Finite Presentation of Homogeneous Graphs, Posets and Ramsey Classes

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Dedicated to Professor Hillel Furstenberg.

Abstract

It is commonly believed that one can prove Ramsey properties only for simple and "well behaved" structures. This is supported by the link of Ramsey classes of structures with homogeneous structures. We outline this correspondence in the context of the Classification Programme for Ramsey classes. As particular instances of this approach one can characterize all Ramsey classes of graphs, tournaments and partial ordered sets and also fully characterize all monotone Ramsey classes of relational systems (of any type). On the other side of this spectrum many homogeneous structures allow a concise description (called here a finitary presentation) by means of all finite models of a suitable theory. Extending classical work of Rado (for the random graph) we find a finitary presentation for each of the above classes where the classification problem is solved: (undirected) graphs, tournaments and partially ordered sets. The main result of the paper is a construction of classes $\mathcal{P}_{\in}$ and $\mathcal{P}_f$ of finite structures which are isomorphic to the generic (i.e. homogeneous and universal)

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partially ordered set. Somehow surprisingly, the structure $\mathcal{P}_\in$ extends Conway's surreal numbers and their linear ordering.

1 Ramsey Classes and Homogeneous Structures

Let $\mathcal{K}$ be a class of objects which is isomorphism closed and endowed with subobjects. Given two objects $A, B \in \mathcal{K}$ we denote by $\binom{B}{A}$ the set of all subobjects A' of B which are isomorphic to A. We say that the class $\mathcal{K}$ has *A-Ramsey property* if the following statement holds:

For every positive integer k and for every $B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that $C \longrightarrow (B)^A_k$. Here the last symbol (*Erdős - Rado partition arrow*) has the following meaning:

For every partition $\binom{C}{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_k$ there exists $B' \in \binom{C}{B}$ and an $i, 1 \leq i \leq k$ such that $\binom{B'}{A} \subset \mathcal{A}_i$.

In the extremal case that a class $\mathcal{K}$ has A-Ramsey property for every its object A we say that $\mathcal{K}$ is a *Ramsey class.*

These notions crystallized in the early seventies, see e.g. [23, 34, 11]. This formalism and the natural questions it motivated essentially contributed to create establish Ramsey theory as a "theory" (as nicely put in the introduction to [12]).

The notion of a Ramsey class is highly structured and in a sense it is the top of the line of the Ramsey notions ("one can partition everything in any number of classes to get anything homogeneous"). Consequently there are not many (essentially different) examples of Ramsey classes known.

Examples of Ramsey classes include

i. The class of all finite ordered graphs;

ii. The class of all finite partially ordered sets (with a fixed linear extension);

iii. The class of all finite vector spaces (over a fixed field F).

iv. The class of all (labeled) finite partitions.

For these results see [11, 12, 32, 27]. We formulate explicitly one of the most general results (for relational structures of a given type):

Let $\Delta = (\delta_i; i \in I)$ be a sequence of natural numbers. Δ is called the type (or signature). For a fixed Δ, we shall consider the class $\mathrm{Rel}(\Delta)$ of all finite *ordered relational structures* of type Δ. These are objects of the form $(X, (R_i; i \in I))$ where X is a non-empty ordered set and $R_i \subseteq X^{\delta_i}$ (i.e. R_i is a δ_i-ary relation). We also put $\underline{A} = X$, $R_i(A) = R_i$. The class $\mathrm{Rel}(\Delta)$ will be considered with embeddings (corresponding to induced substructures): Given two relational structures $(X, (R_i; i \in I))$ and $(X', (R'_i; i \in I))$ of type Δ a mapping $f : X \longrightarrow X'$ is called an *embedding* if it is monotone injection of X into X' satisfying $(f(x_j); j = 1, \ldots, \delta_i) \in R'_i$ iff $(x_j; j = 1, \ldots, \delta_i) \in R_i$. As usual, an inclusion (or bijective) embedding is called *substructure* (or *isomorphism*). Given two ordered relational structures A, B we denote by $\binom{B}{A}$ the class of all substructures A' of B which are isomorphic to A. We have the following

Theorem 1.1 ([30]) *For every choice of a natural number k, of a type Δ and of structures $A, B \in \mathrm{Rel}(\Delta)$ there exists a structure $C \in \mathrm{Rel}(\Delta)$ with the following property: For every partition $\binom{C}{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_k$ there exists $i, 1 \leq i \leq k$, and a substructure $B' \in \binom{C}{B}$ such that $\binom{B'}{A} \subset \mathcal{A}_i$.*

From today's perspective all these results may be obtained (from a few "basic" Ramsey-type results, such as Ramsey theorem itself or its "dual form" Hales–Jewett theorem) by variants of *amalgamation technique* (known also as *Partite Construction*) see [35, 33, 32, 27].

It is interesting that the amalgamation technique is (almost) necessary as we have the following easy but important result observed already in [25], see recent [26, 17]. This can be formulated in surprising generality.

First, we review some basic notions from model theory: Let $\mathcal{K}$ be a class of structures endowed with embeddings. $\mathcal{K}$ is said to be

i. *isomorphism closed* if it contains with every $A \in \mathcal{K}$ any object A' isomorphic to A;

ii. *hereditary* if $B \in \mathcal{K}$ and $A \longrightarrow B$ (in $\mathcal{K}$) implies that $A \in \mathcal{K}$;

iii. *joint embedding property* if for every $A, B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that both A and B can be embedded to C;

iv. $\mathcal{K}$ is said to have an *amalgamation property* if for every $A, B, B' \in \mathcal{K}$ and every choice of embeddings $f : A \longrightarrow B, f' : A \longrightarrow B'$ there exists $C \in \mathcal{K}$ and embeddings $g : B \longrightarrow C, g' : B' \longrightarrow C$ such that $g \circ f = g' \circ f'$. (C, g, g') is called an *amalgamation* of (A, B, B', f, f').

Note that it is not assumed that the sets $g \circ f(\underline{B})$ and $g' \circ f'(\underline{B'})$ intersect in the set $g \circ f(\underline{A}) = g' \circ f'(\underline{A})$. An amalgamation with this additional property is called *strong*.

The key observation in our programme is the fact that Ramsey classes are closed on amalgamation (this has been noted already in [31], see also [17, 27]). We formulate this in a stronger form by means of a class $\preceq$ of "admissible" orderings. We say that a class $\preceq$ of orderings of objects in $\mathcal{K}$ is *rich* if the following holds:

For every object A, B_1, B_2 of $\mathcal{K}$ and with embeddings $f_i : A \longrightarrow B_i$ there exists objects $B'_i \in \mathcal{K}$, embeddings $f'_i : A \longrightarrow B'_i$ and admissible orderings $\leq, \leq_i$ of $A, B'_i, i = 1, 2$, such that f'_i extends f_i (thus B_i is a substructure of B'_i) and such that every amalgamation (in $(\mathcal{K}, \preceq)$) of $((A, \leq)$, $(B'_1, \leq'_1)$, $(B'_2, \leq'_2)$, f'_1, $f'_2)$ is strong.

All classes of admissible orderings which were mentioned above are rich. For example the class of all linear orderings is rich as well as the class of all linear extensions of partial orders.

