On the density of trigraph homomorphisms *

Pavol Hell[1], Jarik Nešetřil[2]

[1] School of Computing Science, Simon Fraser University, Burnaby, B.C., Canada V5A 1S6, pavol@cs.sfu.ca
[2] KAM MFF UK, Malostranské nám 25, Praha, nesetril@kam.mff.cuni.cz

Abstract. An order is dense if $A < B$ implies $A < C < B$ for some C. The homomorphism order of (nontrivial) graphs is known to be dense. Homomorphisms of trigraphs extend homomorphisms of graphs, and model many partitions of interest in the study of perfect graphs. We address the question of density of the homomorphism order for trigraphs. It turns out that there are gaps in the order, and we exactly characterize where they occur.

Key words. Trigraphs, homomorphisms, density, gaps.

1. Introduction

A *trigraph* G consists of a finite set $V(G)$ of vertices, and two (possibly intersecting) edge sets $E_1(G)$ and $E_2(G)$ on $V(G)$, such

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that $E_1(G) \cup E_2(G)$ contains all pairs of (possibly equal) vertices. Thus a trigraph G can be viewed as a superposition of two graphs on the vertex set $V(G)$ - the graph G_1 with the edge set $E_1(G)$, and the graph G_2 with the edge set $E_2(G)$, both standard graphs (without multiple edges but) with loops allowed [7]. Alternately, we may view a trigraph as a relational structure consisting of a set $V(G)$ with two symmetric binary relations $E_1(G)$ and $E_2(G)$. The only restriction we have is that each pair of (possibly equal) vertices is adjacent in at least one of the graphs G_a, i.e., related in at least one of the relations $E_a(G), a = 1, 2$. This 'completeness' restriction substantially affects the situation, and the questions we address here have been answered for relational structures without this restriction [7,9]. (The 'completeness' restriction is in a sense similar to restricting digraphs to tournaments, cf. [8].)

A trigraph G is a *subtrigraph* of a trigraph H if $V(G) \subseteq V(H)$ and $E_a(G) \subseteq E_a(H)$ for $a = 1, 2$. If every two vertices of G have the same relations in G as in H, we say that G is an *induced* subtrigraph of H. Let G and H be any trigraphs. A *homomorphism* f of G to H is a mapping of $V(G)$ to $V(H)$ which preserves both relations, i.e., such that $uv \in E_a(G)$ implies $f(u)f(v) \in E_a(H)$, for $a = 1, 2$. A bijective homomorphism of G to H is an *isomorphism* between G and H, and if $G = H$, it is an *automorphism* of G. Two trigraphs are *homomorphically equivalent* if each admits a homomorphism to the other. A trigraph is a *core* if it is not homomorphically equivalent to any proper subtrigraph. Each trigraph is homomorphically equivalent to a unique (up to isomorphism) subtrigraph which is a core, called *the core of* H. (The details of this, as well as of other statements asserted without proof can be found in [7].)

As noted above, graphs are taken to have no multiple edges, but with loops allowed [7]. A *reflexive graph* has a loop at each vertex, and an *irreflexive graph* has no loop. A graph X can be viewed as a trigraph G by taking X to be G_1, and letting G_2 be the reflexive complete graph on $V(X)$. (In other words, $E_2(G)$ contains all possible pairs of vertices, distinct or not, of X.) If trigraphs G and H arise this way from graphs G_1 and H_1, i.e., if G_2 and H_2 are complete reflexive graphs, then the homomorphisms of trigraphs G to H precisely coincide with the standard homomorphisms of graphs G_1 to H_1 [7]. In this sense the study of trigraph homomorphisms extends the study of graph homomorphisms. (In somewhat more technical terms [7,10], the category of trigraphs and trigraph homomorphisms contains the category of graphs and graph homomorphisms as a full subcategory.)

