A Surprising Permanence of Old Motivations (a not so rigid story)

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Abstract

This is not a survey article. Rather it is a personal statement written for a lifelong friend and collaborator. Still it is an ambition of this article to trace some of the key moments of our development in the past 40 years. In doing so perhaps some evidence has arisen which otherwise seems to be obscured by the hectic day-to-day academic life. Thus the title.*

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*This paper is based on the lecture by the author at the meeting in Victoria devoted to 60 birthday of P.Hell.

Introduction

In 1956 Eugene Wigner wrote an influential article [148] *The unreasonable effectiveness of mathematics in the natural science*. The paper became not only influential but also kind of paradigm for other papers about "unreasonable" or "surprising" effects, [17, 82, 32, 108] (thus this beeing also an evidence how important is to select the right title). I also chosen to paraphrase this title. But by doing so I should stress immediately that I am not analysing the phenomenon in the title *per se* (as Wigner did) but merely describing the situation which became (a little bit surprisingly) apparent when treating the main topic of this paper - the joint work of P. Hell and myself from the contemporary perspective. This may sound overdone. It is not. This paper is mostly about rigid graphs.

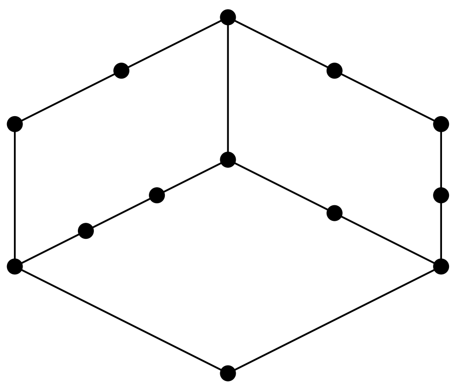


Figure 1: Hedrlín Pultr graph [63]

I.

This is neither a survey of Pavol's work, nor a history of our collaboration. But it begins with a little history. We entered the Faculty of Mathematics and Physics of the Charles University (abbreviated in Czech as MFF UK) in Prague in 1964. As was customary in those days the actual begin of classes was proceeded by an agriculture brigade. In September that year we were harvesting hops and for two weeks we had a great time and some of lasting friendships started there. Immediately after classes began we realized that one of our teachers was very different in his style and approach to us. It was the first year when Zdeněk Hedrlín was teaching Matematická analýza (i.e. Calculus) for freshman and he did it with an enthusiasm and a great ambition. So when he together with Aleš Pultr started a (no credit) seminar where we would "do problems" we all went along - some 30 students in the freshman class, winter term! Well, in the first year you mostly do what you are told to do.

Hedrlín and Pultr were then in their prime as scientists [62, 63, 64, 59, 140] and they had a vision to do graph theory with us. They presented us with the following problem:

Problem 1 (Rigid graph)
Find a graph G such that the identity is the only homomorphism $G \to G$.

This was right simple: What is an undirected graph $G = (V, E)$ we understood quickly (although we never heard about anything like it before) and what is a homomorphism $G \to G$ was also easy as this was very much same as in algebra (just to be on the safe side: a *homomorphism* $G \to G'$ is a mapping $f : V(G) \to V(G')$ which preserves the adjacency of vertices: $xy \in E(G) \Rightarrow f(x)f(y) \in E(G')$). So this seemed to be all too simple task (particularly, if we would accept the trivial solution). But later, as we got deeper into various interesting aspects of the problem, we were thrilled that here is something so simple and yet it could be perhaps new and the beginning of our doing mathematics.

It is my life's conviction that if you want to teach well you have to give the best without reserves. Original problems, fresh ideas, confidence and dreams. All what you know, what you would like to know or you dream it could be true.

And of course in retrospect, it appears that our teachers did not tell us (intentionally) the whole story. They knew the solution [62], see Fig. 1. But they believed that we have to discover things ourselves and that there is enough substance in the problem (being also encouraged by a conversation with P. Erdős who informed them about his probabilistic solution [60]).

We were working on the problem and as the work became more involved (and as of course we had more and more school duties) the group became smaller (but always included V. Chvátal, P. Hell, L. Kučera and the author). There were various examples of rigid graphs found. One of the nicer ones was Pavol's example of a rigid graph, see Fig. 2.

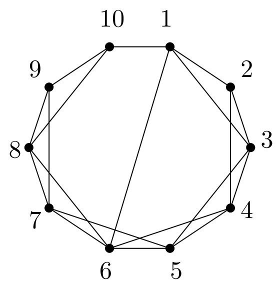
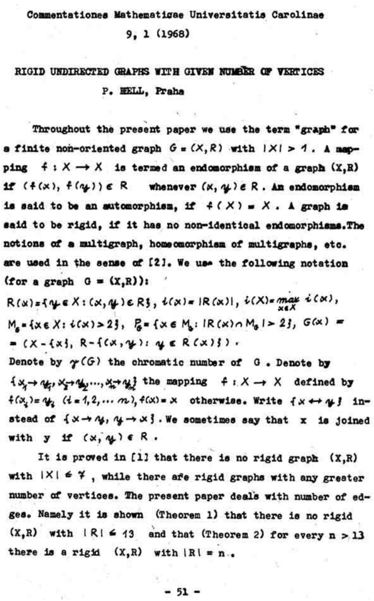


Figure 2: Pavol's rigid graph

This example involved the notion of chromatic number and critical graph. It clearly separated asymmetry from rigidity. And it was not just a singular example, it was a method. This example (and its variants) continues to be useful [128, 53, 39, 21, 111].

Later we were suggested other problems which led to our first publications of Chvátal [18] and myself [99] and the seminar was transformed to a more traditional structure. Pavol was the most active in the original direction of rigid graphs and he wrote his first paper [35] (where he showed that the minimal number of edges of a non-trivial rigid graph is 14, see Fig. 3).



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RIGID UNDIRECTED GRAPHS WITH GIVEN NUMBER OF VERTICES
P. HELL, Praha

Throughout the present paper we use the term "graph" for a finite non-oriented graph $G = (X, R)$ with $|X| > 1$. A mapping $f : X \to X$ is termed an endomorphism of a graph (X,R) if $(f(x), f(y)) \in R$ whenever $(x, y) \in R$. An endomorphism is said to be an automorphism, if $f(X) = X$. A graph is said to be rigid, if it has no non-identical endomorphisms.The notions of a multigraph, homeomorphism of multigraphs, etc. are used in the sense of [2]. We use the following notation (for a graph G = (X,R)):

$R(x) = \{y \in X : (x, y) \in R\}$, $i(x) = |R(x)|$, $i(X) = \max_{x \in X} i(x)$,
$M_G = \{x \in X : i(x) > 2\}$, $P_G = \{x \in M_G : |R(x) \cap M_G| > 2\}$, $G(x) =$
$= (X - \{x\}, R - \{(x, y) : y \in R(x)\})$.

