# HOMOMORPHISMS OF STRUCTURES (CONCEPTS AND HIGHLIGHTS)

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ABSTRACT. In this paper we survey the recent results on graph homomorphisms perhaps for the first time in the broad range of their relationship to wide range applications in computer science, physics and other branches of mathematics. We illustrate this development in each area by few results.

## 1. Introduction

Graph theory receives its mathematical motivation from the two main areas of mathematics: algebra and geometry (topology) and it is fair to say that the graph notions stood at the birth of algebraic topology. Consequently various operations and comparisons for graphs stress either its algebraic part (e.g. various products) or geometrical part (e.g. contraction, subdivision). It is only natural that the key place in the modern graph theory is played by (fortunate) mixtures of both approaches as exhibited best by the various modifications of the notion of graph minor. However from the algebraic point of view perhaps the most natural graph notion is the following notion of a homomorphism:

Given two graphs $G$ and $H$ a *homomorphism* $f$ of $G$ to $H$ is any mapping $f : V(G) \to V(H)$ which satisfies the following condition:

$$[x, y] \in E(G) \text{ implies } [f(x), f(y)] \in E(H)$$

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This condition should be understood as follows: on both sides of the implication one considers the same type of edges: undirected $\{u, v\}$ often denoted just $uv$) or directed ($(u, v)$ often as well just $uv$). It is important that this definition is flexible enough to induce analogous definitions of the homomorphisms for hypergraphs (set systems) and relational systems (with a given signature; that will be specified later).

Homomorphisms arise naturally in various and very diverse situations

- in extremal combinatorics (and particularly in problems related to colorings, partitions and decompositions of graphs and hypergraphs);
- in statistical physics (as a model for partition functions);
- in probability (as a model of random processes, for example random walk);
- in logic (any satisfiability assignment of a formula may be viewed as a homomorphism);
- in Artificial Intelligence (as a model and criterium of satisfaction leading to Constraint Satisfaction Problems);
- finite model theory (as a natural way of compare and classify models);
- theory of algorithms (as an example and reduction tool);
- in complexity theory (and more recently in logic, descriptive complexity in particular);
- in algebraic combinatorics (providing the vital link to algebraic topology);
- in category theory (as a motivating example, a thoroughly studied particular case).

This paper cannot even touch upon all these topics. This is too ambitious even for a monograph. To a certain extent this has been a plan of recently published book in [22]. But the progress has been fast and we want to complement this by giving some highlights of this development. The interested reader can consult also surveys and papers [30, 63, 32, 3, 48].

This paper is a (much) extended version of the talk given by that author at Cargese school *Physics and Computer Science*, October 17-29, 2005. The purpose of this text is to illustrate the rich conceptual framework of the contemporary study of homomorphisms in various mathematical as well as non mathematical context of various related applications. Because of this (and size) we cannot present full proofs and even to define all the related concepts. But we aim to present at least outline of the recent trends and perhaps for the first time we bring together topics which never coexisted together in a paper.

Contents



## 2. Preliminaries

We rely on standard texts such as [2], [41] (for graph theory), [22] (for graphs and homomorphisms), [35, 41] (for general combinatorics). However the field of combinatorial structures related to homomorphisms (which this author likes to call *combinatorics of mappings*, see forthcoming [47]) is currently developing very fast and it is the purpose of this short survey to cover this recent development.

## 3. Counting

3.1. **Hom and hom.** The symbol $Hom(G, H)$ will denote the set of all homomorphisms of $G$ to $H$. The symbol $hom(G, H)$ will denote the number of all homomorphisms of $G$ to $H$. These sets carry much of the information about the structure of graphs $G$ and $H$. Consider for example the simple situation when $G$ is an undirected graph and $H = K_2$. In this case $hom(G, H) = 2^k$ where $k$ is the number of bipartite components of $G$. But this simplicity is an exception. Already when we consider the graph $H = K_2^*$ which consists from two vertices, one edge joining them and one loop at one of the vertices (if the vertices are denoted by $0, 1$ the edges are $01, 11$; sometimes this graph is called a "lollipop" sometimes even"io") then the situation changes dramatically.

What is the meaning of $hom(G, K_2^*)$? This is suddenly much more interesting: a homomorphisms $f : G \longrightarrow K_2^*$ corresponds exactly to an independent subset of vertices of $G$ (a subset $A \subset V(G)$ is said to be independent if it does not contain any edge; the correspondence is easy: we can put $A = f^{-1}(0)$) and thus $hom(G, K_2^*)$ is just an the number of independent sets in the graph $G$. It follows that $hom(G, K_2^*)$ is a difficult parameter related to *hard - core model* in statistical physics. It is a difficult even in simple (and important) cases such as the $d$-dimensional cube (and its determination is known as "Dedekind problem"). $K_2^*$) is of course not an isolated example. The triangle $K_3$ is another ($hom(G, K_3)$ is the number of $3-$ colorings of a graph $G$.)

On the other side the set $Hom(G, H)$ may be endowed not only with the categorial structure (inherited from the category of graphs; this leads to *sums* and *products* as well as to the notion of *power graph* $G^H$) but more recently also by the following geometric structure:

Given graphs $G, H$ we consider all mappings $f : V(G) \longrightarrow \mathcal{P}'(V(H))$ (here $\mathcal{P}'(X)$ denotes the set of all non-empty subsets of $X$) which satisfy

$$xy \in E(G), u \in f(x), v \in f(y) \Rightarrow uv \in E(F).$$

It is natural to call such mapping $f$ a *multihomomorphism*: every homomorphism is a multihomomorphism and, moreover, for every multihomomorphism $f$ every mapping $g : V(G) \to V(H)$ satisfying $g(v) \in f(v)$ for every $v \in V(G)$ is a homomorphism. By abuse of notation (for this moment) denote the set of all multihomomorphisms $G \to H$ also by $Hom(G, H)$. This set may be naturally partially ordered: for multihomomorphisms $f, f'$ we put $f \leq f'$ iff for every $v \in V(G)$ holds $f(v) \subseteq f'(v)$.

This construction is called *Hom complex* and it crystalized in the long and intensive history of coloring special graphs, most notably *Kneser* graphs, see [42, 30, 12, 65]. It plays the key role in this application of topology to combinatorics. Hom complex $Hom(G, H)$ is viewed as an order complex and this in turn as a topological space (in its geometric realization). All these constructions are functorial. ([60] is early study of graphs from the categorical point of view.)

3.2. **Lovász' theorem.** Let $F_1, F_2, F_3, \dots$ be a fixed enumeration of all non-isomorphic finite graphs. The *Lovász vector* of a graph $G$ is $hom(G) = (n_1, n_2, n_3, \dots)$, where $n_k = hom(F_k, G)$.

**Theorem.**[36] Two finite graphs $G$ and $H$ are isomorphic if and only if $hom(G) = hom(H)$.

We include a short proof of this important result.