Theorem 1.2 *Let $\mathcal{K}$ be any hereditary, isomorphism closed class with the joint embedding property. Then we have*

1. *If $\mathcal{K}$ is a Ramsey class (with embeddings as subobjects) then $\mathcal{K}$ has amalgamation property.*

2. *If $\preceq$ is a rich class of admissible orderings of $\mathcal{K}$ and $(\mathcal{K}, \preceq)$ is Ramsey class then $\mathcal{K}$ has strong amalgamation property.*

Proof. We prove 1. first. Let A, B_1, B_2 be objects of $\mathcal{K}$ and let embeddings $f_i : A \longrightarrow B_i, i = 1, 2$, be given. Using the joint embedding property let C be an object for which there are embeddings $g_i : B_i \longrightarrow C, i = 1, 2$. If $g_1 \circ f_1 = g_2 \circ f_2$ the we are done. So assume $g_1 \circ f_1 \neq g_2 \circ f_2$. Let $D \longrightarrow (C)^A_2$. In this situation we define a partition $\binom{D}{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ as follows:

$A' \in \mathcal{A}_1$ iff there exists are embeddings $h : C \longrightarrow D$ such that the embedding $f : A \longrightarrow D$ which represent A' as member of the set $\binom{D}{A}$ can be written as $f = h \circ g_1 \circ f_1$. Otherwise we put $A' \in \mathcal{A}_2$. By the Ramsey property of D there exists $C' \in \binom{D}{C}$ represented by an embedding $h' : C \longrightarrow D$ such that $\{h' \circ f; f \in \binom{C'}{A}\}$ is a subset of $\mathcal{A}_{i_0}$. Now any subobject of D isomorphic to C contains a subobject isomorphic to A which is colored by 1. Thus $i_0 = 1$. Consider the embedding $h' \circ g_2 \circ f_2 \in \binom{D}{A}$. This subobject of D is as well a subobject of C' and thus it is also colored by $i_0 = 1$. Thus there must exist an embedding $h'' : C \longrightarrow D$ corresponding to a subobject $C'' \in \binom{D}{C}$ such that $h'' \circ g_1 \circ f_1 = h' \circ g_2 \circ f_2$. But this means that object D together with embeddings $h'' \circ g_1 : B_1 \longrightarrow D$ and $h' \circ g_2 : B_2 \longrightarrow D$ is an amalgamation of (A, B_1, B_2, f_1, f_2).

To prove 2. we can proceed similarly. We stress only the differences: Given $A, B_1, B_2 \in \mathcal{K}$ and embeddings f_1, f_2 we find objects B_1', B_2', admissible orderings $\leq, \leq_1, \leq_2$ and embeddings f_1', f_2' by the richness of admissible class $\preceq$ (we preserve the notation of the above definition). By the proof of $i.$ (for the class $(\mathcal{K}, \preceq)$ there exists an amalgamation (C, g_1, g_2) of $(A, \leq), (B_1', \leq_1), (B_2', \leq_2), f_1, f_2)$ (in $(\mathcal{K}, \preceq)$). Consider the restrictions $\tilde{g}_i = g_i \mid \underline{B_i}, i = 1, 2$, and denote by $\tilde{C}$ the subobject of C induced by the set $\underline{B_1} \cup \underline{B_2}$. We see easily that $(\tilde{C}, \tilde{g}_1, \tilde{g}_2)$ is a strong amalgamation of (A, B_1, B_2, f_1, f_2) in $\mathcal{K}$.

■

Despite its simplicity this result establishes an important link to model theory and its classification programme for homogeneous structures:

A (finite or infinite) structure A is called *homogeneous* if for any choice of finite substructures B, B' of A, every isomorphism $f : B \longrightarrow B'$ can be extended to an isomorphism $g : A \longrightarrow A$ (i.e. we demand that g restricted to B is f). For an infinite structure A the *age* of A (denoted by $age(A)$) is the class of all (isomorphism types of) finite substructures of A. In this case we also say that A is *universal* for $age(A)$ (although this term is usually reserved for countable universality).

Homogeneity is a very strong symmetry property (generalizing vertex-, edge-, path-transitive graphs) and as a result there are just a handful of finite examples (totally symmetric objects are of course among them). But infinite homogeneous structures are more frequent and present important examples. Note that not every homogeneous structure is Ramsey.

It follows from classical result of Fraïssé ([8], see also [14]) that a class $\mathcal{K}$ is an age of a homogeneous structure S iff it satisfies the above assumptions $i., ii., iii., iv.$:

In addition, for a class $\mathcal{K}$ with $i., ii., iii., iv.$, the homogeneous S satisfying $age(S) = \mathcal{K}$ is up to an isomorphism uniquely determined and it is called the *Fraïssé limit* of $\mathcal{K}$.

The structure $\mathcal{U}$ is called *generic*, or *Fraïssé limit* for the class $\mathcal{K}$. The generic structure is homogeneous and thus highly structured. This is reflected also in the fact that very often $\mathcal{U}$ has an easy description. This correspondence allows us to study (seemingly very diverse) Ramsey classes by means of single objects of high symmetry.

The following is a consequence of Fraïssé Theorem and Theorem 1.2:

Corollary 1.3 *Let $\mathcal{K}$ be a Ramsey class (with embeddings as subobjects) which is hereditary, isomorphism closed and with joint embedding property. Then $\mathcal{K}$ is the age of a generic (homogeneous and universal) structure $\mathcal{U}(\mathcal{K})$.*

This relates two seemingly unrelated things: Ramsey classes and homogeneous structures. This allows us to use known results about homogeneous structures (in the cases when their classification programme is completed) and to check whether the corresponding classes (i.e. their ages) are Ramsey. Schematically we propose (see [26]) to proceed as follows:

I. Ramsey Classes $\Rightarrow$ Amalgamation Classes.

II. Amalgamation Classes $\Rightarrow$ Generic Structures.

III. Generic Structure $\Rightarrow$ Special Generic Structures.

IV. Special Generic Structures $\Rightarrow$ Ramsey Classes.

$$\begin{tikzcd}
\text{Ramsey Classes} \arrow[r, Rightarrow] & \text{Amalgamation Classes} \arrow[d, Rightarrow] \\
\text{Special Structures} \arrow[u, Rightarrow] & \text{Generic Structures} \arrow[l, Rightarrow]
\end{tikzcd}$$

I. is provided by Theorem 1.2, **II.** is provided by the Fraïssé Theorem. The bottleneck of this programme are obviously statements **III.** and **IV.** The statement **III.** symbolizes the *Classification Programme* (of Lachlan-Cherlin; see e.g. [21, 4]). Ramsey theory context **IV.** is presenting also some interesting (and difficult) problems.

Despite of the difficulties we believe that this is a realistic project as also illustrated by the fact that in this way all Ramsey Classes of ordered graphs, partially ordered sets and tournaments were determined [25, 26]. We do not know what are Ramsey classes of relational structures. But we are hopeful that we can deduce a strengthening of Corollary 1.3 in such a way that the classification programme will be easier. This can be illustrated on all monotone Ramsey classes of relational structures.

A class is said to be *monotone* if it is closed under monomorphisms. Explicitly, $\mathcal{K}$ is monotone if $A' = (X', (R'_i; i \in I)) \in \mathcal{K}$ providing there exists $A = (X, (R_i; i \in I)) \in \mathcal{K}$ such that $X' \subseteq X, R'_i \subseteq R_i, i \in I$. In this case we write simply $A' \subseteq A$. For graphs a monotone class corresponds to a class closed on (not necessarily induced) subgraphs.

