FORBIDDEN LIFTS
(NP AND CSP FOR COMBINATORISTS)

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Abstract. We present a definition of the class NP in combinatorial context as the class of languages of structures defined by finitely many forbidden lifted substructures. We apply this to special syntactically defined subclasses and show how they correspond to naturally defined (and intensively studied) combinatorial problems. We show that some types of combinatorial problems like edge colorings and graph decompositions express the full computational power of the class NP. We then characterize Constraint Satisfaction Problems (i.e. H-coloring problems) which are expressible by finitely many forbidden lifted substructures. This greatly simplifies and generalizes the earlier attempts to characterize this problem. As a corollary of this approach we perhaps find a proper setting of Feder and Vardi analysis of CSP languages within the class MMSNP.

1. Introduction

Think of a 3-colorability of a graph $G = (V, E)$. This is a well known hard problem and there is a multiple evidence for this: concrete instances of the problem are difficult to solve (if you want a non-trivial example consider Kneser graphs; [20]), there is an abundance of minimal graphs which are not 3-colorable (these are called 4-critical graphs, see e.g. [13]) and in the full generality (and even for important "small" subclasses such as 4-regular graphs or planar graphs) the problem is a canonical NP-complete problem.

Yet the problem has an easy formulation. A 3-coloring is simple to formulate even at the kindergarten level. This is in a sharp contrast with the usual definition of the class NP by means of polynomially bounded nondeterministic computations. Fagin [5] gave a concise description of the class

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NP by means of logic: NP languages are just languages accepted by an Existential Second Order (ESO) formula of the form

$$\exists P \Psi(S, P),$$

where S is the set of input relations, P is a set of existential relations, the proof for the membership in the class, and Ψ is a first-order formula without existential quantifiers. This definition of NP inspired a sequence of related investigations (see e.g. [12, 29] and these *descriptive complexity* results established that most major complexity classes can be characterized in terms of logical definability of finite structures. Particularly this led Feder and Vardi [6] to their seminal reduction of *Constraint Satisfaction Problems* (shortly CSP) to so called MMSNP (*Monotone Monadic Strict Nondeterministic Polynomial*) problems which also nicely link MMSNP to the class NP in computational sense. This will be explained in some detail in Section 3 which presents one of the main motivations of this paper. Inspired by these results we would like to ask an even simpler question:

Can one express the computational power of the class NP by combinatorial means?

From the combinatorial point of view there is a standard way how to approach (and sometimes to solve) a monotone property P: one investigates those structures without the property P which are *critical*, (or *minimal*) without P. One proceeds as follows: denote by $\mathcal{F}$ the class of all critical structures and define the class $\mathrm{Forb}(\mathcal{F})$ of all structures which do not "contain" any $F \in \mathcal{F}$. The class $\mathrm{Forb}(\mathcal{F})$ is the class of all structures not containing any of the critical substructures and thus it is easy to see that $\mathrm{Forb}(\mathcal{F})$ coincides with the class of structures with the property P. Of course in most cases the class $\mathcal{F}$ is infinite yet a structural result about it may shed some light on property P. For example this is the case with 3-colorability of graphs where 4-critical graphs were (and are) studied thoroughly (historically mostly in relationship to Four Color Conjecture).

Of particular interest (and as the extremal case in our setting) are those monotone properties P of structures which can be described by finitely many forbidden substructures. It has been proved in a sequence of papers [1, 28] that a homomorphism monotone problem is *First Order* (shortly FO) definable if and only if it is *positively* FO definable (shortly FO+ definable), i.e. the formula does not contain any negations (and so implications and inequalitites), and thus alternatively defined as Forb($\mathcal{F}$) for a finite set $\mathcal{F}$ of structures. Although FO-definability is not a rare fact (and extremely useful in database theory), still FO-definability cannot express most combinatorial problems (compare [26],[1] which characterize all CSP which are FO-definable; see also Theorem 1). Thus it may seem to be surprising that the classes of relational structures defined by ESO formulas (i.e. the whole class NP) corresponds exactly to those canonical *lifts* of structures which are defined by a finite set of forbidden substructures. Shortly, finitely many forbidden lifts determine any language in NP. This is being made precise in Section 3. Here, let us just briefly illustrate this by our example of 3-colorability. Instead of a graph $G = (V, E)$ we consider the graph G together with three unary relations C_1, C_2, C_3 which *cover* the vertex set V; this structure will be denoted by G' and called a *lift* of G (G' has one binary and three unary relations). There are 3 *forbidden substructures* or *patterns*: For each $i = 1, 2, 3$ the graph K_2 together with cover $C_i = \{1, 2\}$ and $C_j = \emptyset$ for $j \neq i$ form pattern $\mathbf{F_i'}$ (where the signature of $\mathbf{F_i'}$ contains one binary and three unary relations). The language of all 3-colorable graphs then corresponds just to the language $\Phi(\mathrm{Forb}(\mathbf{F_1'}, \mathbf{F_2'}, \mathbf{F_3'}))$ where Φ is the forgetful functor which transforms G' to G and the language of 3-colorable graphs is just the language of the class satisfying formula $\exists G'(G' \in \mathrm{Forb}(\mathbf{F_1'}, \mathbf{F_2'}, \mathbf{F_3'}))$. This *extended language* (of structures G') of course just expresses the membership of 3-colorability to the class NP. There is more than this that meets the eye. This scheme fits nicely into the mainstream combinatorial and combinatorial complexity research. Building upon Feder-Vardi classification of MMSNP we isolate (in Theorems 5, 7, 9) three computationally equivalent formulations of NP class:

(1) By means of shadows of forbidden homomorphisms of relational lifts (the corresponding category is denoted by $Rel^{cov}(\Delta, \Delta'))$,
(2) By means of shadows of forbidden injections (monomorphisms) of monadic lifts (the corresponding category will be denoted by $Rel^{cov}_{inj}(\Delta, \Delta'))$,
(3) By means of shadows of forbidden full homomorphisms of monadic lifts (the corresponding category will be denoted by $Rel^{cov}_{full}(\Delta, \Delta'))$.

Our results imply that each of these approaches includes the whole class NP. It is interesting to note how nicely these categories fit to the combinatorial common sense about the difficulty of problems: On the one side the problems in CSP correspond and generalize ordinary (vertex) coloring problems. One expects a *dichotomy* here: every CSP problem should be either polynomial or NP-complete (as conjectured in [6] and probabilistically verified in [17]). On the other side the above classes $1.-3.$ model the whole class NP and thus we cannot expect dichotomy there (by a celebrated result of Ladner [16]). But this is in accordance with the combinatorial meaning of these classes: the class 1. expresses coloring of edges, triples etc. and thus it involves problems in Ramsey theory [8, 22]. The class 2. may express vertex coloring of classes with restricted degrees of vertices [14, 10]. The class 3. relates to vertex colorings with a given pattern among classes which appears in many graph decomposition techniques (for example in the solution of the Perfect Graph Conjecture [3]). The point of view of forbidden partitions (in the language of graphs and matrices) is taken for example in [9]. This clear difference between combinatorial interpretations of syntactic restrictions on formulas expressing the computational power of NP is one of the pleasant consequences of our approach.

At this point we should add one more remark. We of course do not only claim that every problem in NP can be polynomially reduced to a problem in one of these classes. This would only mean that each of these classes contains an NP-complete problem. What we claim is that these classes have the *computational power* of the whole of NP, i.e. these classes are *computationally equivalent* to all problems in NP. More precisely, to each language L in NP there exists a language M in any of these three classes such that M is *polynomially equivalent* to L, i.e. there exist *polynomial reductions* of L to M and M to L.