Denote by $\gamma(G)$ the chromatic number of G. Denote by $\{x_1 \to y_1, x_2 \to y_2, \dots, x_n \to y_m\}$ the mapping $f : X \to X$ defined by $f(x_i) = y_i$ $(i = 1, 2, \dots n)$, $f(x) = x$ otherwise. Write $\{x \leftrightarrow y\}$ instead of $\{x \to y, y \to x\}$. We sometimes say that x is joined with y if $(x, y) \in R$.

It is proved in [1] that there is no rigid graph (X,R) with $|X| \leq 7$, while there are rigid graphs with any greater number of vertices. The present paper deals with number of edges. Namely it is shown (Theorem 1) that there is no rigid (X,R) with $|R| \leq 13$ and that (Theorem 2) for every n > 13 there is a rigid (X,R) with |R| = n.

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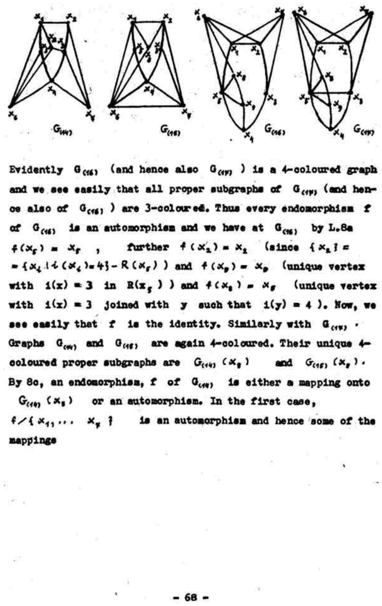
(a)



Evidently $G_{(16)}$ (and hence also $G_{(17)}$) is a 4-coloured graph and we see easily that all proper subgraphs of $G_{(17)}$ (and hence also of $G_{(16)}$) are 3-coloured. Thus every endomorphism f of $G_{(16)}$ is an automorphism and we have at $G_{(16)}$ by L.8a $f(x_5) = x_5$, further $f(x_2) = x_2$ (since $\{x_2\} = \{x_i \mid i(x_i) = 4\} - R(x_5)$) and $f(x_9) = x_9$ (unique vertex with i(x) = 3 in $R(x_5)$)) and $f(x_8) = x_8$ (unique vertex with i(x) = 3 joined with y such that i(y) = 4). Now, we see easily that f is the identity. Similarly with $G_{(17)}$. Graphs $G_{(14)}$ and $G_{(15)}$ are again 4-coloured. Their unique 4-coloured proper subgraphs are $G_{(14)}(x_8)$ and $G_{(15)}(x_8)$. By 8c, an endomorphism, f of $G_{(14)}$ is either a mapping onto $G_{(14)}(x_8)$ or an automorphism. In the first case, $f / \{x_1, \dots x_7\}$ is an automorphism and hence some of the mappings

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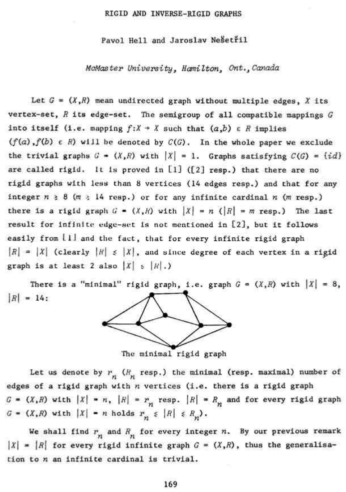
(b)

Figure 3: Mystical examples

The figure from [35] reproduced above contains a series of mystical examples. The first of these is an example of a rigid graph with 8 vertices and 14 edges the smallest rigid graph. This graph is by now known to be unique - the smallest rigid graph. The fact that it is unique is first stated in our conference article [47], see Fig. 4. This example is dear to us and we humbly call it *Our Graph.*



RIGID AND INVERSE-RIGID GRAPHS

Pavol Hell and Jaroslav Nešetřil

McMaster University, Hamilton, Ont., Canada

Let $G = (X,R)$ mean undirected graph without multiple edges, X its vertex-set, R its edge-set. The semigroup of all compatible mappings G into itself (i.e. mapping $f:X \to X$ such that $(a,b) \in R$ implies $(f(a),f(b) \in R)$ will be denoted by $C(G)$. In the whole paper we exclude the trivial graphs $G = (X,R)$ with $|X| = 1$. Graphs satisfying $C(G) = \{id\}$ are called rigid. It is proved in [1] ([2] resp.) that there are no rigid graphs with less than 8 vertices (14 edges resp.) and that for any integer $n \geq 8$ ($m \geq 14$ resp.) or for any infinite cardinal n (m resp.) there is a rigid graph $G = (X,R)$ with $|X| = n$ ($|R| = m$ resp.) The last result for infinite edge-set is not mentioned in [2], but it follows easily from [1] and the fact, that for every infinite rigid graph $|R| = |X|$ (clearly $|R| \leq |X|$, and since degree of each vertex in a rigid graph is at least 2 also $|X| \leq |R|$.)

There is a "minimal" rigid graph, i.e. graph $G = (X,R)$ with $|X| = 8$, $|R| = 14$:

The minimal rigid graph

Let us denote by r_n (R_n resp.) the minimal (resp. maximal) number of edges of a rigid graph with n vertices (i.e. there is a rigid graph $G = (X,R)$ with $|X| = n$, $|R| = r_n$ resp. $|R| = R_n$ and for every rigid graph $G = (X,R)$ with $|X| = n$ holds $r_n \leq |R| \leq R_n$).

We shall find r_n and R_n for every integer n. By our previous remark $|X| = |R|$ for every rigid infinite graph $G = (X,R)$, thus the generalisation to n an infinite cardinal is trivial.

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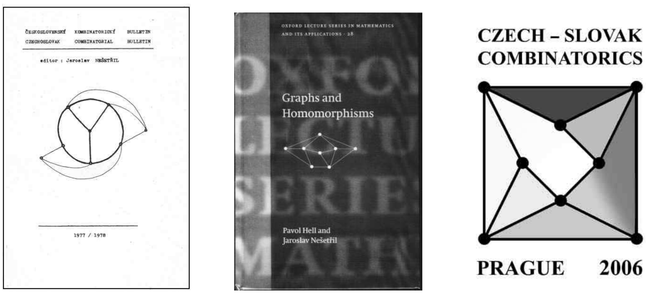
Figure 4: Our Graph

Our Graph has been reproduced many times, see Fig. 5.