Let $\mathcal{F}$ be a class of structures (of type Δ). Denote by $\mathrm{Forb}_{m,\Delta}(\mathcal{F})$ the class of all structures $A \in \mathrm{Rel}(\Delta)$ for which there are no $F \in \mathcal{F}$ with $F \subseteq A$.

We also say that a structure $F \in \mathrm{Rel}(\Delta)$ is said to be *(amalgamation) irreducible* if any two its vertices appear in one of the edges of F. (Alternatively F is irreducible if it is not a "free" amalgam of two proper substructures.)

We have the following:

Theorem 1.4 ([26]) *For every type Δ and every monotone, isomorphism closed class $\mathcal{K}$ of Δ-systems with the joint embedding property the following two statements are equivalent:*

i. The class $(\mathcal{K}, \preceq)$ is a Ramsey Class;

ii. $\mathcal{K} = \mathrm{Forb}_{m,\Delta}(\mathcal{F})$ for a class of irreducible Δ-systems.

Proof.[Sketch of a proof] One can prove easily that for each Fraïssé class $\mathcal{K}$ has the class $(\mathcal{K}, \preceq)$ of all linearly ordered Δ-systems in $\mathcal{K}$ has strong amalgamation. Further, if $\mathcal{K}$ is monotone then $\mathcal{K}$ is determined by a set of forbidden amalgam–irreducible systems, see [26] for details. However any set of amalgam irreducible Δ-systems leads to a Ramsey class as follows from [30]. Combined we get the statement. ∎

Now we show that most of the above examples have a very simple description. So not only Ramsey classes lead to homogeneous structures, often these structures permit a simple (local) description.

2 Concise Description of Homogeneous Structures

We specialize the notions of Section 1 for graphs and partially ordered sets. A countable partially ordered set P is said to be *universal* if it contains any countable partially ordered sets (as an induced subset).

A partially ordered set P is said to be *homogeneous* if every partial isomorphism between finite subsets extends to an isomorphism (of P).

It is a classical model theory result that a homogeneous universal partially ordered set exists and that it is up to an isomorphism uniquely determined. This partially ordered set is naturally called *generic poset* and it will be denoted by $\mathcal{P}$. The main result of this paper is devoted to the study of $\mathcal{P}$. $\mathcal{P}$ can be constructed as Fraïssé limit of all finite partially ordered sets: we start with the singleton partially ordered sets and at n-th step we add new vertices which extend the given partially ordered sets in all possible (consistent) ways to a partially ordered set with $(n+1)$ vertices.

This procedure applies (as proved by Fraïssé) not only to partially ordered sets but to structures in general and thus, in particular, the homogeneous universal (undirected) graph exists. This graph is called *Rado graph* $\mathcal{R}$. We state two of its striking properties which motivate the present paper (see the excellent survey by P. Cameron [3], see also [5, 21]):

1. $\mathcal{R}$ is isomorphic to the following graph $\mathcal{R}_{\in}$: vertices of $\mathcal{R}_{\in}$ are all finite sets (in some countable model of set theory) with edges of the form $\{A, B\}$ where either $A \in B$ or $B \in A$.
2. $\mathcal{R}$ is isomorphic to the following graph $\mathcal{R}_{\mathbb{N}}$: vertices of $\mathcal{R}_{\mathbb{N}}$ are all natural numbers with edges of the form $\{m, n\}$ where the m-th digit in the binary expansion of n is 1.
3. $\mathcal{R}$ is isomorphic to the following graph $\mathcal{R}_{QR}$: vertices of $\mathcal{R}_{QR}$ are all prime natural numbers $x \equiv 1 \mod 4$ with xy forming an edge iff $(\frac{x}{y}) = +1$.

There are other explicit constructions (e.g. by universal sequences, see [3]). It is remarkable that all these seemingly unrelated constructions define the same graph $\mathcal{R}$. The simplicity of these constructions motivated our notion of finitely presented structures:

Definition 2.1 *A countable structure $\mathcal{S}$ is* finitely presented *if there exists a finitely axiomatizable theory $\mathcal{T}$ and moreover, for each relation R in $\mathcal{S}$ of arity k, there exists finitely axiomatizable theory $\mathcal{T}_R$ such that all finite models of $\mathcal{T}$ (or all finite structures axiomatizable by $\mathcal{T}$) in the countable set theory together with relations among them induced by theories $\mathcal{T}_R$ induce a structure isomorphic to $\mathcal{S}$.*

Let us state two particular cases of this definition explicitly:

A graph G is finitely presented if there exist finitely axiomatizable theories $\mathcal{T}_V$ and $\mathcal{T}_E$ such that the following graph $\mathcal{G}$ is isomorphic to G: vertices of $\mathcal{G}$ are all models of M of $\mathcal{T}_V$ (in a countable model of set theory); edges of $\mathcal{G}$ are all pairs of models $\{M, M'\}$ which satisfy $\mathcal{T}_E$.

All three constructions $\mathcal{R}_{\in}$, $\mathcal{R}_{\mathbb{N}}$, $\mathcal{R}_{QR}$ are obviously finite presentations of $\mathcal{R}$.

A partially ordered set P is finitely presented if there exists finitely axiomatizable theories $\mathcal{T}_P, \mathcal{T}_{\leq}$ such that the following partially ordered set $\mathcal{P}$ is isomorphic to P: vertices of $\mathcal{P}$ are all models M of $\mathcal{T}_P$ (in a countable model of finite set theory) with $M \leq M'$ iff (M, M') satisfy $\mathcal{T}_{\leq}$.

For partially ordered sets the situation is more complicated than for graphs. In fact an explicit construction of the generic partially ordered set $\mathcal{P}$ by means of all finite models of a finitely axiomatizable structure was until recently an open problem. The main result of this paper is a construction of a finitely presented partially ordered set $\mathcal{P}_f$ which is isomorphic to $\mathcal{P}$. This is proved in Section 3.

We found the construction of $\mathcal{P}_f$ in the broader context of a study of finite presentation of homogeneous structures, homogeneous undirected and directed graphs, tournaments and partially ordered sets. For these structures the classification programme has been completed in a series of difficult papers (see e.g. [19, 4]) all based on the Fraïssé equivalent definition of homogeneous structures as amalgamation classes of finite structures. Particularly the homogeneous undirected graphs were characterized in [19]. We shall prove that all these graphs are finitely presented. For some graphs on the Lachlan-Woodrow list (all finite examples, equivalences and Turán graphs) this is an easy exercise. For Rado graph this has been stated above. We prove in Section 3 that the generic graphs for the class $\mathrm{Forb}(K_k)$, of all finite graphs which do not contain a complete graph K_k, are finitely presented for every $k \geq 3$. Thus all homogeneous graphs are finitely presented (Corollary 3.10).

For oriented graphs and partial orders (two other structures with solved classification problem) the situation is different in that we have to construct a finite presentation even for the generic oriented graph and for generic partially ordered set. This presented an open problem (see [37]) and this is solved is done in Section 4, Theorem 4.6 and in Section 5, Corollary 6.2.