**Proof.** It is more than evident that if $G \cong H$ then $hom(G) = hom(H)$. Let $hom_i(F,G)$ denote the number of all monomorphisms (injective homomorphisms) of $F$ to $G$. Suppose that $hom(G) = hom(H)$. We claim that then also $hom_i(F,G) = hom_i(F,H)$ for an arbitrary graph $F$. This claim will be proved by induction on the number of vertices of the graph $F$. First, if $|V(F)| = 1$, then $hom_i(F,G) = hom(F,G) = hom(F,H) = hom_i(F,H)$. Suppose $|V(F)| > 1$. Then we can write

$$\begin{aligned} hom(F,G) &= \sum_{\Theta \in Eq(V(F))} hom_i(F/\Theta, G) \\ &= hom_i(F,G) + \sum_{\substack{\Theta \in Eq(V(F)) \\ \Theta \neq \mathrm{id}}} hom_i(F/\Theta, G), \end{aligned}$$

where $Eq(V(F))$ is the set of all equivalence relations on $V(F)$ and $F/\Theta$ is the graph whose vertex set is the set of all equivalence classes of $\Theta$ and an edge connects two classes $c$ and $c'$ if there are vertices $u \in c$ and $u' \in c'$ so that $\{u, u'\}$ is an edge of $F$. (Note that loops may occur in $F/\Theta$.) This is because every homomorphism $f : F \to G$ corresponds to a monomorphism of $F/\Theta$ to $G$ for $\Theta = \{(u,u'); f(u) = f(u')\}$.

Similarly, we get

$$hom(F,H) = hom_i(F,H) + \sum_{\substack{\Theta \in Eq(V(F)) \\ \Theta \neq \mathrm{id}}} hom_i(F/\Theta, H).$$

By induction, we know that for any $\Theta \in Eq(V(F))$, $\Theta \neq \mathrm{id}$,

$$hom_i(F/\Theta, G) = hom_i(F/\Theta, H),$$

since $|V(F/\Theta)| < |V(F)|$. It follows that we have also $hom_i(F,G) = hom_i(F,H)$.

Applying this for the following choices $F = G$ and $F = H$ we get $hom_i(G,H) = hom_i(G,G) \geq 1$ and $hom_i(H,G) = hom_i(H,H) \geq 1$. If there is a monomorphism of $G$ to $H$ and a monomorphism of $H$ to $G$, then (as our graphs are finite) $G$ and $H$ are isomorphic. $\Box$

The Lovász' theorem has a number of interesting (and despite its seeming simplicity, profound) consequences. For example one can prove easily the following cancellation law for products of graphs. (There are many graph products. Here we mean the product $G \times H$ defined by the property that projections are homomorphisms. This is the categorical product of category of all graphs and their homomorphisms.)

3.3. **Corollary.** Let $G$ and $H$ be graphs. If graphs $G \times G = G^2$ and $H \times H = H^2$ are isomorphic then so are graphs $G$ and $H$.

**Proof.** (sketch) Let $F$ be a graph. Every homomorphism $f : F \to G^2$ corresponds to a pair of homomorphisms $(f_1, f_2)$ of $F$ to $G$; if $f(u) = (x_1, x_2)$, then $f_i(u) = x_i$. Moreover, the correspondence is one-to-one (due to the categorical properties of the product). Therefore

$$hom(F,G)^2 = hom(F,G^2) = hom(F,H^2) = hom(F,H)^2$$

and so $hom(F,G) = hom(F,H)$.) $\square$

This particular case was conjectured by Ulam (for finite partially ordered sets)([5]). Along the same lines one can also prove the following:

Let $A$, $B$ and $C$ be graphs, let $C$ have a loop. If $A \times C \cong B \times C$, then $A \cong B$.

These results hold in fact not just for graphs but for arbitrary finite structures (with a mild conditions on the underlying category). For example it is important to observe that the following *dual form* of Theorem 2.2 holds:

If $hom(G, F_i) = hom(H, F_i)$ for every $i = 1, 2, 3, \ldots$ then graphs $G$ and $H$ are isomorphic.

The proof uses (again) the inclusion-exclusion principle. These results form an important part in the Tarski's and Birkhoff's project of *arithmetization* of theory of finite structures (see [43, 5]). It is not generally true that $A \times C \cong B \times C$ implies $A \cong B$. A counterexample: $A$ consists of two isolated loops, $B = C = K_2$. Another counterexample: $A = K_3$, $B = C_6$ (the cycle of length 6), $C = K_2$. With more efforts one can prove that if $A$, $B$ and $C$ are not bipartite, then they have the cancellation property: $A \times C \cong B \times C \implies A \cong B$, [36].

## 4. Weighted Counting, Random and Quantum

In the statistical physics we deal with a structure of (typically) large number of particles each in a finite number of states $\sigma_1, \ldots, \sigma_t$. The particles are positioned in the vertices of a graph $G$ (with vertices $\{1, 2, \ldots, t\}$) and interactions occurs only between neighboring vertices. Two particles $\sigma_i, \sigma_j$ are interacting with energy $\gamma(\sigma_i, \sigma_j)$ and the total energy of the state $\sigma$ of the structure (i.e. of all states of the vertices of the graph $G$) is given by

$H(\sigma) = \sum_{ij\in E(G)} \gamma(\sigma_i, \sigma_j)$. Finally the *partition function* (in a simplified form) is given by

$$Z = \sum_{\sigma} e^{-H(\sigma)}.$$

The partition function relates to the number of *weighted* homomorphisms. This was as developed recently in a series paper by Lovász et al. in a broad spectrum of asymptotic graph theory, random structures and abstract algebra (see e.g. [16, 35, 38, 37, 3, 4]).

Let $G, H$ be graphs (undirected). Additionally, let the vertices and edges of $H$ be weighted:

$$\alpha : V \longrightarrow \mathbb{R}^+, \beta : E \longrightarrow \mathbb{R}.$$

In this situation define the weighted version of $hom(G, (H, \alpha, \beta))$ as follows:

$$hom(G, (H, \alpha, \beta)) = \sum_{\varphi:V(G)\to V(H)} \prod_{v\in V(G)} \alpha_{\varphi(v)} \prod_{uv\in E(G)} \beta_{\varphi(u)\varphi(v)}.$$

Of course if $\alpha$ and $\beta$ are functions identically equal to 1 then the weight $hom(G, (H, \alpha, \beta))$ of homomorphisms is just $hom(G, H)$. The partition function $Z$ may be expressed by this weighted homomorphism function. Towards this end write

$$Z = \sum_{\sigma} e^{-H(\sigma)} = \sum_{\sigma} \prod_{ij\in E(G)} e^{-\gamma(\sigma_i,\sigma_j} = \sum_{\sigma} \prod_{ij\in E(G)} \beta(\sigma_i, \sigma_j).$$

where we put $\beta(\sigma_i, \sigma_j) = e^{-\gamma(\sigma_i,\sigma_j)}$.

It follows that the partition function may be computed as weighted homomorphism function. This has many variants and consequences. For example, in the analogy with number of 3-colorings expressed by $hom(G, K_3)$ and Ising model expressed by $hom(G, K_2^*)$, one can ask which partition functions can be expressed as weighted functions $hom(-, (H, \alpha, \beta))$ for a (finite) weighted graph $(H, \alpha, \beta)$. The surprising and elegant solution to this question was given in [16] and we finish this section by formulating this result. A *graph parameter* is a function $p$ which assigns to every finite graph $G$ a real number $p(G)$ and which invariant under isomorphisms.

4.1. **Theorem.** ([16]) For a graph parameter $p$ are the following two statements equivalent:

1. $p$ is a graph parameter for which there exists a weighted graph $(H, \alpha, \beta)$ such that $p(G) = hom(G, (H, \alpha, \beta))$ for every graph $G$;

2. there exists a positive integer $q$ such that for every $k \geq 0$ the matrix $M(p, k)$ is positive semidefinite and its rank is at most $q^k$.

Motivated by physical context, the parameter $p$ is called *reflection positive* if the matrix $M(p, k)$ (called in [16] the *connection matrix*) is positive semidefinite for every $k$. There is no place to define here the connection matrix, let us just say that it is an infinite matrix induced by values of the parameter $p$ on amalgams of graphs along $k$-element subsets (roots).