(a) (b) (c)

Figure 5: Reproducing Our Graph

The last nice drawing is due to Jiří Fiala and Jan Kratochvíl for the 2006 Prague meeting.

II.

We did not complete our studies at Charles University. This is not the place for a more detailed description of mathematical environment of the Charles University. Let me just say that as undergraduates we had access to excellent teachers of international prominence (as we of course learned only later). Let me just say that lectures by Jaroslav Kurzweil, Jan Mařík, Jindřich Nečas, Ladislav Procházka, Alois Švec, Věra Trnková and Petr Vopěnka, lectures and seminars by Miroslav Katětov, they all proved to be most inspiring intellectually.

Both Pavol and I treasure a memory when, in December 1967, we tool an early exam from Analytic Functions and were allowed by our teacher Vojtěch Jarník to study for the last two lectures from his handwritten notes, as his illness prevented him from delivering the lectures; indeed we were invited to write the exam at his home. These were the last regular classes of Jarník. Our speciality was Mathematical analysis and it consisted from just 11 students [†]. This was regarded at the time as mathematically most theoretically oriented study group.

I have always highly valued the mathematical and educational excellence of MFF UK and I am very proud to be the part of this organization for many years now.

In the winter term 1968 we were both in Vienna where we were accepted as students by the faculty responsible for foreign students which was represented by Edmund Hlavka and F. Schweiger. As a curiosity (certainly from the today point of view) we were admitted and received reasonable scholarship solely on the basis of our two publications [99], [35]. In Vienna

[†]K. Neubauerová-Bendová, J. Blaťák, V. Chvátal, M. Friš, P. Hell, V. Kubát, L. Kučera, M. Kučera, J. Nešetřil S. Verner, J. Zemánek

Figure 6: Prater

we stayed together and we had a lot of free time and it is there where we started to write papers. Shortly before Christmas 1968 we went to Canada as graduate students of Gert Sabidussi who was then at McMaster University. (This was made possible by two facts: Sabidussi's Vienna roots (and H. Izbicki's recommendation), and also by the fact that Aleš Pultr was a visiting professor at McMaster in 1968). We had much less time now, mastering the language and taking classes. Vašek and Jarmila Chvátal were at University of Waterloo and we have much enjoyed our student life in Canada (which was very different from the situation back home).

In the summer 1969 we all took part in the legendary conference in Calgary (which became a template for large combinatorial conferences for many years to come). We gave two lectures and presented two papers to proceedings [47, 111] (one of them with Our Graph mention above). We travelled across Canada by train and continued until Victoria. Vancouver was very different back then (and so we were).

We were still working on rigid graphs [46] and completed the paper [21] with Vašek Chvátal and Luděk Kučera (which is the only souvenir of the entire group from our student days). This paper contains the following.

Figure 7: Vancouver

Theorem 1 ([21])
For every finite graph G there exists a graph H with the following properties:

1. H contains G as an induced subgraph;
2. H is rigid.

The proof given in [21] is constructive and uses rigid graphs from [111], which are themselves relatives of Pavol's graph from figure 2. (Today an alternative proof follows easily from properties of random graphs: Take large graph H at random with the probability $1-\epsilon$ this graph is rigid and also it contains G as an induced subgraph. But this has been shown only later [70].) The construction proved to be useful in the other context [52, 39, 53].

In retrospect out work in 1969 led to the important notion of the core of a graph which we state generally for finite structures:

Definition 1
A structure S is a *core* if every homomorphism $S \to S$ is an automorphism. A substructure S' of S is called *core of* S if S' is a core and there is a homomorphism $S \to S'$.

The nice thing is that core of S (for a general finite structure) is uniquely determined (up to isomorphism) and thus we can speak about the core of S. The core of a structure is useful invariant which captures (and reduces) the complexity of coloring problems see e.g. [51, 38] and recently [81]. It also allows to study finite structures by means of a partial order. Write

$S \leq S'$ if there is a homomorphism $S \to S'$. $\leq$ is a quasiorder (as it many happen $S \leq S' \leq S$ without S and S' beeing isomorphic). But when we restrict $\leq$ to non-isomorphic core structures we get a partial order called the *homomorphism order* [53].

The term core seems to be now generally accepted yet it started as a naive student joke: Our supervisor in Canada was Sabidussi, to whom we endearingly reffered as "dussi", or rather "duši", which is close to the Czech word "duše", meaning the soul, or core. Although we isolated and made use of this concept in 1969 we wrote the paper [52] that became the standard reference much later (for Sabidussi's 60th birthday meeting; there were proved the NP-completeness of the core-decision problem).

Some results which were treated in (the Calgary conference) papers [47, 111] are continuing to be interesting. Let us list two of them: The paper [47] determines (thus extending [35]) the minimal (and maximal) number of edges of a rigid (undirected) graph with n vertices (these numbers appears to be $n + 2$ and $\binom{n}{2} - n + 1$ for $n \geq 20$). The situation is very different for relations (= oriented graphs):

Problem 2 (Minimal rigid relation)
Given a set V of n vertices determine the minimal number $RGD(n)$ of arcs of a rigid relation on V.

Clearly $RGD(n) < n$ and the true value is of the order $n(1 - \frac{1}{\log n})$. However the exact value seems to be a hard problem - homomorphisms are hard to enumerate.

The existence of a rigid relation leads to an important result:

Theorem 2 ([145])
On every set there exists a rigid relation.

This has been proved in a landmark note [145]. Other constructions (which are however related to the original proof) are given in [61] and perhaps the simplest recently in [107].

This paper is about (algebraic aspects) of finite combinatorics but at this point we make a little excursion to infinite graphs. While on every set there exists a rigid relation, it is not clear whether these relations can be made *mutually rigid* (i.e. with no homomorphism between them). In fact Petr Vopěnka conjectured that this cannot be done without help of further set theoretical axioms:

Vopěnka's Axiom (VA) There is no proper class G_α, α ordinal number, of rigid graphs such that there is no homomorphism between any G_α and G_β for $\alpha \neq \beta$.

VA is known to be consistent with ZFC [72] and it has been studied in various context, see e.g. [2]. From the combinatorial point of view it can be equivalently formulated as follows:

Proper class WQO axiom (PCWQO) Any proper class collection of algebraic or relational objects S_α, α ordinal, contains two objects S_α, S_β, $\alpha < \beta$ such that S_α is an induced substructure of S_β.

(Algebraic or relational object means that we bound the arities of relations and operations; without this PCWQO does not hold: consider object of form $(\alpha, \{\alpha\})$, α is ordinal number viewed as set $\{0, 1, \ldots\}$.)

PCWQO can be seen to be equivalent to VA (via Theorem 2 which hold in ZFC) and it presents a deep and general property of infinite graphs (in the spirit of WQO theory for finite objects).

