Art of Graph Drawing and Art

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1 Introduction

Art and Science .... These seems to be key words of many contemporary writings. And scientists certainly are only happy to stress aesthetic qualities of their activity and artists like to justify their endeavour by science and technology. But these connections are very often superficial and often lead only to *impatience how superficially without a deeper understanding and insight these connections are treated* [7].

Here we want to illustrate some conceptual, or rather spiritual and methodological similarities of doing mathematics and art. We stress the word 'doing'. 'Doing' in the true sense of informed and complex pursuit of actual questions and problems of contemporary art and science. (By "Science" we mean here Mathematics in general, but also Computer Science and. of course, Graph Drawing; this is a volume devoted to the Drawing of Graphs.) Although both these areas are manifold and extremely diverse, we believe that they share some common qualities and profound similarities. These similarities are best

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traced by considering together the development and the methods of modern mathematics and modern art. Rather than considering the final products (works of art or theorems) we wish to stress similarities in the *modus operandi* of artists and mathematicians. We wish to stress that some of these similarities can be traced to all levels and moments of the production of both art and mathematics. And we are convinced that these similarities stem mainly from the idealistic character of both pursuits. But one has to search deep and one has to try to avoid superficial similarities and differences. We would like to rephrase the words Kandinsky used when comparing art and music [18]: In our opinion the similarities of art and mathematics are evident but they lie very deep. In fact, in the realm of true activity guided by inspiration and inspiring awareness there seem to be common features in most human activities. We put this forward as a *creative thesis* [29] and we return to this at the end of this article.

But this article is for Graph Drawing and also there is more here than meets the eye. The prime problem of Graph Drawing is to visualize properly and as accurately as possible the information which is given to us in a confusing, sometimes pictorial and sometimes mathematical, or "technical", way. This intended visualization should help us to understand, should attract and should explain. It is interesting to note that on such abstract *gnoseological* level these are the same aspects and problems which are facing an artist when he/she is trying to convey his/her message. Visualization, representation, the reality and its model are the code words. Clearly this is complemented by further aspects which make the whole picture more complicated and stresses the differences of these areas but still the abstract core is similar if not the same. (At this place it is only proper to stress the following: one has to be careful with analogies and parallels. We try to stress only those points which one can see and *prove* beyond doubts. Otherwise we can easily slip to the level of an essay and unjustified speculations.)

This text is based on the article *Art of Drawing* [28] which in turn is based on an invited talk delivered by the author at GD'99. That lecture was conceived as a multimedia show with slides, transparencies and CD projection (which operated fluently thanks to Hubert de Fraysseix). These three parts of the lecture, projected on three different screens, were called *Samples, Stories and Souveniers.* Since then some progress has been made and this lecture has been given in another context. Most notably in March 2001 in the Institute for Art History of the Academy of Sciences of the Czech Republic (thanks to an invitation of Mahulena Nešlehová) and at E.H.E.S.S., Paris, (thanks to an invitation of Pierre Rosenstiehl). Also on the scientific level a progress (a bit surprisingly) has been made. Recently this author devised an invariant, *Combinatorial Entropy*, which could serve as a measure of an aesthetic quality of a scheme and this, after several disscussions with Michel Mendès France, was thoroughly tested together with Jan Adamec on a large number of examples from both Art and Graph Drawing. A research article appeared in [2] and led to an article (of a more philosophical contents) in [30]. This text then complements both of these texts and it contains some new complementary material.

I thank to the editors H. de Fraysseix and J. Kratochvíl for suggestion to include this paper in their volume and for several constructive remarks.

2 Three easy problems

We begin our story by recalling a situation which happens often at IQ (or similar) tests: The person being tested is asked to perform a routine task, task which may be a bit time consuming and which basically tests whether he or she understands a certain notion (i.e., understands under certain mental-time stress). Such a question may read:

1. Draw a curve!
2. Divide a square in two rectangles and two squares!
3. Divide a rectangle into two triangles and two pentagons!

Of course the authors (of such tests, if they exist at all) are seeking an easy answer. But sometimes in their own naivete they perhaps do not know that a dog is hidden under carpet.

The expected answer to Question 1. is of course some "easy" picture of a curve. And as the authors of these tests believe, the easier answer mirrors a deeper understanding. But the complicated history of curves and their theory tells us something else.