We further refine this construction to any oriented homogeneous graph of type $\mathrm{Forb}(T_1, \ldots, T_n)$ where T_i are (forbidden) tournaments. It follows that also all homogeneous tournaments are finitely axiomatizable (Corollary 6.2). As there are continuously many homogeneous oriented graphs we cannot expect that there is a finite presentation of all homogeneous oriented graphs. This shows that there are natural limits to the programme of representing homogeneous structures by means of simple structures.

The classification of homogeneous partially ordered sets is easier than for undirected graphs. The classification was given by Schmerl [39]. Apart from anti-chains, the set $\mathbb{Q}$ of all rationals, disjoint unions and "blow up" $\mathbb{Q}$, the only other homogeneous partially ordered set is the generic one.

Several examples of finitely presented linear orders and partially ordered sets are easy to find:

- The set of all natural numbers $(\mathbb{N}, \leq)$ (proved by von Neumann);
- the set $\mathbb{Q}$ (by standard construction from $\mathbb{N}$);
- $P \times P'$ for finitely presented P and P';
- lexicographic product of P and P' for finitely presented P and P'.

It follows that almost all homogeneous partially ordered sets are finitely presented. The only remaining case is the generic partially ordered set $\mathcal{P}$. However the finite presentation of $\mathcal{P}$ is much more difficult question and it is the main result of this paper. $\mathcal{P}$ is shown to be finitely presented in Section 4 (by means of the structures $\mathcal{P}_{\in}$ and $\mathcal{P}_f$).

It seems that homogeneous structures are likely to be finitely presented. Intuitively a high degree of symmetry (homogeneity) perhaps leads to a "low entropy" and thus in turn perhaps to a concise representation. (Remark that "concise representations" of finite structures were studied from complexity point of view for graphs ([24, 40]) and partially ordered sets ([10, 29]). These considerations and the above results perhaps suggest the following general conjecture:

Conjecture 2.2 *Let $\mathcal{T}$ be a finitely axiomatizable theory. Let $\mathcal{M}$ be the class of all finite models of $\mathcal{T}$. Then the generic (i. e. homogeneous universal) structure corresponding to $\mathcal{M}$, if it exists, is finitely presented.*

On the other hand our main result (the construction of the structures $\mathcal{P}_{\in}$ and $\mathcal{P}_f$) may be viewed as an extension of surreal numbers of Conway [6, 18] to partially ordered sets. In Section 5 we exhibit this connection.

Our research was also motivated by trying to solve several problems related to universal partially ordered sets represented by finite graphs with special properties. It has been proved in a different (category theory) context (see [13, 38]) that the class of all finite graphs ordered by the existence of homomorphism is the universal partially ordered set. (However note that both this partially ordered set and partially ordered sets constructed in [13, 38] are far from being homogeneous.) An alternative combinatorial proof of the result [28] was a starting point of this research.

The techniques of [13, 38] also do not extend to some of the basic subclasses of graphs such as planar graphs or graphs with bounded degrees. In fact these classes do not represent arbitrary subgroups [1] and monoids [2]. Yet by means of the techniques of this paper we can construct universal partially ordered sets for both planar and bounded-degree graphs. This also solves a problem stated in [37]. This together with a deeper analysis of finitely presented homogeneous oriented graphs is going to appear in the sequel [15] of this paper. Here we concentrate mostly on partially ordered sets.

3 Homogeneous Directed Graphs

In this section we find a finite presentation of the homogeneous universal (generic) directed graph $\overrightarrow{\mathcal{R}}$ and also of some other homogeneous graphs. We study the homogeneous graphs by means of extension properties. This we briefly recall for completeness (see [14, 1]).

Definition 3.1 *Let $\mathcal{C}$ be an isomorphism closed class of graphs, G a graph. We say that G has the* extension property *for $\mathcal{C}$ if the following holds: For any pair of finite subgraphs $G', G'' \in \mathcal{C}$ and any embeddings $\varphi' : G' \hookrightarrow G$, $\varphi : G' \hookrightarrow G''$ there exists an embedding $\varphi'' : G'' \hookrightarrow G$ such that $\varphi'' \circ \varphi = \varphi'$.*

(An embedding is an isomorphism onto an induced subgraph.)

The extension property implies both universality and homogeneity of G (see [14]):

Lemma 3.2 *Every graph G having the extension property for the class $age(G)$ is universal and homogeneous.*

This statement is a useful tool in proving that a finitely presented structure is generic. As a warm up we prove this for graph $\mathcal{R}_\in$:

Theorem 3.3 *$\mathcal{R}_\in$ has the extension property for the class of finite undirected graphs. Thus $\mathcal{R}_\in$ is isomorphic to the generic undirected graph $\mathcal{R}$.*

Proof. Let M_0 and M_1 be two disjoint finite sets of vertices of $R_\in$. (The subgraph of $R_\in$ induced by $M_1 \cup M_0$ plays the rôle of G' in the extension property.) We are looking for the vertex X of $\mathcal{R}_\in$ such that $Y \in X$ for every $Y \in M_1$ and $Y \notin X$ for every $Y \in M_0$. It suffices to put $X = M_1 \cup \{x\}$ where $M_1 \notin M_0$ satisfies $M_1 \cup \{x\} \notin M_0$ and $M_0 \notin M_1 \cup \{x\}$.

Thus $\mathcal{R}_\in$ has the extension property and thus it is generic for the class of all finite undirected graphs. ∎

Analogously we will construct the homogeneous universal directed graph $\overrightarrow{\mathcal{R}}$. In order to easily embed our construction into set theory, we first introduce an alternative definition of ordered pair. In the rest of the paper we will use a fixed countable model of set theory $\mathfrak{M}$ containing a single atomic element $\acute{\mathrm{O}}$.

Definition 3.4 *For every set M we put*

$$M_L = \{A; A \in M, \acute{\mathrm{O}} \notin A\};$$

$$M_R = \{A; A \cup \{\acute{\mathrm{O}}\} \in M, \acute{\mathrm{O}} \notin A\}.$$

For any two sets A and B we will denote by $(\, A \mid B\,)$ the set

$$A \cup \{M \cup \{\acute{\mathrm{O}}\}; M \in B\}$$

For any M not containing $\acute{\mathrm{O}}$ holds $(\, M_L \mid M_R\,) = M$. Thus for the model $\mathfrak{M}$, the class of sets not containing $\acute{\mathrm{O}}$ represents the universum of the recursively nested ordered pairs.

Definition 3.5 *Define graph $\overrightarrow{\mathcal{R}}_\in$ as follows: The vertices are all sets not containing $\acute{\mathrm{O}}$. (M, N) is an arc of $\overrightarrow{\mathcal{R}}_\in$ iff either $M \in N_L$ or $N \in M_R$. (Recall, that we consider a fixed countable model $\mathfrak{M}$ of set theory containing an additional atomic element $\acute{\mathrm{O}}$.)*

Theorem 3.6 *$\overrightarrow{\mathcal{R}}_\in$ has extension property and thus it is the homogeneous universal directed graph for the class of all directed graphs. Thus $\overrightarrow{\mathcal{R}}_\in$ is isomorphic to $\overrightarrow{\mathcal{R}}$.*

Proof. We proceed analogously to the proof of Theorem 3.3: let M_-, M_+ and M_0 be three disjoint sets of vertices, where $M_0 \cap (M_- \cup M_+)$ is empty. We need to find vertex M with following properties:

I. For each $X \in M_-$ there is an edge from X to M.

II. For each $X \in M_+$ there is an edge from M to X.

III. For each $X \in (M_- \cup M_+ \cup M_0)$ there are no other edges from X to M or M to X then ones required by I. and II.