Having finitely many forbidden patterns (i.e. forbidden substructures) for a class of structures $\mathcal{K}$ we are naturally led to the question whether $\mathcal{K}$ is the class determined by a finite set of templates, or in other words by the existence of homomorphisms to particular structures. In technical terms (see e.g. [10, 6]) this amounts to the question whether $\mathcal{K}$ is an instance of a *Constraint Satisfaction Problem* (shortly CSP). On the other hand finitely many forbidden patterns lead to the question whether the class $\mathcal{K}$ is not defined by a *finite duality*. This scheme for combinatorial problems goes back to [23], see e.g. [10] and it was studied in situations as diverse as *bounded tree width dualities* [11], *duality of linear programming* [10] and *classes with bounded expansion* [24]. Here we completely characterize (using results of [26]) shadows of finitary dualities in the case where the extension of the language is monadic, i.e. it consists of unary relations (as is the above case of 3-coloring), see Theorem 13. These general results can be used in the investigation of the class MMSNP (to be defined in Section 3). Feder and

Vardi introduced this class as a fragment of SNP in [6]. They proved that the class MMSNP is randomly polynomially equivalent to the class of finite union of CSP languages. This was later derandomized by the first author proving that the classes MMSNP and CSP are computationally equivalent [15]. We will examine these classes from the viewpoint of descriptive complexity theory: Any finite union of CSP languages belongs to MMSNP. But the converse does not hold. Consider for example the language of triangle free graphs: this is an MMSNP language which is not a finite union of CSP languages. Madelaine and Stewart introduced the class of *Forbidden Pattern Problems* (FP) as an equivalent combinatorial version of MMSNP [19], [18]. They gave an effective, yet lengthy process to decide whether an MMSNP language is a CSP language. We give a short and easy procedure to decide whether an MMSNP language is a finite union of CSP languages, and we show that these are exactly those languages defined by forbidden patterns not containing any cycle. This simplicity is possible by translation and generalization of the Feder-Vardi proof of the computational equivalence of finite union of CSP's and MMSNP in the context of category theoretical language of duality.

The paper is organized as follows: In Section 2 we review the basic notions and previous work related to finite structures. Particularly we state two our basic tools: the characterization of *finite dualities* [26, 7] and a combinatorial classique, the *sparse incomparability lemma*. It is here where we introduce two our basic notions of *lifts* and *shadows*. The interplay of corresponding classes (categories) is a central theme of this paper. In Section 3 we introduce the relevant notions of descriptive complexity (mostly taken from [6]) and relate it to our approach. We prove that the class NP is polynomially equivalent with classes of structures characterized by finitely many forbidden lifts (this is proved in three different categories, see Theorems 5, 7 and 9). In Section 4 we study the relationship of lifts and shadows abstractly from the point of view of dualities. Theorem 13 enables us to prove the characterization of shadows of finite dualities (called *lifted dualities*) in lifts and shadows. This, as a corollary, proves the main result of [19]. In Section 5 we return to Feder-Vardi setting and indicate how the polynomial equivalence of classes MMSNP and finite unions of CSP problems emerges naturally in our combinatorial-categorical context.

For more complicated (i.e. nonmonadic) lifts we (of course) have partial results only. Perhaps the next case is that of covering equivalences. This we are still able to handle with our methods and we characterize all CSP languages in this class. But we postpone this to another occasion.

2. Categories of Finite Structures

For a relational symbol R and relational structure $\mathbf{A}$ let $A = X(\mathbf{A})$ denote the universe of $\mathbf{A}$ and let $R(\mathbf{A})$ denote the relation set of tuples of $\mathbf{A}$ which belong to R. Let Δ denote the *signature* (type) of relational symbols, and let $Rel(\Delta)$ denote the class of all relational structures with signature Δ. We will often work with two (fixed) signatures, Δ and $\Delta \cup \Delta'$ (the signatures Δ and Δ' are always supposed to be disjoint). For convenience we denote structures in $Rel(\Delta)$ by $\mathbf{A}, \mathbf{B}$ etc. and structures in $Rel(\Delta \cup \Delta')$ by $\mathbf{A}', \mathbf{B}'$ etc. For convenience we shall denote $Rel(\Delta \cup \Delta')$ by $Rel(\Delta, \Delta')$. The classes $Rel(\Delta)$ and $Rel(\Delta, \Delta')$ will be considered as categories endowed with all homomorphisms. Recall, that a homomorphism is a mapping which preserves all relations. Just to be explicit, for relational structures $\mathbf{A}, \mathbf{B} \in Rel(\Delta)$ a mapping $f : X(\mathbf{A}) \longrightarrow X(\mathbf{B})$ is a *homomorphism* $\mathbf{A} \longrightarrow \mathbf{B}$ if for every relational symbol $R \in \Delta$ and for every tuple $(x_1, \ldots, x_t) \in R(\mathbf{A})$ we have $(f(x_1), \ldots, f(x_t)) \in R(\mathbf{B})$. Similarly we define homomorphisms for the class $Rel(\Delta, \Delta')$. The interplay of categories $Rel(\Delta, \Delta')$ and $Rel(\Delta)$ is one of the central themes of this paper. Towards this end we define the following notions: Let $\Phi : Rel(\Delta, \Delta') \to Rel(\Delta)$ denote the natural *forgetful functor* that "forgets" relations in Δ'. Explicitly, for a structure $\mathbf{A}' \in Rel(\Delta, \Delta')$ we denote by $\Phi(\mathbf{A}')$ the corresponding structure $\mathbf{A} \in Rel(\Delta)$ defined by $X(\mathbf{A}') = X(\mathbf{A})$, $R(\mathbf{A}') = R(\mathbf{A})$ for every $R \in \Delta$. For a homomorphism $f : \mathbf{A}' \longrightarrow \mathbf{B}'$ we put $\Phi(f) = f$. The mapping f is of course also a homomorphism $\Phi(\mathbf{A}') \longrightarrow \Phi(\mathbf{B}')$. This is expressed by the following diagram.

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "\Phi"'] & B' \arrow[d, "\Phi"'] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

These object-transformations call for a special terminology: For $\mathbf{A}' \in Rel(\Delta, \Delta')$ we call $\Phi(\mathbf{A}') = \mathbf{A}$ the *shadow* of $\mathbf{A}'$. Any $\mathbf{A}'$ with $\Phi(\mathbf{A}') = \mathbf{A}$

is called a *lift* of $\mathbf{A}$. The analogous terminology is used for subclasses $\mathcal{C}$ of $Rel(\Delta, \Delta')$ and $Rel(\Delta)$. (Thus, for example, for a subclass $\mathcal{C} \subseteq Rel(\Delta, \Delta'), \Phi(\mathcal{C})$ is the class of all shadows of all structures in the class $\mathcal{C}$.) The following special subclass of $Rel(\Delta, \Delta')$ will be important: denote by $Rel^{cov}(\Delta, \Delta')$ the class of all structures in $Rel(\Delta, \Delta')$ where we assume that all relations in Δ' have the same arity, say r, and that all the r-tuples of an object are contained by some relation in Δ'. The category $Rel^{cov}(\Delta, \Delta')$ is briefly called *covering* or *r-covering category*. Note that the class $Rel^{cov}(\Delta, \Delta')$ is closed under surjective homomorphisms. We will work with two other similar pairs of categories. We denote by $Rel_{inj}(\Delta)$ and $Rel_{full}(\Delta)$ the categories where the subjects are again the relational structures of type Δ, but the morphisms are the injective and full homomorphisms, respectively. We call a mapping a *full homomorphism* if it is relation and non-relation preserving, too. Such mappings have very easy structure, as every full homomorphism which is onto is a retraction. We denote by $Rel^{cov}_{inj}(\Delta, \Delta')$ and $Rel^{cov}_{full}(\Delta, \Delta')$ the subclasses containing the same class of objects than $Rel^{cov}(\Delta, \Delta')$. We only will use these notions in the case when Δ' contains monadic relations.