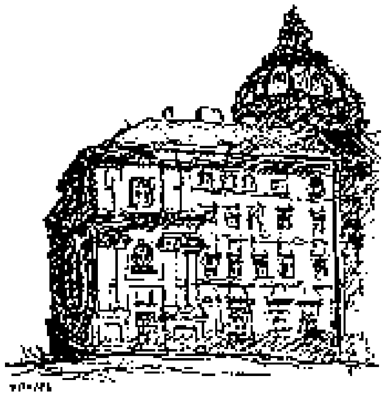
Our of work on VA (unsuccessful work; we wanted to prove it) resulted in the paper [48]. (Note that the essence of VA and PCWQO is the proper class condition. If we instead want to find arbitrary many mutually rigid graphs then this can be done more easily. See [39] for many classes with special properties where this can be done.) This result was later strengthened by Babai and Pultr [7] who showed that k-regular graphs do not represent every finite monoid. This is with contrast with the recent results of Hubička and myself [67] where it is shown that planar graphs with all its degree bounded by 3 (i.e. subcubic graphs) represent every countable poset. The question of representability (and embedding of categories) were at the centre of attention of our teachers at that time ("the Prague school"). We are only touching the subject here and instead refer to a book of Pultr and Trnková [140] or, more recently, our book [53].

Another direction which resulted from [111] was the extension of graph concepts to hypergraphs (which we called then "societies" - a term coined by Hedrlín). The extension was possible by reducing the problem to graphs in today terms using sections or shadows or Gaifman graphs. Even more generally one can consider finite structures S which contain relational and function symbols of prescribed arities from a certain signature set σ. Somewhat more explicitly a relational structure S of type $\Delta = (\delta_i; i \in I)$ is a pair $(X, (R_i; i \in I))$ where $R_i \subseteq X^{\delta_i}$, $i \in I$. Homomorphisms are again defined as mappings preserving all relations of all arities. It was a legacy of our study at MFF UK that *conceptually* the study of homomorphisms is insensitive to structures and that one can aim for the "grand picture" (in today jargon). The structural (or model theoretic) context of homomorphisms gained recently a prominence in the context of Constrained Satisfaction Problem (CSP). The approach goes back to [12, 137] and also to Ivo Rosenberg who initiated in 1972 [141] study of strongly rigid graphs (and relational structures). These are rigid graphs (structures) G in which is satisfied that the only homomorphism $G^k = G\times \ldots \times G \to G$ is a projection. Objects with the later property are now called *projective objects*. Resenberg asked whether almost all relational systems are strongly rigid. This has been verified only recently [86]. This is a very active contemporary context and we shall return to it later.

In the winter 1969 we both wrote our MSc thesis at McMaster University. Neither was about rigid graphs (where we have at the time felt we were the experts). It is interesting (with respect to the later development) to note that Pavol wrote his thesis about Ramsey numbers [36] and parts were published in [37], while my thesis was about asymmetric graphs [100] (i.e. the graphs with the only identical homomorphism). I proved various properties mostly in the relation to Ulam's reconstruction conjecture (which was a very popular subject then); this part appeared in [102]. The second part of my thesis was devoted to the extremal question of asymmetry. To my horror I discovered shortly before the thesis submission that most of the material in this second part was considered by Erdős and Renyi in their classical paper on the subject [26] - the paper which I did not know (in those pre-Google times). Sabidussi's reaction was very nice: this is very good, take it as an encouragement and a proof that you did good things. Nevertheless, I omitted some parts from the thesis and published them separately, see [101]. I was then surprised to receive one (handwritten) reaction to my thesis which was dealing with similar problems. Much later I realized that this was one of the first papers by Saharoni Shelah [144]. After 35 years we collaborated [127] and again in a homomorphism context.

The hastily organized MSc defense marked forever the end of our joint studies (in Prague, Canada and elsewhere).

III.



In 1970, Pavol began his doctoral studies at the Université de Montréal (again with Sabidussi), and I became an Assistant Professor at MFF UK in Prague. Our lives (and worlds) separated, but we never lost contact, and never stopped collaborating. Even through the darkest years we continued writing joint papers [48, 49, 50].

Interestingly enough, we have never had a priority dispute - which is rare, as everybody knows. We wrote doctoral theses on different topics again. Pavol's thesis was on graph homomorphisms and it is rightly seen as the foundation of the theory of graph retracts. The techniques proved useful later on, and the resulting theory is described in Chapter 2 of [53]. In particular, the notion of dismantlability lead to some very nice work of Pavol and various of his coauthors (Hans-Jurgen Bandelt, Ivan Rival, Martin Farber and others [8, 56], and very recently to work of Benoit Larose, Claude Tardif, and Pavol's student Cynthia Loten [81]). But as our lives separated, so did to a large degree our research. While Pavol made (and continues to make) numerous contributions to generalizations of matchings (for instance [43, 44]), various interconnection networks (including [9, 10]), and algorithms for nicely structured graphs (such as interval graphs [42], chordal graphs [45], circular arc graphs [11]), to name a few areas, my interests were, and remain, more on the combinatorial and algebraic side. To keep on the theme of this paper, allow me to focus on the subject as seen from the Prague perspective.

Back home my life changed profoundly in many respects but mathematically the main difference was that I started to work intensively with students (which at the beginning were just a few years my juniors). I founded Kombinatorický seminář (Combinatorial seminar) which I chaired then for many years and which brought me much joy. The Combinatorial seminar was (and I believe is) one of the most active Prague group, which was broadly mathematically based and attracted some of the best talents from the whole country (Czechoslovakia and then Czech Republic). I am not going to report this activity but let me just say that at the beginning I was fortunate to have Vladimír Müller, Jan Pelant and Vojtěch Rödl. This quickly resulted in solutions of open problems [138, 91, 92, 97, 98, 95] and our group started to be well known both abroad and then at home. Paul Erdős was our great teacher and supporter.

In Prague, the principal figure for us at this time was Zdeněk Frolík. Interestingly, I did not know Frolík as a teacher in sixties (he was mostly abroad). But in seventies he was our great supporter and sheltered us from many things. His Winter Schools in Abstract Analysis were for us absolute highlights of each year. Unfortunately Frolík died at an early age, see the volume which we dedicated to him [110]. I believe Frolík would be happy from the development of our "combinatorial group" which is now involved in most of mathematics.

Mathematically (and otherwise) the most important thing I did in seventies and eighties was Ramsey Theory and my collaboration with Vojtěch Rödl. Vojta will be of course forever my most frequent coauthor and the work we did together profoundly influenced my whole career as mathematician and teacher [124]. But this paper is on a different topic. (I will be only happy to return to our collaboration at another occasion, i.e. soon, for example when Vojta will be 60!).