Fix any

$$x \notin \bigcup_{m \in M_- \cup M_+ \cup M_0} m.$$

Obviously vertex $M = (M_- \cup \{x\} \mid M_+)$ has the required properties I.,II.,III.. ∎

Thus generic graphs (both undirected and directed) are finitely presented. We can extend these representations to other homogeneous structures. We start with undirected graphs:

Definition 3.7 *By $\mathcal{R}_{\mathcal{C}}$ we denote homogeneous universal (i. e. generic) graph for class $\mathcal{C}$ of undirected graphs (if it exists). By $\overrightarrow{\mathcal{R}}_{\mathcal{C}}$ we denote homogeneous universal graph for class $\mathcal{C}$ of directed graphs (if it exists).*

We denote by $\mathrm{Forb}(G)$ *the class of all finite graphs not containing G as an induced subgraph.*

We now construct graphs $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$, $k \geq 3$ which are isomorphic to the generic graph $\mathcal{R}_{\mathrm{Forb}(K_k)}$. The construction of graph $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$, $k \geq 3$, is an extension of the construction of $\mathcal{R}_{\in}$ (recall that a finite set S is called *complete* if for any $X, Y \in S$, $X \neq Y$ either $X \in Y$ or $Y \in X$):

Definition 3.8 *Define $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$, $k \geq 3$ as follows:*
The vertices of $R_{\mathrm{Forb}(K_k),\in}$ are all sets which do not contain a complete subset with $k-1$ elements; two vertices of S and S' form an edge of $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ iff either $S \in S'$ or $S' \in S$.

Thus $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ is the restriction of the graph $R_{\in}$ to the class of all sets without complete subset of size $k-1$.

Theorem 3.9 *Graph $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ has the extension property. Consequently $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ is the homogeneous universal undirected graph for the class* $\mathrm{Forb}(K_k)$*.*

Proof. $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ does not contain K_k: For contradiction, let us suppose that $V_1, V_2, \ldots, V_k$ are edges of complete graph. Without loss of generality we may assume that $V_i \in V_{i+1}$ for each $i = 1, 2, \ldots, k-1$. Since K_k is complete graph, $V_i \in V_k$ for each $i = 1, 2, \ldots, k-1$. It follows that $\{V_1, \ldots, V_{k-1}\}$ represents the prohibited subset S. Thus V_k is not vertex of $\mathcal{R}_{\mathrm{Forb}(K_k)}$.

The extension property can be verified in analogous way as we did for $\mathcal{R}$ in Theorem 3.3. (The constructed set satisfies the conditions required by definition $\mathcal{R}_{\mathrm{Forb}(K_k)}$.) ∎

Corollary 3.10 *All homogeneous undirected graphs are finitely presented.*

Proof. Clearly a graph G is finitely presented iff the complementary graph $\overline{G}$ is finitely presented. The statement follows from the discussion in Section 1 and Theorem 3.3 and 3.9. ∎

Finally, we extend our construction of the generic directed graph $\overrightarrow{\mathcal{R}}_{\in}$ to the graphs $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T),\in}$ not containing a tournament T. This is slightly more technical (although it parallels the undirected case).

Put $T = (V, E)$ and for each $v \in V$ put

$$L(v) = \{v' \in V; (v', v) \in E\}$$

$$R(v) = \{v' \in V; (v, v') \in E\}$$

(observe that $L(v) \cup R(v) = V - \{v\}$.)

Our vertices will be sets M which satisfy the following condition $C_v(M)$ (for each $v \in V$).

$C_v(M)$:

There are no sets $X_{v'}$, $v' \in L(v) \cup R(v)$ satisfying the following

I. $X_{v'} \in M_L$ for $v' \in L(v)$;

II. $X_{v'} \in M_R$ for $v' \in R(v)$;

III. For every arc $(v', v'') \in E$, $v', v'' \in L(v) \cup R(v)$ either $X_{v'} \in (X_{v''})_L$ or $X_{v''} \in (X_{v'})_R$.

In the other words $C_V(M)$ holds if the sets $X_{v'}$, $v' \in L(v) \cup R(v)$ do not represent the tournament $T - \{v\}$ in $\overrightarrow{\mathcal{R}}_{\in}$.

Definition 3.11 *Denote by* $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T),\in}$ *the directed graph* $\overrightarrow{\mathcal{R}}_{\in}$ *restricted to the class of all sets* M *which satisfy the condition* $C_v(M)$ *for every* $v \in V$.

Theorem 3.12 $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T),\in}$ *is isomorphic to* $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T)}$, *explicitly* $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T),\in}$ *is the homogeneous universal graph for the class of all directed graphs not containing* T.

Proof. Proof is analogous to the proof of Theorem 3.9. ∎

This can be extended to classes $\mathrm{Forb}(\mathcal{T})$ for any finite set of tournaments (but clearly not to all classes $\mathrm{Forb}(\mathcal{T})$). In Section 6 we shall also prove that all homogeneous tournaments are finitely presented.

4 Universal Homogeneous Structure $\mathcal{P}_\in$

In this section we further modify the finite presentation of $\overrightarrow{\mathcal{R}}_\in$ to a finite presentation of the generic partially ordered set $\mathcal{P}$. We shall proceed in two steps. First we define a partially ordered set $\mathcal{P}_\in$ which extends the definition of $\overrightarrow{\mathcal{R}}_\in$. The definition of $\mathcal{P}_\in$ is recursive and thus not finitely presented. However it is possible to modify the construction of $\mathcal{P}_\in$ to a finite presentation $\mathcal{P}_f$. This is done in the last part of this Section (see Definition 4.8 and Theorem 4.10).

We use the same notation as in Section 2. Particularly we work in a fixed countable model $\mathfrak{M}$ of theory of finite set extended by a single atomic set $\circlearrowleft$. Also recall the following notations:

$$M_L = \{A; A \in M, \circlearrowleft \notin A\};$$

$$M_R = \{A; (A \cup \{\circlearrowleft\}) \in M, \circlearrowleft \notin A\}.$$

The following is our basic construction:

Definition 4.1 *Define the partially ordered set $\mathcal{P}_\in$ as follows:*
The elements of $\mathcal{P}_\in$ are all sets M with the following properties:

1. *(correctness)*
   - *(a)* $\circlearrowleft \notin M$;
   - *(b)* $M_L \cup M_R \subset \mathcal{P}_\in$;
   - *(c)* $M_L \cap M_R = \emptyset$.
2. *(ordering property)* $(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset$ *for each* $A \in M_L, B \in M_R$;
3. *(left completeness)* $A_L \subseteq M_L$ *for each* $A \in M_L$;
4. *(right completeness)* $B_R \subseteq M_R$ *for each* $B \in M_R$;

The relation of $\mathcal{P}_\in$ is denoted by $\leq$ and it is defined as follows: We put $M < N$ if:

$$(\{M\} \cup M_R) \cap (\{N\} \cup N_L) \neq \emptyset$$

We write $M \leq N$ if either $M < N$ or $M = N$.