Where are the random aspects of all this (as claimed by the title of this section) ? For this consider the following:

$$t(G, H) = \frac{hom(G, H)}{|V(H)|^{|V(G)|}}, t(G, (H, \alpha, \beta)) = \frac{hom(G, (H, \alpha, \beta))}{|V(H)|^{|V(G)|}}$$

These quantities are called *homomorphism density*. They express the probability that a random mapping is a homomorphism; or the average weight of a mapping $V(G) \longrightarrow V(H)$. This connection leads to a homomorphism based interpretation of important asymptotic properties of large graphs such as *Szemerédi Regularity Lemma* [66] or properties of *quasirandom graphs* [10, 67]; see [38, 4, 37].

Where are quantum aspects of this? Well it appears that in proving Theorem 4.1 it is both natural and useful to extend the homomorphism function to formal finite linear combinations of graphs. These combinations, called *quantum graphs*, [16], are natural in physical context and they appear a convenient tool in proving 4.1.

## 5. Existence and CSP

Perhaps this section should precede the counting sections. What can be easier than deciding as opposed to seemingly more difficult counting. Well the answer is not so simple and in fact both parts of the theory point to different directions: As we indicated above the counting relates to probability and properties of random structures in general, to partition models of physical phenomena; on the other side the existence problems relate to computational complexity of decision models (such as Constraint Satisfaction Problem (CSP), logic and descriptive complexity and dualities. Some of this will be covered in this and next sections. The sections on counting preceded "existence" sections as they are perhaps conceptually more uniform and they are also closer to this volume (of Cargese school).

We consider here the following decision problem:

5.1. **H-coloring Problem.** Consider the following decision problem (for a fixed graph $H$):

**Instance**: A graph $G$.
**Question**: Does there exists a homomorphism $G \longrightarrow H$.

This problem covers many concrete problems which were and are studied (see [22]):

(i) For $H = K_k$ (the complete graph with $k$ vertices) we get a k-coloring problem;
(ii) For graphs $H = K_{k/d}$ we get *circular chromatic numbers*' see e.g. [69];
(iii) For $H$ Kneser graphs $K\binom{k}{d}$ we get so called *multicoloring*; [22].

Further examples include so called $T$-colorings, see e.g.[69],[62], which in turn are related to the recently popular *channel assignment problem.*

Perhaps the most extensively studied aspect of $H$-coloring problems is its complexity. This is interesting and generally still unresolved. The situation is well understood for complete graphs: For any fixed $k \geq 3$ the $K_k$-coloring problem (which is equivalent to the deciding whether $\chi(G) \leq k$) is NP-complete. On the other hand $K_1$- and $K_2$-coloring problems are easy. Thus, in the undirected case, we can always assume that the graph $H$ is not bipartite.

Some other problems are easy to discuss. For example, if $H = C_5$ then we can consider the following (*arrow replacement*) construction:

For a given graph $G$ let $G^{**}$ be the graph which we obtain from $G$ by replacing of every edge of $G$ by a path of length 3 (these paths are supposed to be internally disjoint). Another way to say this is to consider a subdivision of $G$ where each edge is subdivided by exactly two new vertices. It is now easy to prove that for any undirected graph $G$ the following two statements are equivalent:

(i) $G \longrightarrow K_5$;
(ii) $G^{**} \longrightarrow C_5$.

This example is not isolated (the similar trick may be used e.g. for any odd cycle). Using analogous, but more involved, edge-, vertex- and other replacement constructions (called *indicators, subindicators*, and *edge - subindicators*) the following has been proved in [23]:

5.2. **Theorem.** For a graph $H$ the following two statements are equivalent:

(1) $H$ is non-bipartite;
(2) $H$-coloring problem is NP-complete.

This theorem (and its proof) has some particular features which we are now going to explain:

**1.** The result claimed by the theorem is expected. In fact the result has been a long standing conjecture, but it took nearly 10 years before the conjecture had been verified.

**2.** As the statement of 5.2 is expected, so its proof is unexpected. What one would expect in this situation? Well, we should prove first that $C_{2k+1}$-coloring is NP-complete (which is easy and in fact we sketched this above) and then we would"observe" that the problem is monotone. Formally, iff $H$-coloring problem is NP-complete and $H \subseteq H'$ then also $H'$-coloring problem is NP-complete.

The monotonicity may sound plausible but there is not known a direct proof of it. It is certainly a true statement (by virtue of Theorem 5.2 ) but presently the only known proof is via the Theorem 5.2. In fact there is here more than meets the eye: for oriented graphs the NP-complete instances are not monotone!

**3.** We have to stress that the analogy of Theorem 5.2 for oriented graphs fails to the true.

One can construct easily an orientation $\vec{H}$ of bipartite graph $H$ such that $\vec{H}$ - coloring problem is NP-complete. Even more so, one can construct a balanced oriented graph $\vec{H}$ with the property that $\vec{H}$-coloring problem is NP-complete (an oriented graph is called *balanced* if every cycle has the same number of forward and backward arcs).

One can go even further and (perhaps bit surprisingly) one can omit all cycles. Namely, one has the following [24]:

5.3. **Theorem.** There exist an oriented tree $T$ (i.e. $T$ is an orientation of an undirected tree) such that $T$-coloring problem is NP-complete.

Presently the smallest such tree $T$ has 45 vertices.

## 6. Constrained Satisfaction Problems (CSP)

Every part of mathematics has some typical features which presents both its advantages and its limitations. One of such feature for the study of homomorphisms is the fact that its problems are usually easy to generalize and formulate, that there is the basic *thread* which allows to concentrate on important and "natural" questions (to try to explain this is also the main *motif* of this paper). The $H$-coloring problem (explained in the previous section) is a good example of this. One can formulate it more generally for every finite structure. We consider the general relational structures (so general that they are sometimes called just finite structures):

6.1. **Relational Structures.** A relational structure of a given type generalizes the notion of a relation and of a graph to more relations and to higher (non-binary) arities. The concept was isolated in the thirties by logicians (e.g. Löwenheim, Skolem) who developed logical "static" theory. As we shall see this influenced terminology even today as we find useful to speak about models (of our chosen relational language). In the sixties new impulses came from the study of algebraic categories and the resulting "dynamic" studies called for a more explicit approach, see e.g [61]. We shall adopt here a later notation (with a touch of logical vocabulary).

A type $\Delta$ is a sequence $(\delta_i; i \in I)$ of positive integers. A *relational system* $\mathbf{A}$ of *type* $\Delta$ is a pair $(X, (R_i; i \in I))$ where $X$ is a set and $R_i \in X^{\delta_i}$; that is $R_i$ is a $\delta_i$-nary relation on $X$. In this paper we shall always assume that $X$ and $I$ are finite sets (thus we consider finite relational systems only).

The type $\Delta = (\delta_i; i \in I)$ will be fixed throughout this paper. Note that for the type $\Delta = (2)$ relational systems of type $\Delta$ correspond to directed graphs, the case $\Delta = (2, 2)$ corresponds to directed graphs with blue-green colored edges (or rather arcs).

Relational systems (of type or signature $\Delta$) will be denoted by capital letters **A, B, C,....** A relational system of type $\Delta$ is also called a $\Delta$-system (or a model). If $\mathbf{A} = (X, (R_i; i \in I))$ we also denote the base set $X$ as $A$ and the relation $R_i$ by $R_i(\mathbf{A})$. Let $\mathbf{A} = (X, (R_i; i \in I))$ and $\mathbf{B} = (Y, (S_i; i \in I))$ be $\Delta$-systems. A mapping $f : X \to Y$ is called a *homomorphism* if for each $i \in I$ holds: $(x_1, ..., x_{\delta_i}) \in R_i$ implies $(f(x_1), ..., f(x_{\delta_i})) \in S_i$.