Let $\mathcal{F}'$ be a finite set of structures in the category $\mathcal{C}$ (one of the above categories). By $\mathrm{Forb}(\mathcal{F}')$ we denote the class of all structures $\mathbf{A}' \in \mathcal{C}$ satisfying $\mathbf{F}' \not\longrightarrow \mathbf{A}'$ for every $\mathbf{F}' \in \mathcal{F}'$. (This class is sometimes and perhaps more efficiently denoted by $\mathcal{F}' \not\to$.) Similarly (well, dually), for the finite set of structures $\mathcal{D}'$ in $\mathcal{C}$ we denote by $CSP(\mathcal{D}')$ the class of all structures $\mathbf{A}' \in \mathcal{C}$ satisfying $\mathbf{A}' \longrightarrow \mathbf{D}'$ for some $\mathbf{D}' \in \mathcal{D}'$. (This is sometimes denoted by $\to \mathcal{D}$.) Now suppose that the classes $\mathrm{Forb}(\mathcal{F}')$ and $CSP(\mathcal{D}')$ are equal. Then we say that the pair $(\mathcal{F}', \mathcal{D}')$ is a *finite duality* in $\mathcal{C}$. Explicitly, a finite duality means that the following equivalence holds for every structure $\mathbf{A}' \in \mathcal{C}$:

$\mathbf{F}' \not\longrightarrow \mathbf{A}'$ for every $\mathbf{F}' \in \mathcal{F}'$ iff $\mathbf{A}' \longrightarrow \mathbf{D}'_j$ for some $\mathbf{D}' \in \mathcal{D}'$.

One more definition is needed. In dualities (as well as in most of this paper) we are interested in the existence of a homomorphism (every CSP can be expressed by the existence of a homomorphism to a template; see [6],[10]) . Consequently we can also use the language of partially ordered sets and consider the *homomorphism order* $\mathcal{C}_\Delta$ defined on the class of all structures with signature Δ: we define the order $\leq$ by putting $\mathbf{A} \leq \mathbf{B}$ iff there is a homomorphism $\mathbf{A} \longrightarrow \mathbf{B}$. The ordering $\leq$ is clearly a quasiorder but this becomes a partial order if we either factorize $\mathcal{C}_\Delta$ by the homomorphism equivalence or, perhaps preferably, if we restrict $\mathcal{C}_\Delta$ to non-isomorphic *core* structures. We say that $\mathbf{A}$ is *core* if every homomorphism $\mathbf{A} \longrightarrow \mathbf{A}$ is an automorphism. Every finite structure $\mathbf{A}$ contains (up to an isomorphism) a uniquely determined core substructure homomorphically equivalent to $\mathbf{A}$, see [26, 10]. The following result was recently proved in [7] as a generalization of [26]. It characterizes finite dualities of finite structures., i.e. in the category $Rel(\Delta)$.

Theorem 1. *For every signature Δ and for every finite set $\mathcal{F}$ of (relational) forests there exists (up to a homomorphism equivalence) a uniquely determined set $\mathcal{D}$ of structures such that $(\mathcal{F}, \mathcal{D})$ forms a finite duality. Up to a homomorphism equivalence there are no other finite dualities.*

We did not define what is a forest in a structure(see [26, 7]). For the sake of completeness let us say that a *forest* is a structure not containing any cycle. And a *cycle* in a structure $\mathbf{A}$ is either a sequence of distinct points and distinct tuples $x_0, r_1, x_1, \ldots, r_t, x_t = x_0$ where each tuple r_i belongs to one of the relations $R(\mathbf{A})$ and each x_i is a coordinate of r_i and r_{i+1}, or, in the degenerated case $t = 1$ a relational tuple with at least one multiple coordinate. The *length* of the cycle is the integer t in the first case and 1 in the second case. Finally the *girth* of a structure $\mathbf{A}$ is the shortest length of a cycle in $\mathbf{A}$ (if it exists; otherwise it is a forest).

The study of homomorphism properties of structures not containing short cycles (i.e. with a large girth) is a combinatorial problem studied intensively. The following result has proved particularly useful in various applications. It is often called the *Sparse Incomparability Lemma*:

Lemma 2. *Let k, ℓ be positive integers and let $\mathbf{A}$ be a structure. Then there exists a structure $\mathbf{B}$ with the following properties:*

(1) *There exists a homomorphism $f : \mathbf{B} \longrightarrow \mathbf{A}$;*
(2) *For every structure $\mathbf{C}$ with at most k points the following holds: there exists a homomorphism $\mathbf{A} \longrightarrow \mathbf{C}$ if and only if there exists a homomorphism $\mathbf{B} \longrightarrow \mathbf{C}$;*
(3) *$\mathbf{B}$ has girth $\geq \ell$.*

$$\begin{tikzcd}
B \arrow[d, "f"'] \arrow[dr, dashed] & \\
A \arrow[r, dashed] & C
\end{tikzcd}$$

This result was proved in [25, 27] (see also [10]) by probabilistic methods. In fact in [25, 27] it was proved for graphs only but the proof is the same for finite systems. Of particular interest in this context is the question whether there exists an explicit construction of the structure $\mathbf{B}$. This is indeed possible: In the case of binary relations (digraphs) this was done in [21] and for general systems in [15].

3. NP by means of finitely many forbidden lifts

There is a standard connection between formulae and existence of homomorphisms. This goes back to [2] and it can be formulated as follows:

To every structure $\mathbf{A}$ in $Rel(\Delta)$ we associate the *canonical conjunctive existential formula* $\varphi_{\mathbf{A}}$ as the conjunction of the atoms $R_{\mathbf{A}}(\overline{x})$, where $R \in \Delta$ preceded by existential quantification of all elements of $\mathbf{A}$. Clearly this process may be reversed and thus there is a one-to-one correspondence between canonical conjunctive existential formulae and structures. It is then obvious that the following holds:

There is a homomorphism $\mathbf{A} \longrightarrow \mathbf{B}$ if and only if $\mathbf{B} \models \varphi_{\mathbf{A}}$.

Following Fagin [5], the class SNP consists of all problems expressible by an existential second-order formula with a universal first-order part. The class of problems expressible by an existential second-order formula is exactly the class NP when restricted to finite structures. The class SNP is computationally equivalent to NP. The input of any problem in SNP is a relational structure $\mathbf{A}$ of signature Δ with base set $A = X(\mathbf{A})$ and Π is a set of relations on the same base set A. In this situation it is customary to call the second order relations Π *proof.* Let us be more specific (see [6]): Every language (problem) L in SNP may be equivalently described by a formula of the form

$$\exists \Pi \forall \overline{x} \in X \bigwedge_i \neg\big(\alpha_i \wedge \beta_i \wedge \varepsilon_i\big),$$

where

(1) α_i is a conjunction of atoms or negated atoms involving variables and input relations (i.e. of the form $R(\overline{x})$ and $\neg R(\overline{x})$ for a relational symbol R and $\overline{x}$ a tuple of elements of X),

(2) β_i is a conjunction of atoms and negated atoms involving variables and existential (proof) relations (i.e. of the form $P(\overline{x})$ and $\neg P(\overline{x})$ for $P \in \Pi$ and $\overline{x}$ a tuple of elements of X) and

(3) ε_i is the conjunction of atoms involving variables and inequalities (i.e. of form $x \neq y$).

A formula of this type is called a *canonical formula* of the language L in SNP. It will be denoted by φ_L.

Example: Consider the language containing one binary symbol R with two unary proof relations P_1, P_2 defined by the following formula:
$\exists P_1 \exists P_2 \forall x_1, x_2, x_3, y \in X \bigwedge_k \neg\big[(P_k(x_1) \wedge P_k(x_2) \wedge P_k(x_3)) \wedge (R(x_1, x_2) \wedge R(x_1, x_3) \wedge R(x_2, x_3)) \wedge (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)\big] \wedge \big[\neg(\neg P_1(y) \wedge \neg P_2(y))\big]$.

This formula corresponds to the language of all binary relations whose base set can be covered by two sets in such a way that none of these sets contains linearly ordered set with 3 elements. If we in addition postulate that the relation R is symmetric then these are just graphs which can be vertex partitioned into two triangle free graphs.

Following [6] one can also define three important syntactically restricted subclasses of SNP:

We say that a canonical formula is *monotone* if there are no negations in the α_i's. This implies that more relations lead to less satisfiable formulae. The canonical formula is *monadic* if the relations in Π are all monadic (which means that all proof relations are unary). The canonical formula is said to be *without inequality* if it can described by a canonical formula which does not contain ε_i.