It is hard to tell what a curve is. It is hard to postulate it once for all by means of a self-contained definition (not involving pictures and hand waving) which would nevertheless cover all the variety of shapes of all possible curves. As often happens - a thing which is intuitively clear becomes less clear the more we think about it: A curve may have no derivative (e.g. a sharp peak) at any of its points (Bolzano and Weierstrass), a curve may have an infinite length and yet it may enclose a finite area (Koch and others), a curve may fill the whole plane (Peano and others). Even the notion like the length of a curve is a complicated phenomenon and this question was in fact one of the initial questions which started the Theory of Fractals [20] and led to the notion of the fractal dimension. If we generalize Euclidean plane and space to n-dimension we always have n an integer (as it depicts the number of coordinates). However measuring curves in a *fractal way* we arrive at dimensions which are not necessarily integers. So some curves may have, say, dimension $4/3$. The concept of fractal dimension goes back to the beginning of this century (Hausdorff and Besicovitch). Much later it recently played key role in theory which became truly part of popular science. With all its beautiful and easily generated pictures (and perhaps, because of this, so often quoted misleadingly in an art-historical context; see e.g. [19]).

The fractal theory got mathematics seemingly as close to visual mathematics and even art as possible. As a result of this, the results were popularized, misunderstood and even misused at both ends of the spectrum. On one side these mathematically involved theories with all its interesting visual output are too tempting to the scientific audience and as a result of this they substitute modern art for many. On the other side, the interesting 'mystical' vocabulary is too tempting to art historians and critics alike. There is much evidence that this is done freely, too freely, like a new poetic enchanting word machine. (There is a newly emerging new candidate for this: the complexity theory).

But the fractal theory *is a natural counterpart* of our theory and of our definition of Combinatorial Entropy below.

Returning to our main theme: so just to define a curve is a highly non-trivial matter and thus an answer to question 1. could be certainly less self-confident. An informed individual would have the right to ask 'A curve in which sense?'

Admittedly we cannot use this strategy at questions 2. and 3. Here the questions call for a solution (displaying that one knows the meaning of the concepts). And the answer to 3. comes without any effort at the very moment when one gives up (try a flat rectangle and indicate that you give up by crossing it twice!).

But both these questions involve the construction of a special pavement or tiling and that is in general both an important and complicated question. A question which can be traced from prehistoric times, through mysticism of the Middle Ages until today. And in its combinatorial refinement this area took several surprising twists.

We cannot resist mentioning here the spectacular configuration (non-periodic tiling of the plane using just two building blocks!) due to Penrose which had spectacular influence in as remote areas as algebra, crystalography, molecular chemistry and other areas which had nothing to do with the original intentions.

Now, here is one of the pearls of the tiling theory: In 1903 the German mathematician Max Dehn asked a simple, seemingly innocent question: Can one divide a square into a finite number of squares of different sizes? Unlike in the usual picture (a possible answer to the Question 2) where we divide a square into 4 squares all of which are of the same size. Such a partition of the square is called *perfect square*. Does a perfect square exist? Does a perfect square exist at all? Yes or no? This is one of the beautiful aspects of mathematics, yes or no, there is no other way around. Modern mathematics developed techniques for asking and answering imprecise questions, for relative answers (subject to various models), for answering and (lying) with certain probability. Yet the categorical yes/no question is still one of the main (and proud) characteristics of the field.

The Dehn's problem appeared to be not so easily solvable and after few decades it was solved independently by A. Sprague(1940) and a group of Cambridge undergraduates: A. Brooks, C. Smith, A. H. Stone and W. Tutte (1940), all of whom later became well known researchers. The affirmative answer was not obtained by an elaborate (and primitive) search, but by a search involving the rudiment of a theory. A series of ingenious reductions and reformulations (involving areas such as flows and electrical networks) made the initial problem accessible. Without this the problem would be too difficult to handle. It is interesting to note that the smallest perfect square is uniquelly determined. It presents a division of a square into 21 different squeares. One cannot partition a square in less than 21 different squares and there is only one such partition into 21 squares (up to natural transformation such as rotations and flipping). This unique design was seeked for several decades and Duijvestijn solution finally came in 1978 (partly with the help of a computer). The uniqueness of this configuration of 21 squares is quite astonishing! It is the conviction of this author that the concentration of hard work and ideas makes from this square tiling a similarly unique object as the sensitivity, experience and (yes) hard work of Mondrian. So perhaps these special qualities justify its inclusion here. Actually, Charles Payan (Grenoble) produced an artistic object based on the Duijvestijn pattern and his object shares similarities with some of the Mondrian's paitning, see [30].