Another way to view an irreflexive graph X, is as having two 'edge-sets' - the edges $E_1(X) = E(X)$ and the nonedges $E_2(X) = E(\bar{X})$. A homomorphism of an irreflexive graph X to a trigraph G is similarly defined as a mapping f of $V(X)$ to $V(G)$ such that $uv \in E_a(X)$ implies $f(u)f(v) \in E_a(H)$, for $a = 1, 2$. Homomorphism of graphs to trigraphs are different from the usual homomorphisms of graphs (even if G is a graph), since they have restrictions not only on where the edges of X can map, but also where the nonedges of X can map. They model many partitions arising in the study of perfect graphs [1,6]. Indeed, such a homomorphism f of X to a trigraph G induces a partition of $V(X)$ into parts $f^{-1}(w), w \in V(G)$, each of which is a clique (if $w \in V(G)$ has $ww \in E_1(G) - E_2(G)$), an independent set (if $w \in V(G)$ has $ww \in E_2(G) - E_1(G)$), or an arbitrary set (if $w \in V(G)$ has $ww \in E_1(G) \cap E_2(G)$), such that parts $f^{-1}(w)$ and $f^{-1}(z)$ are joined by all possible edges (if $wz \in E_1(G) - E_2(G)$), or by no edges (if $wz \in E_2(G) - E_1(G)$), or have arbitrary connection (if $wz \in E_1(G) \cap E_2(G)$).

For example a graph X is a *split graph* if and only if it has a homomorphism f to the trigraph S with $V(S) = \{0, 1\}, E_1(S) =$

$\{00, 01\}, E_2(S) = \{01, 11\}$. Indeed, the vertices of $f^{-1}(1)$ form an independent set in X, and the vertices of $f^{-1}(0)$ form a clique in X.

Similarly, a graph X has a *clique cutset* if and only if it has a *surjective* homomorphism to the trigraph C with $V(C) = \{0, 1, 2\}, E_1(C) = \{00, 11, 22, 01, 12\}, E_2(C) = \{00, 22, 01, 12, 02\}$. The surjectivity ensures that the set $f^{-1}(1)$, forming a clique in X, is nonempty, as are the sets $f^{-1}(0)$,$f^{-1}(2)$, which are joined by no edges of X.

Vašek Chvátal [1] identified another kind of partition as potentially important for efficient recognition of perfect graphs, and for a possible proof of the strong perfect graph conjecture. His hunch turned out to be correct in both cases - cf. [2,3]. A *skew partition* of a graph X is a partition of the vertex set of X into four nonempty parts A, B, C, D, such that there are no edges between A and B, and all edges between C and D. This can again be viewed as a surjective homomorphism of X to a suitable trigraph T with four vertices a, b, c, d (corresponding to A, B, C, D). Namely, $E_1(T) = \{aa, bb, cc, dd, ac, bc, ad, bd, cd\}$, $E_2(T) = \{aa, bb, cc, dd, ac, bc, ad, bd, ab\}$. We depicted this trigraph on the left of the enclosed figure. The light edges (and loops) are pairs in both $E_1(T)$ and $E_2(T)$; the heavy edges are in $E_1(T) - E_2(T)$; and the pairs in $E_2(T)$ are absent from the figure. The figure on the right represents another trigraph for which the corresponding partition problem turned out to be useful for the recognition of perfect graphs [2].

Chvátal asked whether the existence of a skew partition can be decided in polynomial time. In [5], the authors found a subexponential algorithm for the list version of the problem. The introduction of lists (which allow recursion to subproblems) turned out to be useful, and eventually a polynomial time algorithm for the existence of skew partition was found [4].

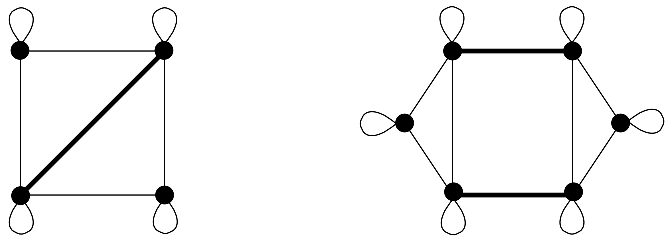

Fig. 1. Trigraphs for some partition problem relevant for perfect graphs.

Such examples are discussed in more detail in, for instance, [5, 7], where other partitions of interest in the study of graph perfection can be found.

It is clear that composition of homomorphisms is also a homomorphism, whether graphs or trigraphs are involved. In particular, we observe the following fact.

Proposition 1. *Suppose the trigraph G admits a homomorphism to the trigraph H. Then each irreflexive graph X which admits a (partition corresponding to a) homomorphism to G also admits a (partition corresponding to a) homomorphism to H.* □

We may define the partial order $<$ on the set of trigraphs by writing $G < H$ just if there is a homomorphism of G to H but not of H to G. As noted above, this partial order extends the partial order of graphs and homomorphisms. One interesting aspect of that order is its density. Indeed, it is known that for any core graphs $X < Y$ (other than the complete graphs $K_1 < K_2$) there exists a graph Z with $X < Z < Y$. The instance $G < H$ for which no K satisfies $G < K < H$ is called a *gap* of the order. (Thus $K_1 < K_2$ is the only gap in the homomorphism order of graphs.) We address here the question of density of the homomorphism order for trigraphs. As in the case of graphs, we may focus on trigraphs that are cores.