Feder and Vardi [6] have proved that the three subclasses of SNP defined by formulae with any two of these syntactical restrictions still have the full computational power of the class NP:

Theorem 3. [6]

(1) *Every problem in NP has a polynomially equivalent problem in monotone SNP without inequality. Moreover, we may assume that the existential relations are at most binary.*
(2) *Every problem in NP has a polynomially equivalent problem in monotone, monadic SNP.*
(3) *Every problem in NP has a polynomially equivalent problem in monadic SNP without inequality.*

(The claim that we may restrict to binary relations in (1) is not stated in [6] but it is clear from the proof.) The class with all the three restrictions is denoted by MMSNP (*Monotone Monadic Strict Nondeterministic Polynomial*). We deal with this class in Section 6.

In this paper we will formulate and prove Theorem 3 in our combinatorial category lift/shadow setting. This will be done in Theorem 5 for item 1., in Theorem 7 for item 2. and in Theorem 9 for item 3. First, we introduce the following: we say that the formula is *primitive* if for every clause $(\alpha_i \wedge \beta_i \wedge \varepsilon_i)$, every variables $x_1, \dots, x_r$ occurring in it and every existential relation $P \in \Pi$ of arity r either the atom $P(x_1, \dots, x_r)$ or its negation is an atom of the clause. We need the following technical lemma.

Lemma 4. *Every language in SNP can be described by a primitive formula. Moreover, if the original formula satisfies some of the restrictions (i.e. if it is either monotone or monadic or without inequality) then so does the primitive formula.*

Proof. Consider the language L and the canonical formula defining L: $\exists P \forall \overline{x} \in S \bigwedge_i \neg(\alpha_i \wedge \beta_i \wedge \varepsilon_i)$. We modify the formula so that for every proof relation R of arity r and variables $x_1, \dots, x_r \in S$ appearing in α_i or β_i either $R(x_1, \dots, x_r)$ or $\neg R(x_1, \dots, x_r)$ is in the appropriate conjunct. In order to have such a formula we can replace $\neg\Big(\alpha_i \wedge \beta_i \wedge \varepsilon_i\Big)$ by $\neg\big(\alpha_i \wedge \beta_i \wedge \varepsilon_i \wedge R(x_1, \dots, x_r)\big) \wedge \neg\big(\alpha_i \wedge \beta_i \wedge \varepsilon_i \wedge \neg R(x_1, \dots, x_r)\big)$, this is equivalent to the original formula. The repetition of this process will terminate in finitely many steps, and it gives an appropriate formula. $\square$

Let $\mathcal{F}'$ be a finite set of structures in $Rel^{cov}(\Delta, \Delta')$. Consider a language L of structures in the class $Rel(\Delta)$ and its canonical primitive formula φ_L (showing that it is in monotone SNP without inequality). We say that L *is the language of* $\Phi(\mathrm{Forb}(\mathcal{F}'))$ if the formula φ_L and the formula $\exists \mathbf{A}'(\mathbf{A}' \in \mathrm{Forb}(\mathcal{F}'))$ are equivalent. Explicitly, $\exists \Pi(\mathbf{A} \models \varphi_L)$ if and only if $\exists \mathbf{A}'(\mathbf{F}' \not\longrightarrow \mathbf{A}'$ for every $\mathbf{F}' \in \mathcal{F}')$. This will be briefly denoted by $L = \Phi(\mathrm{Forb}(\mathcal{F}'))$.

Theorem 5. *For every language $L \in NP$ there exist relational types Δ, Δ' and a finite set $\mathcal{F}'$ of structures in $Rel^{cov}(\Delta, \Delta')$ such that L is computationally equivalent to $\Phi(Forb(\mathcal{F}'))$. Moreover, we may assume that the relations in Δ' are at most binary.*

This theorem presents an equivalent form of item 1. of Theorem 3 by means of homomorphisms and classes $Forb(\mathcal{F}')$. It is interesting to express other conditions 2., 3. of Theorem 3 by means of homomorphisms and classes $Forb(\mathcal{F}')$. These two other versions are stated below as Theorems 7 and 9.

Proof. Consider a language L and the canonical formula φ_L (showing that it is monotone SNP without inequality). The construction of $\mathcal{F}'$ consists of two steps. In the first step we enforce technical conditions on the formula.

Step 1.

We need the technical assumption that all proof relations in Π have the same (at most binary) arity and the formula is primitive. The first condition can be achieved by exchanging relational symbols that are not of maximal arity by new relational symbols of maximal arity (binary would suffice). We can proceed as follows. In every clause of the formula we put new (free) different variables into the new entries in β_i, and we increase the number of variables in $\overline{x}$, too. This new formula is equivalent to the original one. An evaluation satisfies to the new formula exactly iff its restriction to the original variables satisfies to the original formula. By Lemma 4 we may also assume that the new formula is primitive. In the following we denote this formula by φ_L.

In the second step we define lifts.

Step 2.
The type Δ' will contain $2^{|\Pi|}$ relational symbols corresponding to the $2^{|\Pi|}$ possibilities for a subset of proof relations indicating possibilities in which a tuple can be. The pattern $\mathbf{F}'_i$ will correspond to the clause $\alpha_i \wedge \beta_i$. The base set of each structure $\mathbf{F}'_i$ is the set of variables in the clause $\alpha_i \wedge \beta_i$. A tuple $\overline{t}$ of variables is in a relation R (of type Δ) if the atom $R(\overline{t})$ appears in α_i. Every tuple $\overline{t}$ in $\mathbf{F}'_i$ (of appropriate arity) is in exactly one relation from Δ', this is the relation corresponding to the subset of all existential relations $P \in \Pi$ such that the atom $P(\overline{t})$ appears in β_i. Let $\mathcal{F}'$ be the set of all lifts $\mathbf{F}'_i$. These may be disconnected, although we may work with their connected components, see Remark 6.

We prove that for a structure $\mathbf{A} \in Rel(\Delta)$ the formula $\varphi_L(\mathbf{A})$ is satisfiable iff there is a lifted structure $\mathbf{A}' \in Rel(\Delta, \Delta')$ such that no $\mathbf{F}'_i \in \mathcal{F}'$ maps to $\mathbf{A}'$ ($\mathbf{A}'$ is the lift of $\mathbf{A}$ determined by Π). Towards this end we first construct a formula $\varphi_{\mathcal{F}'}$ such that $\varphi_{\mathcal{F}'}(\mathbf{A}') \iff \mathbf{A}' \in \mathrm{Forb}(\mathcal{F}')$. This is easy (and for a single structure it was explained at the beginning of this section). The formula $\varphi_{\mathcal{F}'}(\mathbf{A}')$ will have the form $\forall \overline{x} \in \mathbf{A} \bigwedge_{\mathbf{F}' \in \mathcal{F}'} \neg \varphi_{\mathbf{F}'}(x_1, \ldots, x_{|F'|})$. The formula $\varphi_{\mathbf{F}'}(x_1, \ldots, x_{|F'|})$ is expressing the fact that the set $\{x_1, \ldots, x_{|\mathbf{F}'|}\} \subseteq \mathbf{A}'$ is the homomorphic image of $\mathbf{F}'$. Now $\varphi_{\mathbf{F}'}(x_1, \ldots, x_{|F'|})$ is a conjunction of atomic formulae. If the tuple $\overline{a}$ is in the input relation R then $\varphi_{\mathbf{F}'}(x_1, \ldots, x_{|F'|})$ will have an atom expressing that the image of the tuple is in the relation R.

The formula $\exists \Pi \varphi_{\mathbf{F}'}(x_1, \ldots, x_{|F'|})$ is primitive. In other words for every r-tuple $\overline{y}$ of variables and $P \in \Pi$ either $P(\overline{y})$ or $\neg P(\overline{y})$ is an atom of the formula. Consider the formula $\exists \Pi \forall \overline{x} \in A \bigwedge_{\mathbf{F}' \in \mathcal{F}} \neg \varphi_{\mathbf{F}'}(x_1, \ldots, x_{|F'|})$, this is also a primitive formula.