3 Openings

The formal similarities between art and various aspects of mathematics (mostly geometry) are both classical and contemporary.

The mysticism of the golden ratio, of the pentagon, of the regular polyhedra and of pythagorean numbers draws fascination of many and this is applied with a lot of effort to the analysis of anything from the principles of life to pyramids and Gothic cathedrals. And this is of course more true for complicated mathematical questions of perspective and vision in general. Particularly the Renaissance connection is usually quoted as the prime time of relevance of mathematics and art. There can be no doubts that mathematics at that time did some good service to art. (If only to help to upgrade artists from mere craftsmen to free artists in the same category as thinkers.) Maybe at this time of confusion and (what some call) art crisis, art intuitively seeks again a helping hand or solid base or whatever name you want to use for outwardly directed trends in recent art. But this is not so simple and traps are waiting on this road. All this will succeed only in the case when the mutual relationship (of mathematics and art) is freed from arbitrariness, superficial analogies and passing comments.

Mathematics was always a source (yes, a generator; mathematics is also a logical machine) of complex (for outsiders, mystical) patterns. This is even more true today than ever. With our understanding of randomness, of deterministic chaos and aided by graphical devices attached to computers, this 'artistic potential' of mathematics is growing. It is our conviction that this pattern generation displays mostly formal and superficial similarities to art. This is the case with fractals (mentioned above) and computer graphics in general, with patterns in chip design which resemble both minimalistic works and complex patterns of microbiology and underwater life. These tricks found one of their culminating forms in wonderful commercials and video clips. Complexity of these forms surpasses any expectation and the variety and combination of tools (and energy, and money) is simply amazing. Consider an example from another corner: the work of M.C.Escher displays a dizzying technique and explicitly touches and even anticipates some mathematical problems of tilings (tilings with special symmetries; similar to those we described above). But however intriguing all this may be, we feel that this has no bearing on the mainstream of art, better on the main problems and challenges which art and artists have been solving and are facing and solving today. And in a sense all these forms are based on combinatorially refined byproducts of mathematics and computer science as well.

4 Archeology of Aesthetic

Consider the following two problems:

I. When was the last time that you taught somebody (possibly your own child) how nice this world is and what marvels and beauties it contains?

II. When was the last time that men and women were solving the problems how to make a painting, a drawing or a photograph which would depict accurately (and/or romantically, and/or critically, and/or harmoniously) the surrounding world?

The answers to these questions seem to be very different. An answer to the first is reflecting (everyday) experience of parents and educators and on the abstract (theoretical) level it was treated in numerous books by various theories. These theories seem to be in (general) agreement with the concrete and mainstream educational praxis. The second question seems to be more complex (and it is also a more abstract problem). Somehow it seems (to many) that this is an old fashioned problem which is the subject of early modernism if not renaissance. Yet we believe that with computers this question gains a new momentum. In this paper we address this problem.

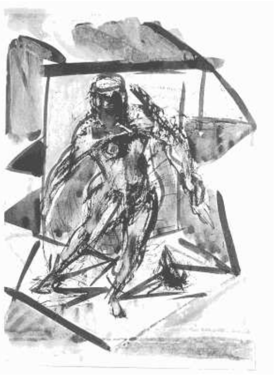
Despite the implicite word *computer* in the title in this paper, we do not address the question of visualization, of picture processing of visual information in the sense of *Computer Graphics*. The information for us is already processed and typically is of a very simple type such as a drawing (however, not necessarily a technical drawing, it may be also a drawing of an artist). What we would like to decide is how to formalize the fact that a picture (drawing) is *harmonious*. We mean harmonious in the sense of beeing aesthetically pleasing. We prefer the word harmonious to aesthetical (which is probably more in common usage) as an aesthetic feeling is probably highly individual and we cannot have an ambition to define it (or even approach it). The long tradition of art history is convincingly telling us this.

We propose an approach which should capture some features of what makes a picture (drawing) harmonious by means of the notion of *Combinatorial Entropy*. This approach is based on the analysis of curves ([25]) which in turn goes back to Steinhaus and Poincaré. The hereditary approach which we introduce below may be viewed as an approach dual to Piaget's analysis of intelligence, see e.g. [33] and compare [30].