The class $\mathcal{P}_\in$ is nonempty. It is of course $M = \emptyset = (\emptyset \mid \emptyset) \in \mathcal{P}_\in$. (Obviously correctness property holds. Because $M_L = \emptyset$, $M_R = \emptyset$, ordering property and completeness properties follow trivially.)

Here are a few examples of non-empty elements of the structure $\mathcal{P}_\in$ are:

$$
\begin{gathered}
(\emptyset \mid \emptyset) \\
(\emptyset \mid \{(\emptyset \mid \emptyset)\}) \\
(\{(\emptyset \mid \emptyset), (\emptyset \mid \{(\emptyset \mid \emptyset)\})\} \mid \emptyset)
\end{gathered}
$$

It is a non-trivial fact that $\mathcal{P}_\in$ is a partially ordered set. This will be proved after introducing some auxiliary notions:

Definition 4.2 *Any element $W \in (A \cup A_R) \cap (B \cup B_L)$ is called a* witness *of the inequality $A < B$.*

The level of $A \in \mathcal{P}_\in$ *is defined as follows:*

$$
\begin{aligned}
l(\emptyset) &= 0; \\
l(A) &= max(l(B); B \in A_L \cup A_R) + 1 \text{ for } A \neq \emptyset.
\end{aligned}
$$

We observe the following facts (which follow directly from the definition of $\mathcal{P}_\in$):

Fact 1 *$X < A < Y$ for every $A \in \mathcal{P}_\in$, $X \in A_L$ and $Y \in A_R$.*

Fact 2 *$A \leq W^{AB} \leq B$ for any $A < B$ and witness W^{AB} of $A < B$.*

Fact 3 *Let $A < B$ and let W^{AB} to be witness of $A < B$. Then $l(W^{AB}) \leq \min(l(A), l(B))$ and either $l(W^{AB}) < l(A)$ or $l(W^{AB}) < l(B)$.*

First we prove transitivity of the strict inequality.

Lemma 4.3 *Relation $<$ is transitive for the class $\mathcal{P}_\in$.*

Proof. Assume that three elements A, B, C of $\mathcal{P}_\in$ satisfy $A < B < C$. We prove that $A < C$ holds as well. Let W^{AB} and W^{BC} to be witnesses of the inequalities $A < B$ and $B < C$ respectively. First we prove that $W^{AB} \leq W^{BC}$. We distinguish four cases (according to the definition of the witness):

1. $W^{AB} \in B_L$ and $W^{BC} \in B_R$.

   In this case it follows from Fact 1 that $W^{AB} < W^{BC}$.

2. $W^{AB} = B$ and $W^{BC} \in B_R$.

   Then W^{BC} is witness of the inequality $B < W^{BC}$ and thus $W^{AB} < W^{BC}$.

3. $W^{AB} \in B_L$ and $W^{BC} = B$.

   Inequality $W^{AB} \leq W^{BC}$ follows analogously to the previous case.

4. $W^{AB} = W^{BC} = B$ (and thus $W^{AB} \leq W^{BC}$).

In the last case B is the witness of the inequality $A < C$. Thus we may assume that $W^{AB} \neq W^{BC}$. Let W^{AC} be a witness of the inequality $W^{AB} < W^{BC}$. Finally we prove that W^{AC} is a witness of the inequality $A < C$. We distinguish three possibilities:

1. $W^{AC} = W^{AB} = A$.

2. $W^{AC} = W^{AB}$ and $W^{AC} \in A_R$.

3. $W^{AC} \in W^{AB}_R$, then also $W^{AC} \in A_R$ from the completeness property.

It follows that either $W^{AC} = A$ or $W^{AC} \in A_R$. Analogously either $W^{AC} = C$ or $W^{AC} \in C_L$ and thus W^{AC} is the witness of inequality $A < C$. ∎

Lemma 4.4 *Relation $<$ is strongly antisymmetric on the class of elements of $\mathcal{P}_\in$.*

Proof. Assume that A and B, $A < B < A$, is a counterexample with minimal $l(A) + l(B)$. Let W^{AB} be a witness of the inequality $A < B$ and W^{BA} a witness of reverse inequality. From Fact 2 it follows that $A \leq W^{AB} \leq B \leq W^{BA} \leq A \leq W^{AB}$. From the transitivity we know that $W^{AB} \leq W^{BA}$ and $W^{BA} \leq W^{AB}$.

Again we shall consider 4 possible cases:

1. $W^{AB} = W^{BA}$.

   From the disjointness of the sets A_L and A_R it follows that $W^{AB} = W^{BA} = A$. Symmetrically $W^{AB} = W^{BA} = B$ which is a contradiction.

2. Either $W^{AB} = A$ and $W^{BA} = B$ or $W^{AB} = B$ and $W^{BA} = A$.

   Then contradiction follows in both cases from the fact that $l(A) < l(B)$ and $l(B) < l(A)$ (by Fact 3).

3. $W^{AB} \neq A$, $W^{AB} \neq B$, $W^{AB} \neq W^{BA}$.

   Then $l(W^{AB}) < l(A)$ and $l(W^{AB}) < L(B)$. Additionally $l(W^{BA}) \leq l(A)$ and $l(W^{BA}) \leq l(B)$ and thus A and B is not the minimal counter example.

4. $W^{BA} \neq A$, $W^{BA} \neq B$, $W^{AB} \neq W^{BA}$.

   The contradiction follows symmetrically to the previous case from minimality of $l(A) + l(B)$.

∎

Theorem 4.5 *$\leq$ is partial ordering on the class of elements of $\mathcal{P}_{\in}$.*

Proof. Reflexivity of the relation follows directly from the definition, transitivity and antisymmetry follows from Lemmas 4.3 and 4.4. ∎

Now we are ready to prove the main result of this section:

Theorem 4.6 *$\mathcal{P}_{\in}$ is the universal and homogeneous partially ordered class.*

First we show the following lemma:

Lemma 4.7 *$\mathcal{P}_{\in}$ has the extension property.*

Proof. Let M be a finite subset of the elements of $\mathcal{P}_{\in}$. We want to extend the partially ordered set induced by M by the new element X. This extension can be described by three subsets of M: M_- containing elements smaller than X, M_+ containing elements greater than X and M_0 containing elements incomparable with X. Since the extended relation is a partial order we have the following properties of these sets:

I. Any element of M_- is strictly smaller than any element of M_+;

II. $B \leq A$ for no $A \in M_-$, $B \in M_0$;

III. $A \leq B$ for no $A \in M_+$, $B \in M_0$;

IV. M_-, M_+ and M_0 form a partition of M.

Put

$$\overline{M_-} = \bigcup_{B \in M_-} B_L \cup M_-;$$

$$\overline{M_+} = \bigcup_{B \in M_+} B_R \cup M_+.$$

We verify that the properties I., II., III., IV. still hold for sets $\overline{M_-}$, $\overline{M_+}$.

ad I. We prove that any element of $\overline{M_-}$ is strictly smaller than any element of $\overline{M_+}$:

Let $A \in \overline{M_-}, A' \in \overline{M_+}$. We prove $A < A'$. By the definition of $\overline{M_-}$ there exists $B \in M_-$ such that either $A = B$ or $A \in B_L$. By the definition of $\overline{M_+}$ there exists $B' \in M_+$ such that either $A' = B'$ or $A' \in B'_R$. By the definition of $<$ we have $A \leq B$, $B < B'$ (by I.) and $B' \leq A'$ again by the definition of $<$. It follows $A < A'$.

ad II. We prove that $B \leq A$ for no $A \in \overline{M_-}$, $B \in M_0$:

Let $A \in \overline{M_-}, B \in M_0$ and let $A' \in M_-$ satisfies either $A = A'$ or $A \in A'_L$. We know that $B \not\leq A'$ and as $A \leq A'$ we have also $B \not\leq A$.

ad III. We prove that $A \leq B$ for no $A \in \overline{M_+}$, $B \in M_0$;

We proceed similarly to ad II.

ad IV. We prove that $\overline{M_-}$, $\overline{M_+}$ and M_0 are pairwise disjoint:

$\overline{M_-} \cap \overline{M_+} = \emptyset$ follows from I. $\overline{M_-} \cap M_0 = \emptyset$ follows from II. $\overline{M_+} \cap M_0 = \emptyset$ follows from III.