The unique Δ' relation covering the appropriate tuple of $\mathbf{F}'$ determines which atoms are negated and which are not. The proper disjoint covering lifts $\mathbf{A}'$ of $\mathbf{A}$ and the Π relational structures on the universe of $\mathbf{A}$ satisfying the formula are in one-to-one correspondence and this correspondence is provided by the forgetful functor. So the constructed formula $\exists \mathbf{A}' \varphi_{\mathcal{F}'}(\mathbf{A}')$ is equivalent to $\varphi_L(\mathbf{A})$. Moreover, if we exchange an atom of the form $R(\overline{x})$, where $R \in \Delta'$ for the appropriate conjunct of $|\Pi|$ atoms or negated atoms we get exactly the original primitive formula φ_L. $\square$

Remark 6. Consider the languages $\Phi(Forb(\mathcal{F}'))$ and $\Phi(Forb(\mathcal{G}'))$. Their union is exactly the language $\Phi(Forb(\mathcal{H}'))$, where $\mathcal{H}' = \{\mathbf{F}' \cup^* \mathbf{G}' : \mathbf{F}' \in \mathbf{Forb}(\mathcal{F}'), \mathbf{G}' \in \mathbf{Forb}(\mathcal{G}')\}$. Hence the languages of the form $\Phi(Forb(\mathcal{F}'))$ are closed under union. In the proof of Theorem 5 we may restrict ourselves to connected lifts when proving that the constructed $\Phi(Forb(\mathcal{F}'))$ is the desired language.

Let us now formulate and prove the two analogous theorems for the class monotone and monadic SNP and for the class monadic SNP without inequality (which correspond to 2. and 3. of Theorem 3). Here we use the categories $Rel^{cov}_{inj}(\Delta, \Delta')$ and $Rel^{cov}_{full}(\Delta, \Delta')$.

Theorem 7. *For every language $L \in NP$ there exists relational types Δ and Δ', where Δ' contains only unary relational symbols and a finite set $\mathcal{F}' \subset Rel^{cov}_{inj}(\Delta, \Delta')$ such that L is computationally equivalent to the class $\Phi(\mathrm{Forb}_{inj}(\mathcal{F}'))$.*

Proof. We proceed analogously as in the proof of Theorem 5 for formulas which are monadic monotone SNP. We stress the differences only. First, using Lemma 4 again, we may suppose that L is defined by a canonical primitive formula. This constitutes the first step as now we do not have problem with the arity of the proof relations since these are all monadic.

Step 2.

We want to enforce for $(\alpha_i \wedge \beta_i \wedge \varepsilon_i)$ and distinct variables x, y appearing in it that $x \neq y$ is an atom of ε_i. If this atom is not in β_i then we exchange $\neg(\alpha_i \wedge \beta_i \wedge \varepsilon_i)$ by the following conjunction: $\neg(\alpha_{i_1} \wedge \beta_{i_1} \wedge \varepsilon_{i_1}) \bigwedge \neg(\alpha_{i_2} \wedge \beta_{i_2} \wedge \varepsilon_{i_2})$, where $\neg(\alpha_{i_1} \wedge \beta_{i_1} \wedge \varepsilon_{i_1})$ is $\neg(\alpha_i \wedge \beta_i \wedge \varepsilon_i)$ except that we replace all occurence of y by x in it, $\alpha_{i_2} = \alpha_i, \beta_{i_2} = \beta_i$ and $\varepsilon_{i_2} = \varepsilon_i \wedge (x \neq y)$. This new formula is equivalent to the original one. In finitely many steps we manage to enforce that all the required atoms of the form $x \neq y$ are there in the appropriate ε_i.

We now define Δ' in the same way as in Theorem 5 (thus Δ' is a monadic type). The set of forbidden lifts $\mathcal{F}'$ is also defined analogously as in Theorem 5 with the only one difference which relates to the construction of formula $\varphi_{\mathbf{F}'}$ which will have now more clauses: the formula $\varphi_{\mathbf{F}'}$ will have all the atom clauses as in Theorem 5 (i.e. $\varphi_{\mathbf{F}'}(x_1, \ldots, x_{|\mathbf{F}'|})$ will contain as atoms all those tuples which express the fact that a tuple $\overline{a}$ is in the homomorphic image of $\mathbf{F}'$) and additionally we will have atoms $x \neq y$ for every pair of different variables. After this change we see easily that the rest of the proof does not depend on which category we work.

□

Remark 8. If we do not enforce the condition that the atom $x \neq y$ appears in every clause containing the variables x and y (Step 1. of the proof) before constructing $\mathcal{F}'$ then we get some weaker characterization. Namely, the language L will be similar to the form of Theorem 7 but we allow partially injective mappings not only injective ones. For every $\mathbf{F}' \in \mathcal{F}'$ and pair $x, y \in \mathbf{F}'$ we may have the plus condition that they can not collapse by a homomorphism. The class defined by such partially injective forbidden lifts still equals to the class of languages of the form $\Phi(\mathrm{Forb}_{inj}(\mathcal{F}'))$: we can do Step 2. in this combinatorial setting, too. Here the transformation means that for any $\mathbf{F}' \in \mathcal{F}'$ and pair $x, y \in \mathbf{F}'$ which may collapse, we exchange $\mathbf{F}'$ by two new forbidden structures. One of the structures is $\mathbf{F}'$ with conditions on the same pairs plus we require that x and y may not collapse. The other is a factor of $\mathbf{F}'$ where we identify x and y, and we have the condition on a pair of elements of new structures not to collapse iff we have it on a pair in their preimages in $\mathbf{F}'$. The iteration of this transformation expresses a language defined in partially injective setting in the fully injective terminology of Theorem 7 (with the same Δ and Δ').

Let us now transform the third syntactic class of Theorem 3.

Theorem 9. *For every language $L \in NP$ there exist relational types Δ and Δ', where Δ' contains only unary relational symbols and a finite set $\mathcal{F}' \subset Rel^{cov}_{full}(\Delta, \Delta')$ such that L is computationally equivalent to the class $\Phi(\mathrm{Forb}_{full}(\mathcal{F}'))$.*

Proof. The proof of Theorem 9 is again a modification of the above proof of Theorem 5 (and of Theorem 7) for formulas in the monadic SNP without inequality. The construction of $\mathcal{F}'$ is even easier: Again, in Step 1., it suffices to assume that φ_L is canonical primitive. We only need to be careful with construction of the formula $\varphi_{\mathbf{F}'}(\mathbf{A}')(x_1, \ldots, x_{|'\mathbf{F}|})$ expressing the fact that the set $\{x_1, \ldots, x_{|\mathbf{F}'|}\} \subseteq \mathbf{A}'$ is the homomorphic image of $\mathbf{F}'$ (recall that all homomorphism are now considered in $Rel_{full}(\Delta, \Delta')$). The formula will contain again more atoms. For every tuple $\overline{a}$ in the input relation R we will have an atom expressing that the image of the tuple is in relation R like in the proof of Theorem 5. But additionally we will have the negation of such an atom for every tuple not contained by an input relation. The rest of the proof is again the same. □

Similarly as above (Remark 8 to Theorem 7) we have the possibility to state a weaker theorem in the notion of partially full mappings. Consider a relational symbol R of arity q. We may have two conditions on an q-tuple in a structure $\mathbf{A}$, either it is in R or not. In the category Rel_{full} this gives some restrictions on the homomorphisms of $\mathbf{A}$ in both cases. We may generalize the class of objects such that for every relation R and q-tuple we have three possibilities (from the viewpoint of mappings to a structure): either the tuple should be mapped to a tuple in R, or to a tuple not in R or we have no restriction. We may define a class of languages in $Rel(\Delta)$ using this enlarged set of forbidden lifts. However this new class of languages is still equal to those of the form $\Phi(\mathrm{Forb}_{full}(\mathcal{F}'))$. This may be seen as follows: a forbidden lift in this new setting may be replaced by a set of forbidden lift in $Rel^{cov}_{full}(\Delta, \Delta')$ as for the set of tuple-relation pairs without any condition we take all possibilities of relation and non-relation conditions. This new set of forbidden lifts defines the same language.

4. Lifts and Shadows of Dualities

Some of the transformations presented in Section 4 lead to deeper results - the lifts and shadows give rise to a life on their own. We prove here two results which will prove to be useful in the next section.