In other words a homomorphism $f$ is any mapping $F : A \to B$ which satisfies $f(R_i(\mathbf{A})) \subseteq R_i(\mathbf{B})$ for each $i \in I$. (Here we extended the definition of $f$ by putting $f(x_1, ..., x_t) = (f(x_1), ..., f(x_t))$.)

For $\Delta$-systems $\mathbf{A}$ and $\mathbf{B}$ we write $\mathbf{A} \to \mathbf{B}$ if there exists a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Hence the symbol $\to$ denotes a relation that is defined on the class of all $\Delta$-systems. This relation is clearly reflexive and transitive, thus induces a quasi-ordering of all $\Delta$-systems. As is usual with quasi-orderings, it is convenient to reduce it to a partial order on classes of equivalent objects: Two $\Delta$-systems $\mathbf{A}$ and $\mathbf{B}$ are called *homomorphically equivalent* if we have both $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{A}$; we then write $\mathbf{A} \sim \mathbf{B}$. For every $\mathbf{A}$ there exists up to an isomorphism unique $\mathbf{A}'$ such that $\mathbf{A} \sim \mathbf{A}'$ and $\mathbf{A}'$ has smallest size $|A'|$. Such $\mathbf{A}'$ is called the *core* of $\mathbf{A}$.

The relation $\to$ induces an order on the classes of homomorphically equivalent $\Delta$-system, which we call the *homomorphism order*. (So this is partial order when restricted to non-isomorphic core structures.)

The homomorphism order will be denoted by $\mathcal{C}_\Delta$ (as it is also called coloring order). We denote by $\mathbf{Rel}(\Delta)$ the class of all finite relational structures of type $\Delta$ and all homomorphisms between them. This category plays a special role in the model theory and theory of categories [61]. It is also central in the branch of Artificial Intelligence (AI) dealing with *Constraint Satisfaction Problems* [11]. The expressive power of homomorphisms between relational structures leads to the following:

6.2. **Theorem.** ([14]) Every Constraint Satisfaction Problem can be expressed as a membership problem for a class $\mathbf{CSP}(\mathbf{B})$ of relational structures (of a certain type $\Delta$) defined as follows:

$$\mathbf{CSP}(\mathbf{B}) = \{\mathbf{A}; \mathbf{A} \longrightarrow \mathbf{B}\}.$$

Recall, that the *membership problem* for a class $\mathcal{K}$ is the following problem: Given a structure $\mathbf{A}$ does $\mathbf{A}$ belong to $\mathcal{K}$? Is it $\mathbf{A} \in \mathcal{K}$? For brevity we call the membership problem for class $\mathbf{CSP}(\mathbf{B})$ simply $\mathbf{B}-$*coloring problem*, or $\mathbf{CSP}(\mathbf{B})$ *problem*. The structure $\mathbf{B}$ is usually called *template* of $\mathbf{CSP}(\mathbf{B})$.

(Generalized) coloring problems cover a wide spectrum (applications rich) problems. This attracted recently a very active research on the boundary of complexity theory, combinatorics, logic and universal algebra. Only some of it will be review in this paper.

However the complexity status of the $\mathbf{CSP}(\mathbf{B})$ problem is solved only for special and rather restricted situations. The following are principal results:

(1) undirected graph coloring (i.e. Theorem 5.2), see [23];
(2) the characterization of complexity of $\mathbf{B}-$coloring problems for structures $\mathbf{B}$ which are *binary* (i.e. for which $|B| = 2$), see [64];
(3) the characterization of complexity of $\mathbf{B}-$coloring problems for structures $\mathbf{B}$ which are *ternary* (i.e. for which $|B| = 3$), see [7].

The last two results may seem to be easy, or limited, but reader should realize that while the size of the $|B|$ may be small (such as 2 or 3) the relational system can in fact be very large as the arities $\delta_i$ of relations $R_i(\mathbf{B})$ may be arbitrary large. Whole book [11] is devoted to the case $|B| = 2$.

Nevertheless in all known instances one proves that the $\mathbf{CSP}(\mathbf{B})$ problem is either polynomial (the class of polynomial problems is denoted by P) or NP-complete. This is remarkable as such *dichotomy* generally does not hold. Of course, there is a possibility that the classes P and NP coincide (this constitutes famous P-NP problem; one of the *millennium problems*). But if these classes are distinct (i.e. if P $\subset$ NP) then there are infinitely intermediate classes (by a celebrated result of Ladner [39]). This (and other more theoretical evidence) prompted Feder and Vardi [14] to formulate the following by now well known problem:

6.3. **Dichotomy Conjecture.** Every $\mathbf{CSP}(\mathbf{B})$ problem is either P or NP-complete.

Although this is open, a lot of work was done. Let us finish this section by formulating two related results.

6.3.1. **Oriented Graphs Suffice.** At the first glance the complexity of the $\mathbf{CSP}(\mathbf{B})$ problem lies in the great variety of possible relational structures. Already in [14] it has been realized that it is not so.

**Theorem.** The dichotomy conjecture follows from the dichotomy of the complexity of $H$-coloring problem where $H$ is an oriented graph.

This is interesting as this positions Theorem 5.2 in the new light and shows a surprising difference between colorings (partitions) of undirected and directed graphs which was not before realized. See [14] for the original proof; see also [22].

6.3.2. **Dichotomy is Asymptotically Almost Surely True.** Relational structures and homomorphisms express various decision and counting combinatorial problems such as coloring, satisfiability, and linear algebra problems. Many of them can be reduced to special cases of a general Constraint Satisfaction Problem $\mathbf{CSP}(\mathbf{B})$. A number of such problems have been studied and have known complexity, e.g., when we deal with undirected graphs or the problem is restricted to small sets $A$ (see [64, 23, 7]). However at this moment we are far from understanding the behavior of $\mathbf{CSP}(\mathbf{B})$ problem even for binary relations (i.e., when relational systems of type $\Delta = (2)$). It seems that the Dichotomy Conjecture holds in a stronger sense:

**Dichotomy Conjecture*.** Most **CSP**(**B**) problems are NP-complete problems with a few exceptions which are polynomial.

For example for undirected graphs **CSP**(**B**)-problem is always hard problem with exactly 3 exceptions: **B** is homomorphically equivalent either to the loop graph, or to the single vertex graph (with no edges) or the symmetric edge, [23]. Results for other solved cases have a similar character supporting the modified Dichotomy Conjecture* (see [64, 7]). One can confirm this feeling by proving that both Dichotomy and Dichotomy* Conjectures are equivalent:

**Theorem.** Let $\Delta = (\delta_i)_{i\in I}$ be such that $\max_{i\in I}\delta_i \geq 2$. Then $\mathbf{CSP}(\mathbf{B})$ is NP-complete for almost all relational systems $\mathbf{B}$ of type $\Delta$. (Note that for $\mathbf{B}$ of type $(1, 1, \ldots, 1)$ the problem $\mathbf{CSP}(\mathbf{B})$ is trivial.)

In order to make the statement of Theorem precise, let $\mathcal{R}(n, k)$ denote a random $k$-ary relation defined on a set $[n] = \{1, 2, \ldots, n\}$, for which the probability that $(a_1, \ldots, a_k) \in \mathcal{R}(n, k)$ is equal to $1/2$ independently for each $(a_1, \ldots, a_k)$, where $1 \leq a_r \leq n$ for $r = 1, \ldots, k$. Assume further that not all $a_i$'s are equal: for $a \in [n]$, we assume $(a, a, \ldots, a) \notin \mathcal{R}(n, k)$. Let $([n], (\mathcal{R}(\delta_i, n)_{i\in I}))$ denote the random relational system of type $\Delta = (\delta_i)_{i\in I}$. In this situation we show that the probability that $([n], (\mathcal{R}(\delta_i, n)_{i\in I}))-$coloring is NP-complete tends to one as either $n$, or $\max_i \delta_i$ tends to infinity.