It follows, that $A = (\,\overline{M_-} \mid \overline{M_+}\,)$ is an element of $\mathcal{P}_\in$ with the desired inequalities to the elements in the sets M_- and M_+.

Obviously each element of M_- is smaller than A and each element of M_+ greater than A.

It remains to be shown that each $N \in M_0$ is incomparable with A. However we run into a problem here: it is possible that $A = N$. We can avoid this problem by first considering the set:

$$M' = \bigcup_{B \in M} B_R \cup M.$$

It is then easy to show that $B = (\,\emptyset \mid M'\,)$ is an element of $\mathcal{P}_\in$ strictly smaller than all elements of M.

Finally we construct the set $A' = (\,A_L \cup \{B\} \mid A_R\,)$. The set A' has the same properties with respect to the elements of the sets M_- and M_+ and differs from any set in M_0. It remains to be shown that A' is incomparable with N.

For contrary, assume for example, that $N < A'$ and $W^{NA'}$ is the witness of the inequality. Then $W^{NA'} \in \overline{M_-}$ and $N \leq W^{NA'}$. Recall that $N \in M_0$. From IV. above and definition of A follows that $N < W^{NA'}$. From *ad* III.

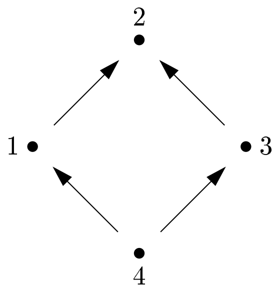

Figure 1: Partially ordered set P

above follows that there is no choice of elements such as $N < W^{NA'}$, a contradiction.

The case $N > A'$ is analogous. ∎

Proof. Proof of Theorem 4.6 follows by combining Lemma 4.7 and Lemma 3.2. ∎

Example 4.1 *The above proof when applied to the partially ordered set P from Figure 1 (with the indicated order of elements) will proceed as follows:*

$$\begin{aligned}
c(1) &= (\,\emptyset \mid \emptyset\,) \\
c(2) &= (\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,) \\
c(3) &= (\,\{(\,\emptyset \mid \emptyset\,), (\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)\} \mid \emptyset\,) \\
c(4) &= (\,\{(\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)\} \mid \{(\,\{(\,\emptyset \mid \emptyset\,), (\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)\} \mid \emptyset\,)\}\,)
\end{aligned}$$

It is not clear from Definition 4.1 that $\mathcal{P}_\in$ has finite representation (Definition 4.1 is recursive). We will now give the finite representation $\mathcal{P}_f$:

Definition 4.8 *Elements of $\mathcal{P}_f$ are all finite partially ordered sets $(P, \leq_P)$ with the greatest element denoted by $m(P, \leq_P)$. In addition each $M \in P$ satisfies properties analogous to elements of $\mathcal{P}_\in$:*

1. *(correctness)*

   (a) $\Apple \notin M$;

   (b) $M_L \cup M_R \subset P$;

   (c) $M_L \cap M_R = \emptyset$.

2. *(ordering property)* $(\{A\}\cup A_R)\cap(\{B\}\cup B_L) \neq \emptyset$ *for each* $A \in M_L, B \in M_R$*;*

3. *(left completeness)* $A_L \subseteq M_L$ *for each* $A \in M_L$*;*

4. *(right completeness)* $B_R \subseteq M_R$ *for each* $B \in M_R$*;*

The relation $<_P$ *is the transitive closure of the set* $\{(A,B); A \in B_L \cup B_R, B \in P\}$.

The relation $\leq_{\mathcal{P}_f}$ *of* $\mathcal{P}_f$ *is defined by comparison of the greatest elements:*

$$(P,<_P) \leq_{\mathcal{P}_f} (P',<_{P'}) \text{ iff } m(P,<_P) \leq m(P',<_{P'}) \text{ in } \mathcal{P}_\in.$$

Lemma 4.9 $P \subset \mathcal{P}_\in$ *for each* $(P,<_P) \in \mathcal{P}_f$.

Proof. Suppose on the contrary that there is $A \in P \in \mathcal{P}_f$ such that $A \notin \mathcal{P}_\in$. Without loss of generality we may assume that there is no $B \in P$, $B \notin \mathcal{P}_\in$ such that $B <_P A$. From the definition of $<_P$ it follows that $C \in \mathcal{P}_\in$ for each $C \in A_L \cup A_R$. Thus for A we have (1) (b) in Definition 4.8 equivalent to the (1) (b) from the Definition 4.1. The rest of the definitions are equivalent too, so we have $A \in \mathcal{P}_\in$. ∎

Theorem 4.10 $\mathcal{P}_f$ *is finitely presented and isomorphic to* $\mathcal{P}_\in$.

For the correctness of the definition of $\mathcal{P}_f$ we have to show that $m(P,<_P)$ are elements of $\mathcal{P}_\in$. This follows from Lemma 4.9.

Clearly the Definition 4.8 is a finite presentation of $\mathcal{P}_\in$. We claim that the correspondence

$$\varphi : (P,<_P) \mapsto m(P,<_P)$$

is isomorphism of $\mathcal{P}_f$ and $\mathcal{P}_\in$.

Clearly it suffices to prove that φ is bijective. This follows from the following two facts:

1. For each $(P,<_P)$ the set P contains all the elements of $\mathcal{P}_\in$ which appear in the construction of $m(P,<_P) \in \mathcal{P}_\in$. (This is the consequence of (1) (b)) and both Definition 4.1 and definition above.)

2. For each $(P,<_P)$ the set P consists only of elements of $\mathcal{P}_\in$ which appear in the construction of $m(P,<_P)$.

   Let $A^1 <_P m(P,<_P)$. By definition of $<_P$ we have $A^1, A^2, \ldots, A^t = m(P,<_P)$ such that $A^i \in A_L^{i+1} \cup A_R^{i+1}$. But as $m(P,<_P) \in \mathcal{P}_\in$ we get also $A \in \mathcal{P}_\in$ by definition 4.1 (2).

So for different sets, the greatest elements are different and each $M \in \mathcal{P}_\in$ can be used as a greatest element to construct an element of $\mathcal{P}_f$.