It follows from the Section 3 that shadows of classes $\mathrm{Forb}(\mathcal{F}')$ (in three categories $Rel^{cov}(\Delta, \Delta')$, $Rel^{cov}_{inj}(\Delta, \Delta')$ and $Rel^{cov}_{full}(\Delta, \Delta')$) include all NP-complete languages. What about finite dualities? A delicate interplay of lifting and shadows for dualities is expressed by the following two statements which deal (for brevity) with classes $Rel^{cov}(\Delta, \Delta')$ only. Despite its formal complexity Theorem 10 is an easy statement:

Theorem 10. *Let $\mathcal{F}'$ be a finite set of structures in $Rel^{cov}(\Delta, \Delta')$. Suppose that there exist a finite set of structures $\mathcal{D}'$ such that $(\mathcal{F}', \mathcal{D}')$ is a finite duality in $Rel^{cov}(\Delta, \Delta')$. Then the following sets coincide: the shadow $\Phi(\mathrm{Forb}(\mathcal{F}')) = \{\Phi(\mathbf{A}') : \mathbf{A}' \in \mathrm{Forb}(\mathcal{F}')\}$ and*
$CSP(\Phi(\mathcal{D}')) = \bigvee_{\mathbf{D}' \in \mathcal{D}'} CSP(\Phi(\mathbf{D}'))$. Explicitly: for every $\mathbf{A} \in Rel(\Delta)$ there exists $\mathbf{A}' \in Rel(\Delta, \Delta'), \Phi(\mathbf{A}') = \mathbf{A}$ and $\mathbf{F}' \not\longrightarrow \mathbf{A}'$ for every $\mathbf{F}' \in \mathcal{F}'$ iff $\mathbf{A} \longrightarrow \Phi(\mathbf{D}')$ for some $\mathbf{D}' \in \mathcal{D}'$.

Note that we do not claim that the pair $(\mathcal{F}, \mathcal{D})$ is a duality in the class $Rel(\Delta)$. This of course does not hold (as shown by our example of 3-colorability in the introduction). But the images of all structures defined by all obstacles of $CSP(\mathcal{D}')$ are forming all obstacles of $CSP(\Phi(\mathcal{D}'))$. We call this *shadow duality.*

It is important that Theorem 10 may be sometimes reversed: shadow dualities may be sometimes "lifted". This is non-trivial and in fact Theorem 11 may be seen as the core of this paper.

Theorem 11. *Let $\mathcal{F}'$ be a finite set of structures in $Rel^{cov}(\Delta, \Delta')$, consider* $\mathrm{Forb}(\mathcal{F}')$ *and suppose that* $\Phi(\mathrm{Forb}(\mathcal{F}')) = CSP(\mathcal{D})$ *(in $Rel(\Delta)$) for a finite set $\mathcal{D}$ of objects of $Rel(\Delta)$. (In other words let the pair $(\mathcal{F}', \mathcal{D})$ form a shadow duality.) Assume also that $CSP(\mathcal{D}) \neq Rel(\Delta)$ and that Δ' contains only unary relations. Then there exists a finite set $\mathcal{D}'$ in $Rel^{cov}(\Delta, \Delta')$ such that* $\mathrm{Forb}(\mathcal{F}') = CSP(\mathcal{D}')$.

Before proving Theorems 10 and 11 we formulate first a lemma which we shall use repeatedly:

Lemma 12. *(lifting) Let $\mathbf{A}, \mathbf{B} \in Rel(\Delta)$, homomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ and $\Phi(\mathbf{B}') = \mathbf{B}$ be given. Then there exists $\mathbf{A}' \in Rel^{cov}(\Delta, \Delta')$, $\Phi(\mathbf{A}') = \mathbf{A}$ such that the mapping f is a homomorphism $\mathbf{A}' \longrightarrow \mathbf{B}'$ in $Rel^{cov}(\Delta, \Delta')$.*

$$\begin{tikzcd} A' \arrow[r, dashed, "f"] \arrow[d, dashed, "\Phi"'] & B' \arrow[d, "\Phi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Proof. Assume that $\mathbf{A}, \mathbf{B} \in Rel(\Delta)$, $\Phi(\mathbf{B}') = \mathbf{B}$ and $f : \mathbf{A} \longrightarrow \mathbf{B}$ are as in the statement. For each $R \in \Delta'$ put $R(\mathbf{A}') = f^{-1}(R(\mathbf{B}'))$. It is easy to see that $\mathbf{A}' \in Rel^{cov}(\Delta, \Delta')$ □

Proof. (of Theorem 10) Suppose that $\mathbf{A} \in CSP(\Phi(\mathcal{D}'))$, say $\mathbf{A} \in CSP(\Phi(\mathbf{D}'))$. Now for a homomorphism $f : \mathbf{A} \longrightarrow \Phi(\mathbf{D}')$ there is at least one lift $\mathbf{A}'$ of $\mathbf{A}$ such that the mapping f is a homomorphism $\mathbf{A}' \to \mathbf{D}'$ (here we use Lifting Lemma 12). By the duality $(\mathcal{F}', \mathcal{D}')$ (in $Rel^{cov}(\Delta, \Delta')$) $\mathbf{F}' \nrightarrow \mathbf{A}'$ for any $\mathbf{F}' \in \mathcal{D}'$ and thus in turn $\mathbf{A} \in \Phi(\mathrm{Forb}(\mathcal{F}'))$.

Conversely, let us assume that $\mathbf{A}' \in \mathrm{Forb}(\mathcal{F}')$ satisfies $\Phi(\mathbf{A}') = \mathbf{A}$. But then $\mathbf{A}' \in CSP(\mathcal{D}')$ and thus by the functorial property of Φ we have $\mathbf{A} = \Phi(\mathbf{A}') \in CSP(\Phi(\mathcal{D}'))$.

□

Proof. (of Theorem 11) Assume $\Phi(\mathrm{Forb}(\mathcal{F}')) = CSP(\mathcal{D})$. Our goal is to find $\mathcal{D}'$ such that $\mathrm{Forb}(\mathcal{F}') = CSP(\mathcal{D}')$. This will follow as a (non-trivial) combination of Theorems 1 and 2. By Theorem 1 we know that if $\mathcal{F}'$ is a set of (relational) forests then the set $\mathcal{F}'$ has a dual set $\mathcal{D}'$ (in the class $Rel^{cov}(\Delta, \Delta')$). So assume the contrary that one of the structures, say $\mathbf{F}'_0$, fails to be a forest (i.e. we assume that one of the components of $\mathbf{F}'_0$ has a cycle). We proceed by a refined induction (which will allow us to use more properties of $\mathbf{F}'_0$). Let us introduce carefully the setting of the induction.

We assume shadow duality $\Phi(\mathrm{Forb}(\mathcal{F}')) = CSP(\mathcal{D})$. Let $\mathcal{D}$ be fixed throughout the proof. Clearly many sets $\mathcal{F}'$ will do the job and we select the set $\mathcal{F}'$ such that $\mathcal{F}'$ consists of cores of all homomorphic images (explicitly: we close $\mathcal{F}'$ on homomorphic images and then take the set of cores of all these structures). Among all such sets $\mathcal{F}'$ we take a set of minimal cardinality. It will be again denoted by $\mathcal{F}'$. We proceed by induction on the size $|\mathcal{F}'|$ of $\mathcal{F}'$.

The set $\mathrm{Forb}(\mathcal{F}')$ is clearly determined by the minimal elements of $\mathcal{F}'$ (minimal in the homomorphism order). Thus let us assume that one of these minimal elements, say $\mathbf{F}'_0$, is not a forest. By the minimality of $\mathcal{F}'$ we see that we have a proper inclusion $\Phi(\mathrm{Forb}(\mathcal{F}' \setminus \{\mathbf{F}'_0\})) \supset CSP(\mathcal{D})$. Thus there exists a structure $\mathbf{S}$ in the difference. But this in turn means that there has to be a lift $\mathbf{S}'$ of $\mathbf{S}$ such that $\mathbf{F}'_0 \longrightarrow \mathbf{S}'$ and $\mathbf{S} \not\to \mathbf{D}$ for every $\mathbf{D} \in \mathcal{D}$. In fact not only that: as $\mathbf{F}'_0$ is a core, as $\mathrm{Forb}(\mathcal{F}')$ is homomorphism closed and as $\mathcal{F}'$ has minimal size we conclude that there exist $\mathbf{S}$ and $\mathbf{S}'$ such that any homomorphism $\mathbf{F}'_0 \longrightarrow \mathbf{S}'$ is a monomorphism (i.e. one-to-one, otherwise we could replace $\mathbf{F}'_0$ by a set of all its homomorphic images - $\mathbf{F}'_0$ would not be needed).