Note that $\mathbf{B}-$coloring problem for relational system $\mathbf{B}$ is NP-complete, provided it is NP-complete for the system $(B, R_{i_0}(\mathbf{B}))$ of type $(\delta_{i_0})$ for some $i_0 \in I$. (The converse implication, in general, does not hold). Thus, we prove our result for 'simple' relational systems which consist of just one $k$-ary relation.

**Theorem.** For a fixed $k \geq 2$,

$$\lim_{n\to\infty} \Pr\Big(([n], \mathcal{R}(n, k)) \text{ is NP complete }\Big) = 1,$$

while for a given $n \geq 2$,

$$\lim_{k\to\infty} \Pr\Big(([n], \mathcal{R}(n, k)) \text{ is NP complete }\Big) = 1.$$

The proof uses properties of random graphs together with an algebraic approach to the dichotomy conjecture (based on the analysis of clones of polymorphisms) which was pioneered by [28, 8].

## 7. Dualities

From the combinatorial point of view there is a standard way how to approach (and sometimes to solve) a monotone property $P$: one investigates those structures without the property $P$ which are *critical*, (or *minimal*) without $P$. One proceeds as follows: denote by $\mathcal{F}$ the class of all critical structures and define the class $\mathrm{Forb}(\mathcal{F})$ of all structures which do not "contain" any $\mathbf{F} \in \mathcal{F}$. The class $\mathrm{Forb}(\mathcal{F})$ is the class of all structures not containing any of the critical substructures and thus it is easy to see that $\mathrm{Forb}(\mathcal{F})$ coincides with the class of structures with the property $P$. Of course in most cases the class $\mathcal{F}$ is infinite yet a structural result about it may shed some light on property $P$. For example this is the case with 3-colorability of graphs where 4-critical graphs were (and are) studied thoroughly (historically mostly in relationship to Four Color Conjecture).

Of particular interest (and as the extremal case in our setting) are those monotone properties $P$ of structures which can be described by finitely many forbidden substructures. The object of the theory of homomorphism duality is to characterize a family $\mathcal{F}$ of obstructions to the existence of a homomorphism into a given structure $\mathbf{B}$. In a large sense, such a class $\mathcal{F}$ always exists; for instance, the class of all the structures not admitting a homomorphism to $\mathbf{B}$ has this property. However, it is desirable to seek a more tractable family of obstructions to make this characterization meaningful. The classical examples of graph theory makes this point clear. A graph is bipartite if and only if it does not contain an odd cycle; hence, the odd cycles are a family of obstructions to the existence of a homomorphism into the complete graph $K_2$. However, the class of directed graphs provides a much more fertile ground for the theory, and numerous examples of tree dualities and of bounded treewidth dualities are known (see [24]).

When the family $\mathcal{F}$ of obstructions is finite (or algorithmically "well behaved"), then such theorems clearly provide an example of *good characterizations* (in the sense of Edmonds). Any instance of such good characterization is called a *homomorphism duality.* This concept was introduced by [52] and applied to various graph theoretical good characterizations. The simplest homomorphism dualities are those where the family of obstructions consists from only singletons (i.e. single structures). In the other words such homomorphism dualities are described by a pair $\mathbf{A}$, $\mathbf{B}$ of structures as follows:

*(Singleton) Homomorphism Duality Scheme*

A structure $\mathbf{C}$ admits a homomorphism into $\mathbf{B}$ if and only if $\mathbf{A}$ does not admit a homomorphism into $\mathbf{C}$.

Despite the fact that singleton homomorphism dualities are scarce for both undirected and directed graphs, for more general structures (such as oriented matroids with suitable version of strong maps) the (singleton) homomorphism duality may capture general theorems such as Farkas Lemma (see [25]). In [52] are described all singleton homomorphism dualities for undirected graphs. As a culmination of several partial results all homomorphism dualities for general relational structures were finally described in [57]. This is not the end and more recently the homomorphism dualities emerged as an important phenomena in new context. This we will briefly describe.

7.1. **Generalized CSP Classes.** For the finite set of structures $\mathcal{D}$ in $\mathbf{Rel}(\Delta)$ we denote by $\mathbf{CSP}(\mathcal{D})$ the class of all structures $\mathbf{A} \in \mathbf{Rel}(\Delta)$ satisfying $\mathbf{A} \longrightarrow \mathbf{D}$ for some $\mathbf{D} \in \mathcal{D}$. Thus $\mathbf{CSP}(\mathcal{D})$ is the union of classes $\mathbf{CSP}(\mathbf{D})$ for all $\mathbf{D} \in \mathcal{D}$. This definition (of *generalized CSP class* is sometimes more convenient and in fact generalized CSP classes are polynomially equivalent to the classes described syntactically as MMSNP ( [14]; the equivalence is non trivial and follows from [31]).

These classes are sometimes called *color classes* and denoted by

$$\longrightarrow \mathcal{D} = \mathbf{CSP}(\mathcal{D}).$$

7.2. **Forb Classes.** Let $\mathcal{F}$ be a finite set of structures of $\mathbf{Rel}(\Delta)$. Denote by $\mathbf{Forb}(\mathcal{F})$ the class of all structures $\mathbf{A} \in \mathbf{Rel}(\boldsymbol{\Delta})$ which do not permit a homomorphism $\mathbf{F} \longrightarrow \mathbf{A}$ for any $\mathbf{F} \in \mathcal{F}$. Formally:

$$\mathbf{Forb}(\mathcal{F}) = \{\mathbf{A} \mid \text{there is no } f : \mathbf{F} \to \mathbf{A} \text{ with} \mathbf{F} \in \mathcal{F}\}$$

Let us remark that these classes are sometimes denoted by

$$\mathbf{Forb}(\mathcal{F}) = \mathcal{F} \not\longrightarrow .$$

7.3. **Finite Duality.** *Finite duality* is the equation of two classes: of the class $\mathbf{Forb}(\mathcal{F})$ and of the class $\mathbf{CSP}(\mathcal{D})$ for a particular choice of *forbidden set* $\mathcal{F}$ and *dual set* $\mathcal{D}$. Formally:

$$\mathbf{Forb}(\mathcal{F}) = \mathbf{CSP}(\mathcal{D}).$$

We also say that $\mathcal{D}$ *has finite duality.* Finite dualities were defined in [52]. They are being intensively studied from the logical point of view, and also in the optimization (mostly CSP) context.