The research activity related to rigid objects and homomorphisms continued. With Vladimir Müller and Jan Pelant we published several papers [95, 96] on tournament algebras (and simple tournaments investigated independently at the same time by Paul Erdős and Eric Milner [27]). With Lászlo Babai [5, 6] we extended Theorem 1 to infinite graphs and with Mike Adams and Jiří Sichler [1] we investigated images of rigid graphs (where the situation is not completely clarified yet). I also investigated influence of orientations on automorphisms and homomorphisms [103, 49] (earlier Chvátal and Sichler [22] investigated a similar problem for colored graphs). But perhaps in this context most importantly I decided around 1975 to write a Czech book on graph theory which was otherwise badly needed and which will be "homomorphism based" or better say "influenced". In doing so I rethought many things we did earlier and some new pattern emerged.

I want to single out three particular notions which appear in [105] and which the book certainly helped to crystalize.

Definition 2
A graph is said to be *productive* if the following holds: $G \times G' \nrightarrow H$ providing $G \nrightarrow H$ and $G' \nrightarrow H$ where $G \times G'$ is a *direct product* of graphs G and G'.

The famous *product conjecture* (Hedetniemi conjecture [53]) asserts in this language that every complete graph is productive. In [62] we justified this definition by this connection and established some basic properties, including the productivity of directed cycles of prime length; we conjectured that all directed cycles of prime power length were also productive. Interestingly, Pavol came independently to ask similar questions about 10 years later, unaware at first of our paper [61]. By the time their paper [34] was published, Pavol and his authors (Roland Häggkvist, Donald Miller, and Victor Neumann Lara) knew about our paper and realized that they have proved our conjecture on directed cycles of prime power length. (Their proof uses a topological lemma; a beautiful direct combinatorial proof due to Xuding Zhu [150] is reproduced in [53].) They have used the term multiplicative graphs, which has now become standard [53]. Claude Tardif [146] recently proved that there are multiplicative graphs with circular chromatic number arbitrary close to 4. K_4 is the smallest graph which is not known to be multiplicative. Tardif's proof uses categorical machinery (adjoints) the study of which (for relational structures) was originated by Pultr [139].

Another concept which was in fact the leitmotiv of the whole book [105] was the concept of *homomorphism duality*. Here the genesis is more complicated. Some of the seminal papers of modern computational complexity theory are the work of Jack Edmonds [24, 25]. He anticipated the complexity classes P and NP and coined the term *good characterization* of a decision problem. The class of problems with a good characterization (on the abstract level) coincides with later introduced class $NP \cap coNP$. The good characterizations became very popular in the beginning of 70ies by work of Chvátal, Lovász and others as a paradigm for solving combinatorial problems. I very much liked Vašek Chvátal's paper [20] where he popularized good characterizations by a nice story. I reproduced a similar story in [105] and was thinking hard about the right approach to good characterization for coloring problems (in today's terminology CSP). This led to the notion of homomorphism duality which in its simplest form can be stated as follows:

Definition 3
Let F, D be structures. Denote by $F \nrightarrow$ the class of all structures S for which there is no homomorphism $F \rightarrow S$. Similarly denote by $\rightarrow D$ the class of all structures S for which there is a homomorphism $S \rightarrow D$. A *(singleton) duality* is the equation of classes

$$F \nrightarrow = \rightarrow D$$

In this case (F, D) is called *dual* pair, D *is dual of* F.

In today notation one would write $Forb(F)$ for the class $F \nrightarrow$ and $CSP(D)$ for the class $\rightarrow D$. It is also clear how to extend this to finite families $\mathcal{F}$ and $\mathcal{D}$. We then speak about *finite dualities* [29, 109, 130].

I strongly believed that by choosing appropriate morphisms and structures one can capture all good characterizations. (This belief materialized: Linear programming duality (Farkas lemma) may be rephrased as duality of oriented matroids, see my papers with Winfried Hochstätter [65, 66] and all CSP problems fit to dualities in the context of recent papers with Gábor Kun [78, 79].). In the book [105] I rephrased most of the main min-max theorems in terms of dualities. With Aleš Pultr we wrote shortly after the paper [118] with a self explanatory title *On classes of relations and graphs determined by subobjects and factorobjects*. There we derived some general properties and showed that there are no nontrivial dualities for undirected graphs:

Theorem 3
Up to the homomorphism equivalence there is only one trivial dual pair (K_2, K_1).

However already for directed graphs (not to speak about other structures) we have not found a characterization. For the case of directed graphs this was completed later by my student Pavel Komárek [74, 75]. The full generality of relational structures was considered and solved together with Claude Tardif [130]:

Theorem 4 ([130])
For a finite relational structure F the following two statements are equivalent:

1. F is a tree structure;
2. F has a dual D.

There is a much recent activity surrounding this theorem, see e.g. [3, 81, 29]. But here we are jumping too much in time. Some of the last development is reviewed at the end of this article.

Let us just mention, that another homomorphism concept which originated around the same time in [105] (in the Ramsey theory context (!) [122]) was the notion of the *dimension* of an undirected graph [119, 85, 123].

Out of my work with Babai [5, 6] originated two interesting problems:

Problem 3 (Linear representation of monoids)
Does there exists $c > 0$ such that every monoid M with n points can be represented by the monoid of endomorphism of a graph with at most cn vertices?

Recall that Babai earlier proved that every group with n points can be represented by a graph with $2n$ vertices (with few exceptional cases).

Problem 4 (Chromatically optimal rigid graphs)
Let G be a graph. Does there exists a rigid graph H containing G as an induced subgraph if and only if $\chi(G) > \omega(G)$?

(The condition is clearly necessary; that goes back to 1964.) With a little experience one sees easily that both problems are related to rigid graphs led to the following two results by Václav Koubek, Vojtěch Rödl and author:

Theorem 5 ((Mutually rigid graphs) [76])

1. Asymptotically almost all graphs are rigid. Thus the number of non isomorphic rigid graphs with n vertices is

$$\frac{2^{\binom{n}{2}}}{n!}(1 - o(1))$$

2. The number of mutually rigid non-isomorphic graphs with n vertices is asymptomatically equal to

$$\frac{1}{n!}\binom{\binom{n}{2}}{\lfloor \frac{\binom{n}{2}}{2} \rfloor}(1 - o(1))$$

This allowed the authors of [76] to give a negative answer to the Problem 3: there are monoids M for which every graph G with $End(G) \cong M$

needs at least $|M| \log |M|$ vertices ($End(G)$ is the endomorphism monoid of the graph G).

The stability of rigid structures (reflected in Theorems 5 and 10) may provide an answer to the permanence of rigid graph motivation. Rigid structures are everywhere. Like stones they are all around us. But to find a nice stone (which would fit to your own garden) is another, often non-trivial, thing.

