarly, if $D < X$, then $T + D \le X$. Hence $[D, T + D] \cup [T + D, \top] = [D, \top]$ and so $T + D$ is a cut-point. $\square$

No cut-free subset can contain cut-points; and there are cycles and odd cycles in many duals. Thus $T + D$ is a cut-point belonging to several of the classes defined above, and hence we have:

3.6 Corollary. *None of the classes $\mathcal{C}(\Delta)_1$, $\mathcal{C}(\Delta)_3$ and $\mathbb{D}_5$ is cut-free.* $\square$

4 Splitting antichains

In this section, we prove that many finite maximal antichains in $\mathcal{C}(\Delta)$ split. To do so, we construct a partition $(\mathcal{F}, \mathcal{D})$ for any finite antichain $\mathcal{A}$ and show that this partition is often a splitting of $\mathcal{A}$; that is, in many cases $\mathcal{A}^{\uparrow} = \mathcal{F}^{\uparrow}$ and $\mathcal{A}^{\downarrow} = \mathcal{D}^{\downarrow}$. So let us reveal our construction of the partition.

4.1 Splitting a finite antichain. Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be a finite maximal antichain in $\mathcal{C}(\Delta)$. Recursively, define the sets $\mathcal{F}_0$, $\mathcal{F}_1$, ..., $\mathcal{F}_n$ in this way:

1. Let $\mathcal{F}_0 = \emptyset$.
2. For $i = 1, 2, \dots, n$: check whether there exists a Δ-structure X satisfying
   (i) $A_i < X$,
   (ii) $F \nleq X$ for any $F \in \mathcal{F}_{i-1}$, and
   (iii) $A_j \nleq X$ for any $j > i$.

   If such a structure X exists, let $\mathcal{F}_i = \mathcal{F}_{i-1} \cup \{A_i\}$, otherwise let $\mathcal{F}_i = \mathcal{F}_{i-1}$.
3. Finally, let $\mathcal{F} = \mathcal{F}_n$ and $\mathcal{D} = \mathcal{A} \setminus \mathcal{F}$.

We are just about to prove that $(\mathcal{F}, \mathcal{D})$ defined above is a splitting of $\mathcal{A}$ unless $\mathcal{A}$ lies "at the bottom" of the order. To formalise "the bottom", consider D^*: the Δ-structure, whose components are exactly all trees with at most one edge of each kind. So T is a component of D^* if and only if T is a Δ-tree and $|R_i(T)| \leq 1$ for all $i \in I$.

A Δ-structure X is *small* if either $X \leq D^*$ or there exists $Y \leq D^*$ such that $Y < X$ and whenever $Y < Z < X$ then $Z \leq D^*$.

4.2 Theorem. *Let $\mathcal{A}$ be a finite maximal antichain in the homomorphism poset $\mathcal{C}(\Delta)$. Let $\mathcal{F}$, $\mathcal{D}$ be defined as in 4.1. If no element of $\mathcal{F}$ is small, then $(\mathcal{F}, \mathcal{D})$ is a splitting of $\mathcal{A}$, that is $\mathcal{A}^{\uparrow} = \mathcal{F}^{\uparrow}$ and $\mathcal{A}^{\downarrow} = \mathcal{D}^{\downarrow}$.*

Idea of proof. The construction in 4.1 ensures that $\mathcal{A}^{\uparrow} = \mathcal{F}^{\uparrow}$. So it remains to prove that $\mathcal{A}^{\downarrow} = \mathcal{D}^{\downarrow}$. We will assume that there is a Δ-structure $Y \in \mathcal{A}^{\downarrow} \setminus \mathcal{D}^{\downarrow}$ and prove that then $\mathcal{F}$ contains a small element.

By definition of Y, there exists $F \in \mathcal{F}$ such that $Y \leq F$. Using a variant of sparse incomparability, it can be shown that each element of $\mathcal{F}$ is homomorphic to a Δ-tree (it is *balanced*). Therefore Y is also balanced.

Next comes a cycle-growing trick. We consider forbidden paths (whose precise definition is technical and we omit it here) and construct big unbalanced structures W from them. Some considerations show that for each such W there is $F \in \mathcal{F}$ such that $F \leq W + Y$. Then it follows that $F \leq P + Y$, where P is the appropriate forbidden path. But since $F \nleq Y$, we have $P \nleq Y$.

Then we prove that a connected Δ-structure, to which no forbidden path is homomorphic, admits a homomorphism to a Δ-tree with at most one edge of each kind. Hence $Y \leq D^*$ whenever $Y \in \mathcal{A}^{\downarrow} \setminus \mathcal{D}^{\downarrow}$.

So we have $Y \leq F$ for some $F \in \mathcal{F}$. If $F \to D^*$, then F is small and we are done. Otherwise suppose $Y \leq X \leq F$. Then each X such that $Y < X < F$ satisfies that $X \leq D^*$, therefore F is again small. □

In conclusion, we remark that splitting finite antichains are homomorphism dualities in disguise. Indeed, the splitting $(\mathcal{F}, \mathcal{D})$ of a finite maximal antichain is a homomorphism duality. Conversely, let $(\mathcal{F}, \mathcal{D})$ be a homomorphism duality and let

$$\mathcal{A} = \mathcal{F} \cup \{D \in \mathcal{D} : D \nleq F \text{ for any } F \in \mathcal{F}\}.$$

Then $\mathcal{A}$ is obviously a finite maximal antichain in $\mathcal{C}(\Delta)$. For relational structures with one or two relations all maximal antichains, even the non-splitting ones, are created from finite dualities in this way. For structures with more than two relations this is currently unknown.

5 Conclusion and extensions

Theorems 3.3 and 3.4 do not cover all the properties for all the classes $\mathbb{D}_k$, $\mathcal{C}(\Delta)_k$. We would like to find out which classes have which properties. In particular, the question is interesting for structures that contain cycles only in some components, and for structures with balanced cycles.

Another class of digraphs is introduced in [3]: digraphs that contain a directed cycle. This subset has the finite antichain extension property and is cut-free. This class can be generalised to Δ-structures by looking at their *directed shadows* (these are directed multigraphs constructed by replacing each tuple $(x_1, \ldots, x_t)$ of the Δ-structure with the directed path $x_1 \to x_2 \to \cdots \to x_t$). This class is an upward loose kernel in $\mathcal{C}(\Delta)$ and it has the finite antichain extension property. Is it also a downward loose kernel in $\mathcal{C}(\Delta)$?

In Proposition 3.5 we describe infinitely many cut-points in the homomorphism order. Currently we do not know any other cut-points; however, it remains open to give the complete characterisation. In particular, can such a characterisation yield a new proof of the splitting property (recall that a finite maximal antichain splits if it contains no cut-points)?

Finally, the connection between gaps and dualities is not restricted to the homomorphism order. A similar theory was developed in [17] for Heyting lattices. Thus it is worth asking what additional axioms enable deriving simple conditions for maximal antichains to split.

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