We say that a class $\mathcal{K} \subset \mathbf{Rel}(\Delta)$ is *First Order definable* if there exists a first order formula $\phi$ (i.e. quantification is allowed only over elements) such that the class $\mathcal{K}$ is just the class of all structures $\mathbf{A} \in \mathbf{Rel}(\Delta)$ where $\phi$ is valid. Formally:

$$\mathcal{K} = \{\mathbf{A}; \mathbf{A} \models \phi\}.$$

It has been recently showed [1, 63] that if the class $\mathbf{CSP}(\mathcal{D})$ is first order definable then it has finite duality. (This is a consequence of the solution of an important *homomorphism preservation conjecture* solved in [63].) On the other hand the finite dualities in categories $\mathbf{Rel}(\Delta)$ were characterized in [15] as an extension of [57]. By combining these results we obtain:

**Theorem.** *For a finite set $\mathcal{D}$ relational structures in $\mathbf{Rel}(\Delta)$ are the following statements equivalent:*

(i) *The class $\mathbf{CSP}(\mathcal{D})$ is first order definable;*
(ii) *$\mathcal{D}$ has finite duality; explicitly, there exists a finite set $\mathcal{F}$ such that $\mathbf{Forb}(\mathcal{F}) = \mathbf{CSP}(\mathcal{D})$;*
(iii) *$\mathbf{Forb}(\mathcal{F}) = \mathbf{CSP}(\mathcal{D})$ for a finite set $\mathcal{A}$ of finite forests.*

We did not define what is a forest in a structure(see [57, 15]). For the sake of completeness let us say that a *forest* is a structure not containing any cycle. And a *cycle* in a structure $\mathbf{A}$ is either a sequence of distinct points and distinct tuples $x_0, r_1, x_1, \ldots, r_t, x_t = x_0$ where each tuple $r_i$ belongs to one of the relations $R(\mathbf{A})$ and each $x_i$ is a coordinate of $r_i$ and $r_{i+1}$, or, in the degenerated case $t = 1$ a relational tuple with at least one multiple coordinate. The *length* of the cycle is the integer $t$ in the first case and 1 in the second case. Finally the *girth* of a structure $\mathbf{A}$ is the shortest length of a cycle in $\mathbf{A}$ (if it exists; otherwise it is a forest).

In a sharp contrast with that, there are no finite dualities for (general) finite algebras. It has been recently shown [33] that there are no such dualities at all. Namely, one has

**Theorem.** *For every finite set $\mathcal{A}$ of finite algebras of a given type $(\delta_i)_{i\in I}$ and every finite algebra $\mathbf{B}$ there exists a finite algebra $\mathbf{A}$ such that $\mathbf{A} \in \mathbf{Forb}(\mathcal{A})$ and $\mathbf{A} \notin \mathbf{CSP}(\mathbf{B})$.*

(This concerns the standard homomorphisms $f : (X, (\alpha_i)_{i\in T}) \to (X', (\alpha'_i)_{i\in T})$ satisfying

$$(*) \qquad x = \alpha_i(x_1, \ldots, x_{n_i}) \Rightarrow f(x) = \alpha'_i(f(x_1), \ldots, f(x_{n_i}))\text{. )}$$

## 8. Restricted Dualities

8.1. **Special Classes.** In this section we deal with graphs only. We motivate this section by the following two examples.

**Example1**. Celebrated Grötzsch's theorem (see e.g. [2]) says that every planar graph is 3-colorable. In the language of homomorphisms this says that for every triangle free planar graph $G$ there is a homomorphism of $G$ into $K_3$.

Using the partial order terminology (for the homomorphism order $\mathcal{C}_\Delta$) the Grötzsch's theorem says that $K_3$ is an upper bound (in the homomorphism order) for the class $\mathcal{P}_3$ of all planar triangle free graphs. As obviously $K_3 \notin \mathcal{P}_3$ a natural question (first formulated in [48]) suggests: Is there yet a smaller bound? The answer, which may be viewed as a strengthening of Grötzsch's theorem, is positive: there exists a triangle free 3-colorable graph $H$ such that $G \to H$ for every graph $G \in \mathcal{P}_3$. Explicitly:

$$K_3 \not\longrightarrow G \Leftrightarrow G \longrightarrow H$$

for every planar graph $G$. Because of this we call such theorem *restricted duality.* A restricted duality asserts the duality but only for structures in a restricted class of graphs. The (non-trivial) existence of graph $H$ above has been proved in [49] (in a stronger version for proper minor closed classes). The case of planar graphs and triangle is interesting in its own as it is related to the Seymour conjecture and its partial solution [18], see [44]; it seems that a proper setting of this case is in the context of *$TT$-continuous mappings*, see [55]. Restricted duality results have been generalized since to other classes of graphs and to other forbidden subgraphs. In fact for every "forbidden" finite set of connected graphs we have a duality restricted to any proper subclass $\mathcal{K}$ of all graphs which is minor closed, see [49]. This then implies that Grötzsch's theorem can be strengthened by a sequence of ever stronger bounds and that the supremum of the class of all triangle free planar graphs does not exist.

**Example2**. Let us consider all sub-cubic graphs (i.e. graph with maximum degree $\leq 3$). By Brooks theorem (see e.g. [2]) all these graphs are 3-colorable with the single connected exception $K_4$. What about the class of all sub-cubic *triangle free* graphs? Does there exists a triangle free 3-colorable bound? The positive answer to this question is given in [20]. In fact for every finite set $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ of connected graphs there exists a graph $H$ with the following properties:

- $H$ is 3-chromatic;

- $G \longrightarrow H$ for every subcubic graph $G \in \mathbf{Forb}(\mathcal{F})$.

It is interesting to note that while sub-cubic graphs have restricted dualities (and, more generally, this also holds for the classes of bounded degree graphs) for the classes of degenerated graphs a similar statement is not true (in fact, with a few trivial exceptions, it is never true).

Where lies the boundary for validity of restricted dualities? We clarify this after introducing the formal definition.

**Definition.** A class $\mathcal{K}$ of graphs has a *all restricted dualities* if, for any finite set of connected graphs $\mathcal{F} = \{F_1, F_2, \dots, F_t\}$, there exists a finite graph $D_{\mathcal{F}}^{\mathcal{K}}$ such that $F_i \not\longrightarrow D_{\mathcal{F}}^{\mathcal{K}}$ for $i = 1, \dots, t$ and such that for all $G \in \mathcal{K}$ holds. Explicitly:

$$(F_i \not\longrightarrow G), i = 1, 2, \dots, t, \iff (G \longrightarrow D_{\mathcal{F}}^{\mathcal{K}})$$

.

It is easy to see that using the homomorphism order we can reformulate this definition as follows: A class $\mathcal{K}$ has restricted dualities if for any finite set of connected graphs $\mathcal{F} = \{F_1, F_2, \dots, F_t\}$ the class $\mathbf{Forb}(\mathcal{F}) \cap \mathcal{K}$ is bounded in the class $\mathbf{Forb}(\mathcal{F})$.

The main result of [50] can be then stated as follows:

**Theorem.** Any class of graphs with bounded expansion has all restricted dualities.

Of course we have to yet define bounded expansion (and we do so in the next section). But let us just note that both proper minor closed classes and bounded degree graphs form classes of bounded expansion. Consequently this result generalizes both Examples 1. and 2. In fact the seeming incomparability of bounded degree graphs and minor closed classes led the authors of [50] to the definition of bounded expansion classes.

8.2. **Bounded expansion classes.** Recall that the *maximum average degree* $\mathrm{mad}(G)$ of a graph $G$ is the maximum over all subgraphs $H$ of $G$ of the average degree of $H$, that is $\mathrm{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$. The *distance* $d(x, y)$ between two vertices $x$ and $y$ of a graph is the minimum length of a path linking $x$ and $y$, or $\infty$ if $x$ and $y$ do not belong to same connected component.

We introduce several notations:

- The *radius* $\rho(G)$ of a connected graph $G$ is:
$$\rho(G) = \min_{r \in V(G)} \max_{x \in V(G)} \mathrm{d}(r, x)$$

- A *center* of $G$ is a vertex $r$ such that $\max_{x\in V(G)} \mathrm{d}(r,x) = \rho(G)$.

**Definition** Let $G$ be a graph. A *ball* of $G$ is a subset of vertices inducing a connected subgraph. The set of all the families of pairwise disjoint balls of $G$ is noted $\mathfrak{B}(G)$.