The Problem 4 on chromatically optimal rigid graphs has positive solution. The key ingredient in this is the following result which was first isolated by Rödl and myself in [122]. The result holds for general finite structures. We formulate it just for graphs.

Theorem 6 (Sparse incomprarability lemma SIL [122])
Let k,ℓ be positive integers. Then for every graph G there exists a graph G' with the following properties:

1. G' contains no cycles of length $\leq \ell$
   (i.e. the girth of G' is $> \ell$)

2. $G' \to G$

3. For every graph H with at most k vertices $G' \to H$ iff $G \to H$.

Putting intuitively, despite of the fact that G' is much sparse than G, it cannot be distinguished from G by the existence of homomorphism into small graphs. (Note that we do not consider counting analogs of this result. This leads to different theory [13] which goes back to Lovász pioneering paper [83]. This in turn inspired both Lovász [84] and Müller [91] work on Ulam's conjecture.).

Sparse incomparability lemma holds (with the analogous proof) for relational structures and has many applications (see recent [23]). For example it yields an easy proof of

Theorem 7 (Graph density)
Let G_1,G_2 be graphs satisfying $G_1 \to G_2$ and $G_2 \nrightarrow G_1$ (i.e. $G_1 < G_2$ in the homomorphism order). Let G_2 be non-bipartite. (Thus we are, up to the homomorphism equivalence, excluding the single case: $G_1 = K_1, G_2 = K_2$.) Then there exists a graph G such that

$$G_1 \to G \to G_2,$$
$$G_2 \nrightarrow G \nrightarrow G_1$$

This means that in the homomorphism order of undirected graphs there are no gaps $G_1 < G_2$ (except of $K_1 < K_2$). Density theorem was proved in [147] by Emo Welzl. Later a much shorter proof was found independently by Micha Perles and the author (see [106, 53] and also [31]). Here is another short proof using Sparse incomparability lemma.

Proof. Let G_1,G_2 be as above applying SIL find G_2' such that $G_2' \to G_2$, G_2' has girth $> |V(G_2)|$ (we do not optimize here) and $G_2' \to H$ iff $G_2 \to H$ whenever $|H| \leq |V(G_1)|$. Particularly $G_2' \nrightarrow G_1$ and thus we can put $G = G_1 + G_2'$. (It is $G_2 \nrightarrow G$ as G_2 contains an odd cycle.). $\square$

Sparse incomparability lemma was studied intensively and it was also generalized and strengthened [90, 135, 77]. P. Erdős asked often for a construction of combinatorial objects whose existence is guaranteed by probabilistic method. One such question was whether one can construct uniquely k-colorable graphs without short cycles. The problem was solved by Vlǎda Müller [92, 93] (see also [71]), in a more general form where he proved a remarkable theorem about graphs extending a given set of colorings (on a fixed subset of vertices). We call this result *Müller's extension theorem* (MET).

In the course of generalizations of SIL we recently found with Xuding Zhu a characterization when MET holds:

Theorem 8 ([135])
For a core graph H, the following statements are equivalent:

I. For any choice of a finite set A and distinct mappings $f_1, f_2, \cdots, f_t : A \to V(H)$ there exists a graph $G = (V, E)$ such that the following holds:

i. A is a subset of V;

ii. For every $i = 1, 2, \cdots, t$ there exists unique homomorphism $g_i : G \to H$ such that g_i restricted to the set A coincides with the mapping f_i;

iii. For every homomorphism $f : G \to H$ there exists $i, 1 \leq i \leq t$ and a homomorphism $h : H \to H$ such that $h \circ f_i = f$;

iv. G has girth $> l$.

II. The graph H is projective;

But there we are jumping again. Going back to 80ies I believe these were some of the most intense years for myself. Mathematics was very nice, I had wonderful group of students and collaborators with whom we shared life in general. We even had a time for our mathematical theatre as we recently reported with V. Müller in [94]. The Combinatorics seminar was wonderful,

$$\begin{tikzcd} H \arrow[r, "f"] & H' \\ G' \arrow[u, "c"] \arrow[ur, "g"'] & \end{tikzcd}$$

Figure 8:

perhaps reaching its peak in 80ies also with S. Poljak, J. Kratochvíl, J. Matoušek, R. Thomas, I. Kříž, P. Komárek, M. Loebl, J. Witzany, O. Zýka. It is hard to say, this statement is perhaps not even true as the seminar was all the time high quality and a pure joy (and my pride in otherwise tense situation), I considered it the most important thing I did. And we tried to do all mathematics. But this is another story and far from the “surprising rigid permanence” I am covering here.

IV.

In 1986 I visited Pavol at SFU for the first time. Although we maintained contacts and we of course knew about our work and activities, it was a very new and inspiring moment to meet again. We started to work instantly, as if we had never been separeted, only with more maturity and experience. Soon after we met we were fortunate to complete together what we started to contemplate independently: Recall that H*-coloring* of a graph G is just a homomorphism $G \to H$ (H is called *template*). H-coloring problem is the following decision problem:

Input: graph G

Question: does there exist $G \to H$?

In $1986 - 1987$ we proved the following:

Theorem 9 ((H-coloring) [51])
H-coloring problem is NP-complete iff H is a non-bipartite graph.

This result is one of the inspirations for the celebrated Dichotomy Conjecture of [28].

Dichotomy Conjecture [28] The H-coloring problem (even when generalized to relational structures) is always either polynomially solvable or NP-complete.

H-coloring covers a broad class of problems. Every constraint satisfaction problem (CSP) may be interpreted as an H-coloring problem ([28])

for relational structures. The other general cases when the dichotomy conjecture is know to hold are the cases when the template has 2 [143] or 3 vertices [14] (in the case of relational structures). Theorem 9 is even more striking as the general Dichotomy Conjecture can be reduced to the H coloring problem for oriented graphs. The proof of Theorem 9 is interesting (and presently non-trivial). (This is true also about the second proof published recently by Bulatov [15].) The proof does not follow by a subgraph argument (and this cannot be expected as the NP-completeness fails to be a monotone property in general). But it is possible to say that the proof uses again experience gained in rigid graph constructions (particularly the *replacement*, or *indicator construction*, see [53]. The paper [51] proved to be much more important then we originally thought and it became our most quoted paper.

Pavol introduced me to his then postdoc and most active collaborator Xuding Zhu (fresh PhD from Calgary, Norbert Sauer supervisor), with whom he investigated various homomorphism problems [58, 57] including the "path dualities". We quickly started to work together and produced [54] where we defined Bounded Tree Width Dualities (BTWD) which can be defined as follows:

Definition 4 (BTWD)
We say that H-coloring problem of graphs has bounded tree width duality if there exists a positive integer k such that the following statements are equivalent for any graph G in $\mathcal{K}$:

1. $G \to H$
2. For any graph T with treewidth(T)$\leq k$ holds: If $T \to G$ then T is H-colorable.