Corollary 4.11 *All homogeneous partially ordered sets are finitely presented.*

Proof. Using the Schmerl classification [39] and by remarks in Section 1 all homogeneous non-generic partially ordered sets are finitely presented. The generic partially ordered set $\mathcal{P}$ is isomorphic to $\mathcal{P}_\in$ by Theorem 4.10. ∎

5 Conway's surreal numbers

Recall the definition of surreal numbers, see [6, 18]. (For recent generalization see [7]). Surreal numbers are defined recursively together with their linear order. We now briefly indicate how partial order $\mathcal{P}_\in$ fits to this scheme.

Definition 5.1 *A surreal number is a pair $x = \{x^L|x^R\}$, where every member of the sets x^L and x^L is a surreal number and every member of x^L is strictly lower than every member of x^R.*

We say that a surreal number x is less than or equal to the surreal number y if and only if y is not less than or equal to any member of x^L and any member of y^R is not less than or equal to x.

We will denote the class of surreal numbers by $\mathbb{S}$.

$\mathcal{P}_\in$ may be thought as a subset of $\mathbb{S}$. The recursive definition of $A \in \mathcal{P}_\in$ leads to the following order which we define explicitely:

Definition 5.2 *For elements $A, B \in \mathcal{P}_\in$ we write $A \leq_{\mathbb{S}} B$, when there is no $l \in A_L$, $B \leq_{\mathbb{S}} l$ and no $r \in B_R$, $r \leq_{\mathbb{S}} X$.*

$\leq_{\mathbb{S}}$ is a linear order of $\mathcal{P}_\in$ ($\leq_{\mathbb{S}}$ is the restriction of Conway's order). It is in fact linear extension of our partial order $\leq$ of $\mathcal{P}_\in$:

Theorem 5.3 *For any $A, B \in \mathcal{P}_\in$ $A < B$ implies $A <_{\mathbb{S}} B$.*

Proof. We proceed by induction on $l(A) + l(B)$.

For empty A and B the theorem holds as they are not comparable by $<$.

Let $A < B$ and W^{AB} be the witness. In the case $W^{AB} \neq A, B$, then $A <_{\mathbb{S}} W^{AB} <_{\mathbb{S}} B$ by induction. In the case $A \in B_L$, then $A <_{\mathbb{S}} B$ from definition of $<_{\mathbb{S}}$. ∎

6 Concluding remarks

1. In Theorem 4.6 we presented what we believe to be the first finite presentation of the generic (i. e. homogeneous and universal) partially ordered set.

One should stress that even the finite presentation of universal partially ordered set is a non-trivial question which presented a problem. This weaker problem has been solved in category-theory context by [13] and [38]. However none of these structures is homogeneous. For example the extension properties of the class of finite graphs with the homomorphism order do not hold and also some difficult combinatorial problem (such as the product conjecture) may be expressed as particular extension properties.

2. We can also consider oriented graphs (i. e. antisymmetric relations). Let $\mathcal{O}$ denote the generic oriented graph. $\mathcal{O}$ has finite presentation $\mathcal{O}_\in$ which we obtain as a variant of $\overrightarrow{\mathcal{R}}_\in$: we say that M a is vertex of $\mathcal{O}_\in$, $M \in \overrightarrow{\mathcal{R}}_\in$ which satisfies $M_L \cap M_P = \emptyset$. (see Definition 3.5).

Further results of Section 2 may be modified accordingly.

3. Analogously to the relation $\mathcal{R}_\in$ and $\mathcal{R}_\mathbb{N}$ we can represent the ordered pairs by integers:

Definition 6.1 *Let M be any set not containing $\acute{\emptyset}$. By* code $c(M)$ of M *we denote the integer*

$$c(M) = \sum_{A \in M_L} 2^{2c(A)} + \sum_{A \in M_R} 2^{2c(B)+1}.$$

Notice that c is a bijection between the sets not containing $\acute{\emptyset}$ and the integers. The predicate $X \in M_L$ is equivalent to test whether $2c(X)$-th digit of binary representation of $c(m)$ is 1 and similarly for $X \in M_P$. Thus all our constructions involved in construction of $\overrightarrow{\mathcal{R}}_\in$ based on these predicates can be expressed arithmetically.

4. The finite presentation of generic directed graph $\overrightarrow{\mathcal{R}}$ and of the generic oriented graph $\mathcal{O}$ may be used for finite presentation of the generic tournament $\mathcal{T}$.

Let $\mathcal{O}_\in$ be the finite presentation of $\mathcal{O}$ constructed in Remark 3. Denote by $\mathcal{O}_\mathbb{N}$ the arithmetic presentation of $\mathcal{O}_\in$. Explicitly, an integer n is a vertex of $\mathcal{O}_\mathbb{N}$ iff there exists an element M of $\mathcal{O}_\in$ such that $n = c(M)$. (Thus in addition to Definition 6.1 we have that n does not contain 1's on both positions $2i$ and $2i+1$, $i \geq 1$.) Let n and n' be vertices of $\mathcal{O}_\in$. There is an edge from n to n' if and only if there are sets M and M' such as $c(M) = n$ and $c(M') = n'$ and there is edge from M to N' in $\mathcal{O}_{\in}$. Alternatively there is an edge from n to n' if there is 1 on $2n'$-th place of binary representation of n or on $(2n+1)$-th place of binary representation of n'.

We use the finite presentation $\mathcal{O}_{\mathbb{N}}$ of generic oriented graph $\mathcal{O}$ for the construction of a finite presentation $\mathcal{T}_{\mathbb{N}}$ of the generic tournament $\mathcal{T}$: An integer n is vertex of $\mathcal{T}_{\mathbb{N}}$ iff n is a vertex of $\mathcal{O}_{\mathbb{N}}$. The arcs of $\mathcal{T}_{\mathbb{N}}$ will be all arcs of $\mathcal{O}_{\mathbb{N}}$ together with pairs (n, n'), $n \leq n'$ for which (n', n) is not an arc of $\mathcal{O}_{\mathbb{N}}$.

$\mathcal{T}_{\mathbb{N}}$ is obviously a tournament. $\mathcal{T}_{\mathbb{N}}$ has the extension property by the same proof as above for Theorem 3.6 (the construction vertex M has the same properties in $\mathcal{T}_{\mathbb{N}}$ as in $\overrightarrow{\mathcal{R}}_{\mathbb{N}}$).

Thus we have:

Corollary 6.2 *All homogeneous tournaments are finitely presented.*

Proof. According to Lachlan's classification [20] (see also [4]) all homogeneous tournaments are C_3, $\mathbb{Q}$ (dense linear order), $S(2)$ (dense local order) and the generic tournament. Only $S(2)$ needs to be considered.

Intuitively, the tournament $S(2)$ can be seen as a circuit with edges forming a dense countable set of chords. The orientation is chosen in such a way that shorter chords are oriented clockwise.

One can check that $S(2)$ may be equivalently described as follows: The vertices of $S(2)$ are all rational numbers with an odd denominator q, $0 \leq q < 1$. There is an arc (a, b) in $S(2)$ iff either $a < b < a + \frac{1}{2}$ or $a - 1 < b < a - \frac{1}{2}$. ∎

5. The classification of finitely presented oriented graphs is open and it motivated our general Conjecture 2.2. It remains to be seen whether the relationship between dynamic systems and Ramsey theory (as exhibited by seminal work of Furstenberg and his school [9]) has a bearing on finite presentation and homogeneous objects. Some indication for this are given in [17] relate Ramsey classes to extremaly amenable groups and minimal flows with some interesting applications to admissible orderings.

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