Let $\mathcal{P} = \{V_1, \dots, V_p\}$ be a family of pairwise disjoint balls of $G$.

- The *radius* $\rho(\mathcal{P})$ of $\mathcal{P}$ is $\rho(\mathcal{P}) = \max_{X\in\mathcal{P}} \rho(G[X])$
- The *quotient* $G/\mathcal{P}$ of $G$ by $\mathcal{P}$ is a graph with vertex set $\{1, \dots, p\}$ and edge set $E(G/\mathcal{P}) = \{\{i,j\} : (V_i \times V_j) \cap E(G) \neq \emptyset \text{ or } V_i \cap V_j \neq \emptyset\}$.

We introduce several invariants that generalize the one of maximum average degree:

**Definition** The *greatest reduced average density (grad)*of $G$ with rank $r$ is

$$\nabla_r(G) = \max_{\substack{\mathcal{P}\in\mathfrak{B}(G)\\ \rho(\mathcal{P})\leq r}} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$$

The following is our key definition:

**Definition** A class of graphs $\mathcal{K}$ has *bounded expansion* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{K}$ and every $r$ holds

$$\nabla_r(G) \leq f(r). \tag{1}$$

$f$ is called the expansion function. Proper minor closed classes have expansion function bounded by a constant, regular graphs by the exponential function, geometric graphs such as $d$-dimensional meshes have polynomial expansion. Expansion function can grow arbitrary fast.

Finally note that bounded expansion classes have many applications. Some of them are included in [51].

8.3. **Lifts and Shadows.** We return to the general relational structures. We restrict by generalizing to a particular situation. Duals of structures are a fascinating subject (see e.g. [34, 58, 56]). In closing these sections on dualities we want to briefly stress the following aspect which is on the first glance surprising.

Consider again the general duality scheme: structure $\mathbf{C}$ admits a homomorphism to $\mathbf{D} \in \mathcal{D}$ if and only if $\mathbf{F}$ does not admit a homomorphism into $\mathbf{C}$ for any $\mathbf{F} \in \mathcal{F}$; formally:

$$\mathbf{Forb}(\mathcal{F}) = \mathbf{CSP}(\mathcal{D}).$$

One is somehow tempted to think that the left side of the definition is somewhat more restrictive, that the finitely many obstacles make the problem easy if not trivial. After all, while the class $\mathbf{CSP}(\mathbf{D})$ may be complicated and have membership problem NP-complete even for a simple graph $\mathbf{D}$ (such as $K_3$) the left side is always polynomially decidable (for every finite set $\mathcal{F}$). But in a way this is a misleading argument. The expressing power of the classes $\mathbf{Forb}(\mathcal{F})$ for finite sets $\mathcal{F}$ is very large. This follows from a recent work [32] which we now briefly describe. We start with an example.

Think of 3-coloring of graph $G = (V, E)$. This is a well known hard problem and there is a multiple evidence for this: concrete instances of the problem are difficult to solve (if you want a non-trivial example consider Kneser graphs; [42]), there is an abundance of minimal graphs which are not 3-colorable (these are called 4-critical graphs, see e.g. [29]) and in the full generality (and even for important "small" subclasses such as 4-regular graphs or planar graphs) the problem is a canonical NP-complete problem.

Yet the problem has an easy formulation. A 3-coloring is simple to formulate even at the kindergarten level. This is in a sharp contrast with the usual definition of the class NP by means of polynomially bounded non-deterministic computations. Fagin [13] gave a concise description of the class NP by means of logic: NP languages are just languages accepted by an Existential Second Order (ESO) formula of the form

$$\exists P \Psi(S, P),$$

where $S$ is the set of input relations, $P$ is a set of existential relations, the proof for the membership in the class, and $\Psi$ is a first-order formula without existential quantifiers. This definition of NP inspired a sequence of related investigations and these *descriptive complexity* results established that most major complexity classes can be characterized in terms of logical definability of finite structures. Particularly this led Feder and Vardi [14] to their seminal reduction of *Constraint Satisfaction Problems* to so called MMSNP (*Monotone Monadic Strict Nondeterministic Polynomial*) problems which also nicely link MMSNP to the class NP in computational sense. Inspired by these results we would like to ask an even simpler question:

Can one express the computational power of the class NP by combinatorial means?

It may seem to be surprising that the classes of relational structures defined by ESO formulas (i.e. the whole class NP) are polynomially equivalent to canonical *lifts* of structures which are defined by a finite set of forbidden substructures.

Shortly, finitely many forbidden lifts determine any language in NP.

Let us briefly illustrate this by our example of 3-colorability: Instead of a graph $G = (V, E)$ we consider the graph $G$ together with three unary relations $C_1, C_2, C_3$ which *cover* the vertex set $V$; this structure will be denoted by $G'$ and called a *lift* of $G$ ($G'$ has one binary and three unary relations). There are 3 *forbidden substructures* or *patterns*: For each $i = 1, 2, 3$ the graph $K_2$ together with cover $C_i = \{1, 2\}$ and $C_j = \emptyset$ for $j \neq i$ form pattern $\mathbf{F}_i$ (where the signature of $\mathbf{F}_i$ contains one binary and three unary relations). The class of all 3-colorable graphs then corresponds just to the class $\Phi(\mathrm{Forb}(\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3))$ where $\Phi$ is the forgetful functor which transforms $G'$ to $G$ and the language of 3-colorable graphs is just the language of the class satisfying formula $\exists G'(G' \in \mathrm{Forb}(\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3))$. This *extended language* (of structures $G'$) of course expresses the membership of 3-colorability to the class NP.

Let us define lifts and shadows more formally: We will work with two (fixed) signatures, $\Delta$ and $\Delta \cup \Delta'$ (the signatures $\Delta$ and $\Delta'$ are always supposed to be disjoint). For convenience we denote structures in $Rel(\Delta)$ by $\mathbf{A}, \mathbf{B}$ etc. and structures in $Rel(\Delta \cup \Delta')$ by $\mathbf{A}', \mathbf{B}'$ etc. For convenience we shall denote $Rel(\Delta \cup \Delta')$ by $Rel(\Delta, \Delta')$. The classes $Rel(\Delta)$ and $Rel(\Delta, \Delta')$ will be considered as categories endowed with all homomorphisms. The interplay of categories $Rel(\Delta, \Delta')$ and $Rel(\Delta)$ is here the central theme. Towards this end we define the following notion: Let $\Phi : Rel(\Delta, \Delta') \to Rel(\Delta)$ denote the natural *forgetful functor* that "forgets" relations in $\Delta'$. Explicitly, for a structure $\mathbf{A}' \in Rel(\Delta, \Delta')$ we denote by $\Phi(\mathbf{A}')$ the corresponding structure $\mathbf{A} \in Rel(\Delta)$ defined by $X(\mathbf{A}') = X(\mathbf{A})$, $R(\mathbf{A}') = R(\mathbf{A})$ for every $R \in \Delta$ (for homomorphisms we have $\Phi(f) = f$).

These object-transformations call for a special terminology: For $\mathbf{A}' \in Rel(\Delta, \Delta')$ we call $\Phi(\mathbf{A}') = \mathbf{A}$ the *shadow* of $\mathbf{A}'$. Any $\mathbf{A}'$ with $\Phi(\mathbf{A}') = \mathbf{A}$ is called a *lift* of $\mathbf{A}$. The analogous terminology is used for subclasses $\mathcal{K}$ of $Rel(\Delta, \Delta')$ and $Rel(\Delta)$.