Now we apply (the second non-trivial ingredient) Theorem 2 to structure $\mathbf{S}$ and an $\ell > |X(\mathbf{F}'_0)|$: we find a structure $\mathbf{S}_0$ with the following properties: $\mathbf{S}_0 \longrightarrow \mathbf{S}$, $\mathbf{S}_0 \longrightarrow \mathbf{D}$ if and only if $\mathbf{S} \longrightarrow \mathbf{D}$ for every $\mathbf{D} \in \mathcal{D}$ and $\mathbf{S}_0$ contains no cycles of length $\leq \ell$. It follows that $\mathbf{S}_0 \notin CSP(\mathcal{D})$. Next we apply Lemma 12 to obtain a structure $\mathbf{S}'_0$ with $\mathbf{S}'_0 \longrightarrow \mathbf{S}'$. Now we use that all relations in Δ' are unary and we see that $\mathbf{S}'_0$ does not contain cycles of length $\leq \ell$. Now for any $\mathbf{F}' \in \mathcal{F}'$, $\mathbf{F}' \neq \mathbf{F}'_0$ we have $\mathbf{F}' \nrightarrow \mathbf{S}'_0$ as $\mathbf{S}'_0 \to \mathbf{S}'$ and $\mathbf{F}' \nrightarrow \mathbf{S}'$. As the only homomorphism $\mathbf{F}'_0 \longrightarrow \mathbf{S}'$ is a monomorphism the only (hypothetical) homomorphism $\mathbf{F}'_0 \longrightarrow \mathbf{S}'$ is also monomorphism. But this is a contradiction as $\mathbf{F}'_0$ contains a cycle while $\mathbf{S}'_0$ has no cycles of length $\leq \ell$. This concludes the proof. $\square$

5. MMSNP and forbidden patterns

Madelaine [18] introduced the class FP. Every language of the class FP is defined by *forbidden patterns* which are defined as follows: Consider the finite relational type Δ, the finite set T and the set of pairs $(\mathbf{F}_1, \varphi_1), \ldots, (\mathbf{F}_n, \varphi_n)$, where each $\mathbf{F}_i \in Rel(\Delta)$ and $\varphi_i : \mathbf{F_i} \to \mathbf{T}$ is a mapping ($i = 1, \ldots, n$). The language L belongs to the class FP if there are patterns $(\mathbf{F}_1, \varphi_1), \ldots, (\mathbf{F}_n, \varphi_n)$ such that L is the class of all structures $\mathbf{A} \in Rel(\Delta)$ for which there exists a mapping $\varphi : A \to T$ such that for all $i = 1, \ldots, n$ no homomorphism $\alpha : \mathbf{F}_i \to \mathbf{A}$ satisfies $\varphi \circ \alpha \neq \varphi_i$. Formally: $L = \{\mathbf{A} \in Rel(\Delta) : \exists \varphi : \mathbf{A} \to T \text{ such that } \forall i, \alpha : \mathbf{F}_i \to \mathbf{A} \text{ homomorphism } \varphi \circ \alpha \neq \varphi_i\}$.

This is a special case of our approach and the class FP may be equivalently defined as follows (using lifts and shadows): we say that the set $L \subseteq Rel(\Delta)$ is an FP language if there exist a finite type Δ' of monadic (unary) relational symbols and a language $L' \in Rel^{cov}(\Delta, \Delta')$ such that $L = \Phi(Forb(\mathcal{F}'))$ for a finite set $\mathcal{F}' \subseteq Rel^{cov}(\Delta, \Delta')$. (Thus Δ' is a partition on every $\mathbf{F}' \in \mathcal{F}'$.) The equivalence is clear: we consider the signature (relational type) Δ' that contains the unary symbol u_t for every element $t \in T$. To every pattern $(\mathbf{F}_i, \varphi_i)$ we correspond the relational structure $\mathbf{F}'_i \in Rel(\Delta, \Delta')$ with the shadow $\mathbf{F}_i$ such that the element $x \in \mathbf{F}_i$ is in the relation $u_{\varphi_i(x)}$.

In other words the class FP is the class of languages defined by forbidden monadic lifts of the class $Rel(\Delta)$.

It has been proved in [18] that the classes FP and MMSNP are equal. This also follows from Theorem 5: every MMSNP problem (as any NP problem) can be considered as the shadow of a language $\mathrm{Forb}(\mathcal{F}')$ in a lifted category $Rel^{cov}(\Delta, \Delta')$. It follows from the above proof of Theorem 5 that in the case of the class MMSNP these lifted relations (i.e. Δ') are all unary. And for unary relations we use the preceding remark which claims that unary and forbidden patterns are equivalent.

Madelaine and Stewart [19] gave a long process to decide whether an FP language is a finite union of CSP languages. We use Theorems 10 and 11 and the description of dualities for relational structures [7] to give a short characterization of a more general class of languages.

Theorem 13. *Consider the language L determined by forbidden monadic lifts. Explicitly, $L = \Phi(Forb(\mathcal{F}'))$ for a set $\mathcal{F}' \subset Rel(\Delta, \Delta')$ (with Δ' monadic). If no $\mathbf{F}' \in Forb(\mathcal{F}')$ contains a cycle then there is a set of finite structures $\mathcal{D} \subseteq Rel(\Delta)$ such that $L = CSP(\mathcal{D})$. If one of the lifts $\mathbf{F}'$ in a minimal subfamily of $\mathcal{F}'$ contains a cycle in its core then the language L is not a finite union of CSP languages.*

Proof. If no $\mathbf{F}' \in Forb(\mathcal{F}')$ contains a cycle then the set $\mathcal{F}'$ has a dual $\mathcal{D}'$ in $Rel^{cov}(\Delta, \Delta')$ by [7], and the shadow of this set $\mathcal{D}'$ gives the dual set $\mathcal{D}$ of the set $\Phi(Forb(\mathcal{F}'))$ (by Theorem 10). On the other side if one $\mathbf{F}' \in Forb(\mathcal{F}')$ contains a cycle in its core and if $\mathcal{F}'$ is minimal (i.e. $\mathbf{F}'$ is needed) then $Forb(\mathcal{F}')$ does not have a dual in $Rel^{cov}(\Delta, \Delta')$. The shadow of the language $Forb(\mathcal{F}')$ is the language L and consequently this fails to be a finite union of CSP languages by Theorem 11 (as every monadic shadow duality can be lifted). □

6. Understanding Feder - Vardi

Now we give a proof one of the principal results of [6] by tools which we developed in previous sections. Feder and Vardi have proved that the classes MMSNP and CSP are random equivalent, this was later derandomised. Here we discuss the deterministic part of the Feder-Vardi proof. It seems that our setting streamlines some of the earlier arguments. A structure $\mathbf{A}$ is *biconnected* if it contains a cycle and every point deleted substructure is connected, in other words for every three distinct elements x, y and z there is a path connecting x and y that avoids z. Note that a biconnected structure with more than one relational tuple contains a cycle. Inclusion maximal biconnected substructures are called *biconnected components* (in graph theory they are often called blocks). For the set of relational structures $\mathcal{D}$ we denote by $CSP_{girth>k}(\mathcal{D})$ the language of structures in $CSP(\mathcal{D})$ with girth larger than k. We will prove the following theorem.