(Note that the duality (F, D) can be also expressed in these terms: $F \nrightarrow D$ and $G \to D$ iff $F \to G$ implies $F \to D$.)

We proved that BTWD implies that the problem is polynomially tractable [54]. We were not aware of an independent work done by Tomás Feder and Moshe Vardi [28]. But this was a very inspiring connection which led to a great enrichment of our research and to the important collaboration of Feder and Hell. Many of the methods and problems which we were considering found a proper setting of the complexity of CSP in terms of universal algebra and structures of a more general type. Pavol understood correctly that here is a very rich field and analyzed the complexity of H-coloring problem thoroughly with many coauthors: oriented cycles, semicomplete graphs, list homomorphisms and lately M-partitions. This activity is reflected well by [16, 38, 41] and outlined in [53].

Soon after we met we started to contemplate "writing a book". Well most people do contemplate such a thing but it took us nearly decade [53].

Despite having written many papers I do not write books easily. Subconciuosly, I am perhaps too ambitions. To write a book is a duty (for after the initial optimism it becomes a selfimposed duty). I try to use to organize, to rethink the whole material again, better and basically from the scratch. This is not a very efficient method: Czech Graph Theory [105] is perhaps too original and I had never enough courage and time to transform it in the English. Our book with Jirka Matoušek: Invitation to Discrete Mathematics had 4(!) Czech published iterations before it was finally done in English [89] (and since then to other languages [89]). And with the book Graphs and Homomorphism [53] this was similar. We were not satisfied with the purely algebraic (category theory) motivation and wanted to understand better the combinatorial core of whatever we wanted to include. On the other hand we wanted to keep and stress the flexibility of the homomorphism language and not to write a purely "graph theory" book. I believe we (modestly) succeeded but it took a long time. We selected (a little unusual) collection of algebraic theorems (including the often neglected Freyd-Vinárek characterization of concrete categories) and blended it with combinatorial analysis of various graph operations, complexity and applications (to various types of graph colorings in the context of the Channel assignment problem). What came out of blue and is perhaps the chief novelty of the book was the various aspects and properties of the homomorphism order. Here we get a helping hand from Claude Tardif (another former Sabidussi student) with whom we worked on dualities. We not only proved Theorem 4 but we also showed that dualities may be characterized equivalently in order theoretic terms by means of gaps (i.e. intervals $S < S'$ in the homomorphism order not containing any other structure) and minimal cuts, i.e. maximal antichains, (of size 2). The correspondence is very general and holds in Heyting posets [120].

Dual structures of trees are truly amazing. This is indicated also by the fact that several constructions of duals were discovered in different context. Currently we have the following constructions:

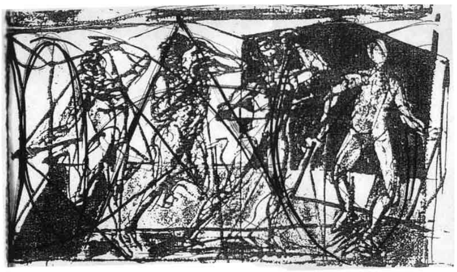
- using gaps (i.e. predecessors) and power graph construction [130];
- "bear construction" via neighbourly mappings [131];

- deletion method (a generalization of Komárek's construction [75]);
- model theoretic construction via monadic lifts, implicit in [28] and [87];
- specialization the universal construction [69].

It is known that duals have exponential size cores, even almost all oriented paths are exponential core duals [80], and that they have a small diameter [129]. They can be recognized and even belong to NP [29, 129, 131, 132, 81].

I believe that our book [53] is not only the first book on graphs and homomorphisms but it is also perhaps the first book which combines algebraic graph theory with complexity and structural methods.

V.



It seems that [53] was well received. It also came in the right time as presently we are witnessing an explosion of research related to homomorphisms. Even to outline the main questions which are considered would be to extensive. So in the spirit of this paper let us finish this paper by restricting ourselves to our main topic - rigid graphs. Indeed they seems to be a persistent flower (or weed?).

With Tomasz Łuczak [86] we recently verified Dichotomy conjecture for almost all templates (over general signature). This is based on the following (which is yet another manifestation of Erdős-Rényi stability of rigidity):

Theorem 10
Asymptotically almost all structures with a given signature are strongly rigid. More precisely this means two thing:

1. Asymptotically almost all structures on large sets are strongly rigid.
2. Asymptotically almost all structures on a fixed universum (of size > 1) with large enough arities are strongly rigid.

(The complexity result is based on the algebraic approach to complexity reduction theorems - see e.g. [86], [16]. It follows that H-coloring problem is NP-complete whenever H is strongly rigid.). Core structures also played a key role in the following recent result in mathematical logic:

Benny Rossman [142] solved an old problem proving that a homomorphism closed class $\mathcal{K}$ of structures is First Order (FO) definable if and only if it is also positively FO definable. The later means that there are finitely many structures $\mathcal{S} = \{S_1, \ldots, S_t\}$ such that $\mathcal{K}$ consists from all structures S satisfying $S_i \to S$ for some i. Where do we get this finitness? The homomorphism order is (countably) universal even for simplest structures (a striking result in this direction is [68]: the homomorphism order restricted to orientations of finite paths is universal; solving a problem and extending [136].).

This of course implies that the homomorphism order is on the opposite side of the spectrum than say WQO posets (which are typically used in a finite basis arguments). The finitness is a consequence of the following definition [112] which goes back to [127] (and perhaps earlier):

A *tree complex* is any subcomplex of the chain complex corresponding to a finite rooted tree (i.e. a branching viewed as poset). Given relational structure $S = (X, (R_i; i \in I))$ the *tree depth* $td(S)$ of S is the minimal height of a rooted tree T such that all tuples in relations of S are contained (as sets) in the tree complex of T. $td(S)$ is well defined (as any S is contained in the tree complex of any chain on X). [112] (and more recent [113]) contains the following:

Theorem 11 (Finitness)
For all fixed positive integer k the class of all structures S with $td(S) \leq k$ (with a given signature) has only finitely many cores.