The following combinatorial characterization of NP was recently proved in [32]:

**Theorem.** For every language $L \in NP$ there exist relational types $\Delta, \Delta'$ and a finite set $\mathcal{F}'$ of structures in $Rel(\Delta, \Delta')$ such that $L$ is computationally equivalent to $\Phi(Forb(\mathcal{F}'))$. Moreover, we may assume that the relations in $\Delta'$ are at most binary.

We omit the technical details (which are involved) but let us add the following: There seems to be here more than meets the eye. This scheme fits nicely into the mainstream combinatorial and combinatorial complexity research. Building upon Feder-Vardi classification of MMSNP we can isolate three computationally equivalent formulations of NP class:

(1) By means of shadows of forbidden homomorphisms of relational lifts (the corresponding category is denoted by $Rel^{cov}(\Delta, \Delta')$),
(2) By means of shadows of forbidden injections (monomorphisms) of monadic lifts (i.e. with type $\Delta'$ consisting from unary relations only),
(3) By means of shadows of forbidden full homomorphisms of monadic lifts (full homomorphisms preserve both edges and also non-edges).

Our results imply that each of these approaches includes the whole class NP. It is interesting to note how nicely these categories fit to the combinatorial common sense about the difficulty of problems: On the one side the problems in CSP correspond and generalize ordinary (vertex) coloring problems. One expects a dichotomy here. On the other side the above classes $1.-3.$ model the whole class NP and thus we cannot expect dichotomy there. But this is in accordance with the combinatorial meaning of these classes: the class 1. expresses coloring of edges, triples etc. and thus it involves problems in Ramsey theory [19, 45]. The class (2) may express vertex coloring of classes with restricted degrees of its vertices (which is difficult restriction in a homomorphism context) . The class (3) relates to vertex colorings with a given pattern among classes which appears in many graph decomposition techniques (for example in the solution of the Perfect Graph Conjecture [9]). The point of view of forbidden partitions (in the language of graphs and matrices) is taken for example in [17]. This clear difference between combinatorial interpretations of syntactic restrictions on formulas expressing the computational power of NP is one of the pleasant consequences of this approach. See [32] for details and other related problems.

## 9. Homomorphism Order

Recall that $\mathcal{C}_\Delta$ denote the homomorphism quasiorder of all relational structures of type $\Delta$:

$$\mathbf{A} \leq \mathbf{B} \Leftrightarrow \mathbf{A} \longrightarrow \mathbf{B}.$$

There are surprising close connections between algorithmic questions (which motivated dualities) and order theoretic properties of $\mathcal{C}_\Delta$. We mention two such results (characterization theorems).

9.1. **Gaps and Density.** A pair $(\mathbf{A}, \mathbf{B})$ of structures is said to be a *gap* in $\mathcal{C}_\Delta$ if $\mathbf{A} < \mathbf{B}$ and there is no structure $\mathbf{C}$ such that $\mathbf{A} < \mathbf{C} < \mathbf{B}$.

Similarly, for a subset $\mathcal{K}$ of $\mathcal{C}$, a pair $(\mathbf{A}, \mathbf{B})$ of structures of $\mathcal{K}$ is said to be a *gap in* $\mathcal{K}$ if $\mathbf{A} < \mathbf{B}$ and there is no structure $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} < \mathbf{C} < \mathbf{B}$.

The *Density Problem* for a class $\mathcal{K}$ is ask for the description of all gaps of the class $\mathcal{K}$. This is a challenging problem even in the simplest case of the class of all undirected graphs. This question has been asked first in the context of the structure properties of classes of languages and grammar forms. The problem has been solved in [68]:

**Theorem.** The pairs $(K_0, K_1)$ and $(K_1, K_2)$ are the only gaps for the class of all undirected graphs. Explicitly, given undirected graphs $G_1, G_2$, $G_1 < G_2$, $G_1 \neq K_0$ and $G_1 \neq K_1$ there is a graph $G$ satisfying $G_1 < G < G_2$.

The density problem for general classes $\mathbf{Rel}(\Delta)$ was solved only in [57] in the context of the characterization of finite dualities.

**Theorem.** For every class $\mathbf{Rel}(\Delta)$ the following holds:

(1) For every (relational) tree $\mathbf{T}$ there exists unique structure $\mathbf{P_T}$ *predecessor of* $\mathbf{T}$ such that the pair $(\mathbf{P_T}, \mathbf{T})$ is a gap in $\mathbf{Rel}(\Delta)$;
(2) Up to homomorphism equivalence there are no other gaps in $\mathbf{Rel}(\Delta)$ of the form $\mathbf{A} < \mathbf{B}$ with $\mathbf{B}$ connected.

The importance of this lies in the next result (a gap of the form $\mathbf{A} < \mathbf{B}$ with $\mathbf{B}$ connected is called a *connected gap*).

**Theorem.** For every category $\mathbf{Rel}(\Delta)$ there is a one to one correspondence between connected gaps and singleton dualities.

In fact this theorem holds in a broad class of posets called *Heyting posets*, [53]. The characterization of gaps in the subclasses of structures present a difficult problem.

9.2. **Maximal Antichains.** Let $\mathcal{P} = (P, \leq)$ be a poset. We say that a subset $Q$ of $P$ is an *antichain* in $\mathcal{P}$, if neither $a \leq b$ nor $b \leq a$ for any two elements $a$, $b$ of $Q$ (such elements are called *incomparable* (this fact is usually denoted by $a \| b$). A finite antichain $Q$ is called *maximal*, if any set $S$

such that $Q \subsetneqq S \subseteq P$ is not an antichain. One can determine maximal antichains in classes $\mathbf{Rel}(\Delta)$.

Consider a duality

$$\mathbf{Forb}(\mathcal{F}) = \mathbf{CSP}(\mathcal{D})$$

and consider the set $\mathcal{M} = \mathcal{F} \cup \mathcal{D}$. Then $\mathcal{M}$ has the property that any other structure $\mathbf{A} \in \mathbf{Rel}(\Delta)$ is comparable to one of its elements (as any structure $\mathbf{A} \in \mathbf{Rel}(\Delta)$ either satisfies $\mathbf{F} \to \mathbf{A}$ for an $\mathbf{F} \in \mathcal{F}$ or $\mathbf{A} \to \mathbf{D}$ for an $\mathbf{D} \in \mathcal{D}$. One can prove the converse of this statement:

**Theorem.** Let $\Delta = (k)$. There is a one-to-one correspondence between generalized dualities and finite maximal antichains in the homomorphism order of $\mathbf{Rel}(\Delta)$.

9.3. **Universality.** The homomorphism order $\mathcal{C}_\Delta$ has spectacular properties. One of them is related to the following notion:

A countable partially ordered set $\mathcal{P}$ is said to be *universal* if it contains any countable poset (as an induced subposet).

A poset $\mathcal{P}$ is said to be *homogeneous* if every partial isomorphism between finite subposets extends to an isomorphism (of the whole poset). It is a classical model theory result that universal homogenous poset exists and that it is uniquely determined. Such universal homogeneous poset can be constructed in a standard model theoretic way as Fraïssé limit of all finite posets.

The poset $\mathcal{C}_\Delta$ fails to be homogeneous (due to its algebraic structure) but it is universal [21]. The oriented graphs create here again a little surprise: Denote by $Path$ the partial order of finite oriented paths (i.e. "zig-zags") with the homomorphism ordering. It seem that the order of paths is an easy one:

- Paths can be coded by $0-1$ sequences;
- One can decide whether $P \leq P'$ (by an easy rewriting rules);
- Density problem for paths has been solved, [59].

We finish this paper with the following non-trivial result which found immediately several applications [26, 27]:

**Theorem.** The partial order $Path$ is universal.

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