Another partial order which extends the homomorphism order of graphs is the homomorphism order of digraphs [7]. Without going into much detail, density is less prevalent for digraphs, and it is known that a connected digraph Y has a *predecessor* X (i.e., a digraph for which there is no Z with $X < Z < Y$) if and only if Y is an oriented tree, see [9], where the existence of predecessors is also described for disconnected digraphs.

2. Density

As noted above, in discussing density, we can focus on cores. Indeed, we have $A < B$ if and only if the cores A', B' of A, B respectively satisfy $A' < B'$.

A *digon* of a trigraph G is a pair of distinct vertices u, v such that $uv \in E_1(G) \cap E_2(G)$. A *diloop* is a vertex u with $uu \in E_1(G) \cap E_2(G)$. Note that a core trigraph with a diloop must have a single vertex.

Proposition 2. *Let H be the trigraph with one vertex and a diloop. Then for any core trigraph G with $G < H$ there exists a trigraph K with $G < K < H$.*

Proof. Since $G < H$, the trigraph G must not have diloops. We now construct G^+ by taking two disjoint copies of G and joining them with all possible edges (each vertex of the first copy has an edge in both E_1 and E_2 to each vertex of the second copy). It is now clear that $G < G^+ < H$, so G is not a predecessor of H.

Thus from now on we may focus on trigraphs without diloops. It turns out that digons also play an important role. Let G be any trigraph; we denote by G^* the subtrigraph of G induced by all vertices incident to digons of G. (Note that G^* may contain edges which are not parts of a digon.) Note that any homomorphism f of a trigraph G to a trigraph H maps digons to digons, and hence maps G^* to H^*. We denote by G_f the subtrigraph of G induced by all vertices x with $f(x)$ in H^*. We note that G_f contains G^*.

Proposition 3. *Let f be a homomorphism of a core trigraph G to a core trigraph H. If f is injective on G_f, then f is injective on G.*

Proof. Suppose $f(x) = f(y)$; then by assumption, at least one of x, y is not in G_f, and hence $f(x) = f(y)$ is not in H^*. This implies that for any vertex z in G we have $xz \in E_a$ if and only if $yz \in E_a$, since this can happen only if $f(x)f(z) = f(y)f(z) \in E_a$, for $a = 1, 2$. This contradicts the fact that G is not a core, as x could be mapped to y.

Our main theorem in this section is the following.

Theorem 1. *Suppose G and H are cores without diloops such that $G < H$.*

If

1. *G is subtrigraph of H with $G^* = H^*$,*
2. *every homomorphism of G to H is injective, and*
3. *the core of each proper subtrigraph of H which contains G is equal to G,*

then $G < H$ is a gap. Otherwise, there exists a trigraph K with $G < K < H$.

Proof. Note that condition 2 implies that every homomorphism of G to H must take G^* to H^* isomorphically. If all the conditions are satisfied, then consider a trigraph K with homomorphisms $f : G \to K, g : K \to H$. By assumption 2, the composition gf

is injective, taking G^* to H^* isomorphically. This means that f is also injective, and that g is injective on $f(G)$, taking $f(G^*)$ isomorphically to H^*. Consider now the subtrigraph $g(K)$ of H. Taking $f(G)$, and one vertex of K for each x in $g(K) - g(f(G))$, we form a subtrigraph K' of K isomorphic to $g(K)$. Note that K admits a homomorphism to K', thus as K is a core we have $K = K'$. But by condition 3, $g(K)$ is either H or G, and K does not satisfy $G < K < H$, i.e., $G < H$ is a gap.

It remains to show that if any of the three conditions is violated, we can find a trigraph K with $G < K < H$. Surprisingly, such a K can be canonically defined. We begin by replacing each vertex x of H by s vertices x_i: if $xx \in E_a$, then each $x_i x_j \in E_a$ for $a = 1, 2$ (including the case $i = j$). (Recall that each vertex has either a loop in E_1 or in E_2.) If x, y are distinct vertices of H, the relation (E_1 or E_2) of the edge $x_i y_j$ is decided as follows.