(While the number of cores of graphs with tree depth $\leq k$ is finite this number grows very rapidly even in the simplest case: For undirected graphs the number is bounded by the power function only.). This finitness result is the basis of the Rossman proof [142] as well as the recent work I have been doing with Patrice Ossona de Mendez [113, 114, 116] on *Bounded Expansion (BE) classes*. We are not going to define the classes here and instead refer to the original articles [113, 114, 116]. But it suffice to say that BE classes contain all proper minor classes (i.e. classes defined by forbidding K_k as a minor) and also classes of graphs with all its degrees bounded by k. The classes of BE are related to dualities as follows:

Let $\mathcal{K}$ be a class of graphs (or structures). A *restricted duality* is the equation of classes

$$Forb(\mathcal{F}) \cap \mathcal{K} = CSP(\mathcal{D}) \cap \mathcal{K}$$

Explicitly for every $G \in \mathcal{K}$ we have the following disjoint alternatives: either $F \to G$ for some $F \in \mathcal{F}$ or $G \to D$ for some $D \in \mathcal{D}$.

($\mathcal{F}$ and $\mathcal{D}$ are finite sets of structures - not necessarily subsets of $\mathcal{K}$). We say that the class $\mathcal{K}$ has *all restricted dualities* if for any finite set $\mathcal{F}$ there is a finite set $\mathcal{D}$ such that $(\mathcal{F}, \mathcal{D})$ form restricted duality. This notion was first considered in [112, 113]. The following has been proved in [115, 117].

Theorem 12 (all restricted dualities for finite structures)
Every class $\mathcal{K}$ of structures with bounded expansion has all restricted dualities.

As a corollary every class $Forb(\mathcal{F}) \cap \mathcal{K}$ is equal to the restriction of a class $CSP(\mathcal{D})$ restricted to $\mathcal{K}$. Viewing the characterization of dualities for finite structures (Theorem 4 [130]) Theorem 12 gives a surprising richness of restricted dualities and this also nicely complements the descriptive complexity result [4]. This theorem is a culmination of the earlier results, particularly of the result [33] by Roland Häggkvist and Pavol Hell about graphs with bounded degrees. Note that we may restrict the set $\mathcal{F}$ to a set of core structures, but we cannot use incomparable cores (only if the structures are connected).

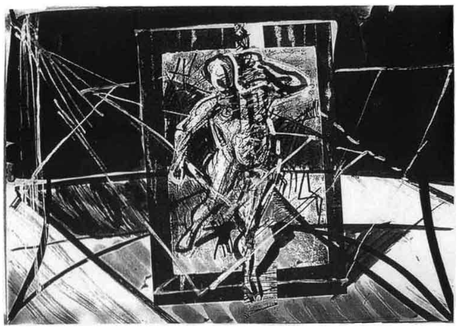
Despite of the generality of Theorem 12 it may seem that the classes $Forb(\mathcal{F})$ are very special. Indeed, what non-trivial can you express by finitely many forbidden substructures? While $CSP(H)$ is a very complicated class even for a simple graph H (think of a triangle), the class $Forb(\mathcal{F})$ seems to be very simple (for a finite set $\mathcal{F}$). But the situation drastically changes if we allow extensions of our signature (which defines the structures under consideration) and projections. This was done recently together with Gábor Kun [78, 79] by means of lifts and shadows.

What is proved, is that, any NP language L is polynomially equivalent to a language of the following form:

$$\Phi(Forb(\mathcal{F}'))$$

where $\mathcal{F}'$ is a finite set of structures with signature $\sigma \cup \sigma'$ and Φ is the forgetful functor which assigns to any structure $S' \in Forb(\mathcal{F}')$ with signature $\sigma \cup \sigma'$ the corresponding structure S with signature σ (by forgetting the relations from σ').

VI.



Rigid and core structures came a long way. Still some beautiful and simple formulated problems remain. Let us finish by listing three of them:

Problem 1 (Minimum asymmetric graphs)
Is it true that every asymmetric orientation $\overrightarrow{G}$ of a graph G contains a vertex $x \in V(G)$ such that $\overrightarrow{G} - x$ is again asymmetric?

This is true for acyclic orientations [149]. A similar problem for undirected graphs was considered by Gert Sabidussi, Jerome Gagnon and myself [125] [30], see [73].

Problem 5 (Maximal antichain)
Let G_1, G_2 be countable graphs, $G_1 \nrightarrow G_2 \nrightarrow G_1$. Assume that any other countable graph is comparable by a homomorphism with either G_1 or G_2. Is then one of the graphs finite?

This is formulated in [127] where it is proved that K_1,K_2 and K_ω are the only maximal antichains of size 1 for the homomorphism order of countable graphs.

Problem 6 (Infinite rigid)
Does there exists positive integer k such that on every set X there exists a rigid relation whose symmetrization does not contain a subdivision of the graph K_k (i.e. K_k as a topological subgraph)?

This is an interesting problem. Babai asked whether there exists a locally planar rigid graph on every set. But we could ask even less formal question: try to find a new construction of a rigid graph on every set which would not be based on the (ordinal number) technique of [145] (and [61, 107]).

The homomorphisms of graphs and more generally of finite structures gained a momentum recently. The various factors which influenced it are too involved to be covered here, so let us just list several texts and books [73, 126, 134], and of course [53], which reflect the various aspect of this development.

This is only a text which reflects our life long collaboration with Pavol Hell. We both believe that our collaboration will continue, e.g. [55]. But what is perhaps evident is a surprising persistence of old motivations. With all modesty, I believe that this permanence is a sign of a true quality and of a beauty of mathematics. Like an everlasting gem ...

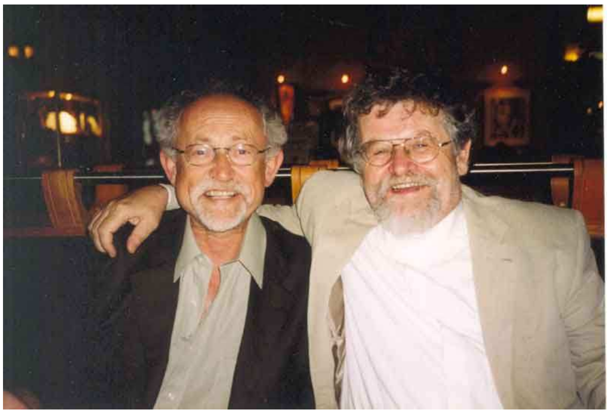
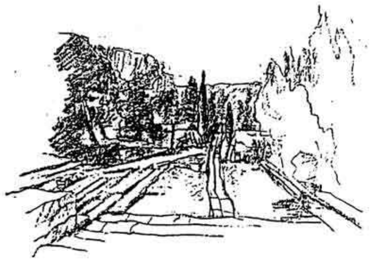


Figure 9: Paris by Helena

Acknowledgement

The included photos are from the archive of the author. The drawings III. and VII. are due to author, the drawings V. and VI. are taken from J. Načeradský, J. Nešetřil: Antropogeometry I, II, Rabasova galerie, Rakovník, 1999. I thank to David Hartman and Pavol Hell for help in producing this article.

VII.



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