Theorem 14. [6] *For every MMSNP language L there is a finite set of relational structures $\mathcal{D}$ (of different type) and a positive integer k such that the following hold.*

1. *L can be polynomially reduced to $CSP(\mathcal{D})$.*
2. *The language $CSP_{girth>k}(\mathcal{D})$ can be polynomially reduced to L.*

Proof. We construct the set $\mathcal{D}$. Consider a class $Forb(\mathcal{F}')$ of monadic lifts of structures satisfying $L = \Phi(Forb(\mathcal{F}'))$ (it exists by Theorem 5). We may suppose that all structures in $\mathcal{F}'$ are cores (in $Rel^{cov}(\Delta, \Delta')$) and there is no homomorphism between them. If all the structures in $\mathcal{F}'$ have no cycles then L itself is a finite union of CSP languages by Theorem 13. Otherwise we will add some forbidden lifts without changing the language L (the reason will be clear later). These additional forbidden lifts will be induced by biconnected components. Consider the set $\mathcal{BC}$ of structures in $Rel(\Delta)$ which are the shadow of some biconnected component of a forbidden lift $\mathbf{F}' \in Forb(\mathcal{F}')$. Observe that each of the structures in $\mathcal{BC}$ is biconnected (as all lifts are monadic). Let $\mathbf{F}'$ be a forbidden lift and suppose that there is a $\mathbf{G}' \in Rel(\Delta, \Delta')$ such that all biconnected components of $\mathbf{G}$ are in $\mathcal{BC}$, and there is a homomorphism $\alpha : \mathbf{F}' \to \mathbf{G}'$. Moreover, we suppose that $\mathbf{G}'$ is minimal in the sense that if we remove the relations in one of its biconnected components then α is no longer a homomorphism. For such an $\mathbf{F}'$ and $\mathbf{G}'$ we add $\mathbf{G}'$ to the list of forbidden lifts. For every original lift $\mathbf{F}'$ we add only finitely many $\mathbf{G}'$, so we have finitely many in the end. Denote this larger set of forbidden lifts by $\mathcal{G}'$. We shall now prove that we have again $L = \Phi(Forb(\mathcal{G}'))$. Clearly $\mathcal{G}' \supseteq \mathcal{F}'$, proven by the identical homomorphisms of forbidden lifts. This yields $Forb(\mathcal{G}') \subseteq Forb(\mathcal{F}')$. On the other hand every $\mathbf{G}' \in \mathcal{G}'$ is the homomorphic image of some $\mathbf{F}' \in \mathcal{F}'$, hence the two classes are equal.

Consider the following relational type β. For every structure in $\mathcal{BC}$ there is a relational symbol of arity of the size of the component, the type β consists of these relational symbols. Consider the functor $\Psi : Rel(\Delta) \to Rel(\beta)$ that assigns to a structure $\mathbf{A}$ the following structure $\Psi(\mathbf{A})$: the base set of $\Psi(\mathbf{A})$ and $\mathbf{A}$ are the same, and the tuple $(a_1, \ldots, a_l)$ is in the relation $B \in \beta$ iff the appropriate structure induced by the set $\{b_1, \ldots, b_l\}$ (belonging to $\mathcal{BC}$) can be mapped homomorphically to $\mathbf{A}$ such that b_i maps to a_i. Now Ψ induces a functor $\Psi' : Rel(\Delta, \Delta') \to Rel(\beta, \Delta')$. We will need another functor $\Theta : Rel(\beta) \to Rel(\Delta)$. We define $\Theta(\mathbf{A})$ on the same base set. The set of relations on $\Theta(\mathbf{A})$ is constructed in such a way that we replace every tuple in relation $B \in \beta$ by the biconnected component in $\mathcal{BC}$ corresponding to B. The functors Ψ, Ψ', Θ and Θ' are really functorial, i.e. $\mathbf{A} \to \mathbf{B}$ implies $\Psi(\mathbf{A}) \to \Psi(\mathbf{B})$, and the same holds for Ψ', Θ and Θ'. Clearly $\Theta \circ \Psi(\mathbf{A}) = \mathbf{A}$ for every $\mathbf{A} \in \mathbf{Rel(\Delta)}$.

We will work with the following set of forbidden lifts $\mathcal{H}'$. For every $\mathbf{G}' \in \mathcal{G}'$ replace every biconnected component of $\mathbf{G}'$ by the appropriate relational tuple in β, denote this by $\mathbf{H}'_{\mathbf{G}}$. The set $\mathcal{H}'$ consists of these structures. It is easy to see that all of these structures are trees. Now there is a finite dual $\mathcal{D}'$ of $\mathcal{H}'$, i.e. $Forb(\mathcal{H}') = CSP(\mathcal{D}')$ holds. And there is a shadow duality $(\mathcal{H}', \mathcal{D})$, where $\mathcal{D} = \Phi(\mathcal{D}')$.

First we prove (1). We will show that $\mathbf{A} \in \mathbf{L} \iff \exists \mathbf{A}' \mathbf{\Psi}(\mathbf{A}) \in \mathbf{Forb}(\mathcal{H}')$ holds for every $\mathbf{A} \in \mathbf{Rel}(\mathbf{\Delta})$. It suffices to prove for every $\mathbf{A}' \in \mathbf{Rel}(\mathbf{\Delta}, \mathbf{\Delta}')$ that $\mathbf{A}' \in \mathbf{Forb}(\mathcal{G}') \iff \mathbf{\Psi}'(\mathbf{A}') \in \mathbf{Forb}(\mathcal{H}')$. If $\Psi'(\mathbf{A}') \notin \mathbf{Forb}(\mathcal{H}')$ then the forbidden lift $\mathbf{H}'_{\mathbf{G}}$ maps to $\Psi'(\mathbf{A}')$. Now $\Theta'(\mathbf{H}'_{\mathbf{G}}) = \mathbf{G}'$ maps to $\Theta'(\Psi'(\mathbf{A}')) = \mathbf{A}'$. On the other hand suppose that $\mathbf{G}' \to \mathbf{A}'$ holds for some $\mathbf{G}' \in \mathcal{G}'$. Now $\Psi'(\mathbf{G}') \to \mathbf{\Psi}'(\mathbf{A}')$. The forbidden lift $\mathbf{H}'_{\mathbf{G}}$ has the same base set as $\Psi'(\mathbf{G}')$, and it contains less relations, hence $\mathbf{H}'_{\mathbf{G}} \to \mathbf{\Psi}'(\mathbf{A}')$.

In order to prove (2) consider a relational structure $\mathbf{A} \in Rel(\beta)$ with girth larger than the size of the largest forbidden lift. We will prove $\mathbf{A} \in \mathbf{\Phi}(\mathbf{Forb}(\mathcal{H}')) \iff \mathbf{\Theta}(\mathbf{A}) \in \mathbf{L}$. It suffices to show $\mathbf{A}' \in \mathbf{Forb}(\mathcal{H}') \iff \mathbf{\Theta}'(\mathbf{A}') \in \mathbf{Forb}(\mathcal{G}')$. First suppose that $\mathbf{G}' \to \mathbf{\Theta}'(\mathbf{A}')$ holds for some $\mathbf{G}' \in \mathcal{G}'$. Here we may assume by the choice of $\mathcal{G}'$ that the image of each biconnected component of $\mathbf{G}'$ is isomorphic to a structure in $\mathcal{BC}$, and the restriction of the mapping to this component is an isomorphism. By the large girth condition this yields that the image of this component is contained by a structure in $\mathcal{BC}$ corresponding to one single tuple of $\mathbf{A}'$, moreover we may suppose that these are equal. Hence the same mapping is actually a $\mathbf{H}'_{\mathbf{G}} \to \mathbf{A}'$ homomorphism. Secondly suppose that $\mathbf{H}'_{\mathbf{G}} \to \mathbf{A}'$, where $\mathbf{H}'_{\mathbf{G}} \in \mathcal{H}'$. Now $\mathbf{G}' = \mathbf{\Theta}'(\mathbf{H}'_{\mathbf{G}}) \to \mathbf{\Theta}'(\mathbf{A}')$. This completes the proof of the theorem. $\square$

The remaining part is the reduction of CSP with large girth to CSP. Feder and Vardi proved a randomized reduction, this was later derandomized.

Lemma 15 ([15])**.** *For every finite set of relational structures $\mathcal{D}$ and integer $k > 0$ the language $CSP(\mathcal{D})$ can be polynomially reduced to $CSP_{girth>k}(\mathcal{D})$.*

The essence of this reduction is the Sparse Incomparability Lemma 2. This polynomial reduction was proved with expanders in the case of digraphs [21]. The reduction in the case of general relational structures needed a generalization of expanders called expander (relational) structures. The notion of expander relational structures was introduced in [15] [14], and also a polynomial time construction of such structures with large girth is given there.

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