Let B be a fixed bipartite graph with white vertices $1, 2, \ldots, s$ and black vertices $1', 2', \ldots, s'$ satisfying the following property: between any s/n white and s/n black vertices of B, there exists both an edge and a nonedge of B. (It follows from the proof of the following lemma that if s is sufficiently large, such a bipartite graph B exists.) If $xy \in E_1 - E_2$, we set $x_i y_j \in E_1$, and similarly, if $xy \in E_2 - E_1$, we set $x_i y_j \in E_2$. If $xy \in E_1 \cap E_2$, we set $x_i y_j \in E_1$ if ij' is an edge of B, and $x_i y_j \in E_2$ if ij' is not an edge of B. Call this trigraph H^-, and denote by F the natural homomorphism of H^- to H, which takes each x_i to its corresponding x; finally, let f be any homomorphism of G to H. We let K be obtained from a disjoint union of G and H^- where the edges between vertices of G and vertices of H^- are decided as follows: if $x \in V(G)$ and $y \in V(H^-)$, then $xy \in E_a$, where E_a is *one* of E_1, E_2, such that $f(x)F(y) \in E_a$. (Note that $f(x)F(y)$ may be in both E_1, E_2, but we still arbitrarily choose just one of $a = 1, 2$ for xy.)

We now observe that G admits a homomorphism to its copy in K, and that the mapping ϕ equal to f on G and F on H^- is a homomorphism of K to H. If s is chosen large enough, the existence of a homomorphism of H^- to G would imply the existence of a homomorphism of H to G, contrary to $G < H$. (See the Sparse Incomparability Lemma below.) Thus K has no homomorphism to G either, i.e., $G < K$. It remains to ask whether we also have $K < H$. Thus suppose there is a homomorphism $g : H \to K$. In that case, ϕg is a homomorphism of H to itself, and since H is a core, we may assume that g was chosen so that ϕg is the identity on H, and, in particular, the identity when restricted to H^*. This means that g is injective and ϕ surjective. Note that $g(H^*)$ is a subtrigraph of K, i.e., of G^* (since all digons are in G^*). Therefore ϕ (and so f) takes G^* to H^*.

We now claim that f is injective on G^* (in other words, ϕ is injective on the copy of G^* in K). In fact, consider the trigraph G_f defined above. If G_f properly contains $g(H^*)$, then we can define the following mapping of G to a proper subtrigraph of itself: map each x of $G - G_f$ to x, and map each vertex y of G_f to $g(f(y))$. This is a homomorphism, as is easily checked by considering an edge xy where x is in $G - G_f$ and y is in G_f, and noting that $f(y) = f(g(f(y)))$. This contradicts the fact that G is a core. Thus $G_f = g(H^*)$ and hence $G_f = G^* = g(H^*)$, i.e., G^* and H^* are isomorphic. Therefore f is injective on G_f, and by Proposition 3 also on G. Since f was an arbitrary homomorphism of G to H, we obtain the conditions 1 and 2. Of course, if condition 3 is not satisfied then any core K properly containing G and properly contained in H satisfies $G < K < H$.

We now prove the existence of an integer s as required by the proof. In fact, we extract the construction of H^-, as we find it a useful tool of general interest. Thus we formulate our *Sparse Incomparability Lemma* as follows.

Lemma 1. *For every trigraph H and for every positive integer n there exists a trigraph H^- with the following properties:*

- *$H^- \to H$,*
- *H^- does not contain a digon, and*
- *for each trigraph G with at most n vertices, we have $H^- \to G$ if and only if $H \to G$.*

Proof. The construction of the trigraph H^- is described in the previous proof; in particular, it involves replacing each vertex of H by s new vertices. The first two statements are obvious from the construction. Thus suppose G is a trigraph with at most n vertices, and assume s/n is sufficiently large. Note that $H \to G$ implies $H^- \to G$ by the first statement. On the other hand, if $\mu : H^- \to G$ is a homomorphism, we can define $\nu : H \to G$ by letting $\nu(x)$ be the majority value of $\mu(x_i)$. If $xy \in E_1 \cap E_2$, then since s/n is large, it follows by the standard properties of random graphs that between any s/n white and s/n black vertices of B there will be both an edge and a nonedge. It follows that $\nu(x)\nu(y) \in E_1 \cap E_2$, and hence ν is a homomorphism of H to G. Specifically, the probability that two sets of size s/n will miss an edge or a nonedge is at most $2(1/2)^{(s/n)^2}$, and the number of choices of such subsets is $\binom{s}{s/n}^2$. We now note that $2(1/2)^{(s/n)^2}\binom{s}{s/n}^2$ converges to zero.

An interesting variation on the question of density would be to consider trigraphs allowed to have vertices without loops.

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