Combinatorial Entropy is an invariant with respect to scaling and rotations, and it is a very robust parameter. This is an important feature as a perception of harmony (and of aesthetic pleasure) is a robust feeling. Moreover Combinatorial Entropy can be computed for a large class of drawing and pictures. It is routine to apply it to scanned information and no analytic description is needed. Perhaps this parameter could aid in the hierarchical approach to graph visualization in selecting a particular model which suits the best a given specific context. And as it can be scanned routinely it can be applied to vast data of molecular biology, digital libraries and even museum collections (particularly graphics and drawings) as an easily available descriptor. After these introductory lines let us begin by describing our initial situation in greater detail.

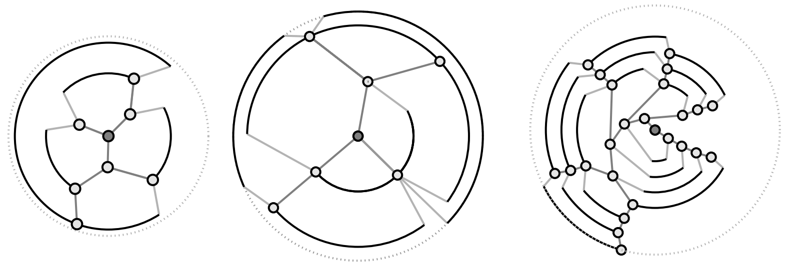
5 On Invariants for Graph Drawing

What makes the following pictures similar and what makes them different? (The substance and the context are important artistic aspects. However we are interested here only in formal similarities of these pictures; one picture is a musical score - a sketch by Janáček [35], the other picture is one of the *Moduli* - a sketch by Načeradský and the author [31].)



And, very simply, can we distinguish or order in some systematic way the following three figures (graph drawings using a program due to [9])?

The modern version of these questions is *not* how to teach a gifted and collaborating child what is nice and beautiful. Instead we need to teach an individual which is not collaborating at all and who takes



every our information deadly seriously and exploits it to the last bit - a computer. People usually do not react this way (and if so, then only in comedies like [13] or [14]; the fact that these great novels have a military setting is then not an accident). In order to "teach" a computer (and even without ambition for teaching, just dealing with it) we need a precision. And precision in the other words calls for some concrete measures of our phenomena, in the other words for *invariants*. The purpose of this note is to suggest such an invariant and to document some experiments which we performed.

Consequently, the traditional principal problem of aesthetics (and art history) - to explain and to predict artistic and aesthetically pleasing - took recently an unexpected twist. We do not explain and deal with individual instances, we have to *classify* a vast amount of data and we have to *design* procedures with likely harmonious output. This problem in its manifold variety is interesting already when our objects are well defined compositions composed from simple building blocks such as lines, squares, sticks,.... This in fact is a familiar exercise and training ground of schools of design and architecture and (traditional) art academies. This illustrates difficulty and variety of solutions even of simple situations. This should be not surprising if we realize how many simple lines needed, say Rembrandt or Picasso, to produce full images (for example drawings; 50 lines or even less!).

For our "simple composition from simple building blocks" we would like to create an *invariant* which would help us to categorize and order these compositions. It is difficult to say even on this simple level what it is an invariant, but we can certainly state which properties such an invariant should have:

i. invariant should be an (easy) *computable* aspect of the structure;

ii. invariant should be *consistent* (or invariant, meaning it should not change) under chosen modification of structure;

iii. invariant should be *useful* in that it can be used to catalogue, to order (which structure is "better"), to classify, to distinguish.

We propose here an invariant - called *hereditary combinatorial entropy* - to measure an aesthetic quality of a visual data (drawing, scheme, painting, note score, molecular data output and others). This invariant is presented in the next section.

6 Hereditary Combinatorial Entropy of a Drawing

Before defining the invariant we want to specify the rules under which the invariant should remain unchanged. Such rules were specified several times and are folklore in the visualization of scientific results. For example the book [5] lists (in section "Aesthetics") the following 11 graphic properties (called *aesthetics*) which are commonly adopted:

Crossings, Area, Total Edge Length, Maximal Edge Length, Uniform Edge Length, Total Bends, Maximum Bends, Uniform Bends, Angular Resolution, Aspect Ration and Symmetry.

The names of these criteria are self-explanatory and together they form a very good paradigm for drawing of graphs. However most of them are specific for drawings of graphs (or similar structures) and they do not apply generally (for example to artistic drawings or sketches). Yet we believe that the aesthetic quality of visual algorithms should be tested on *aesthetically charged* objects. This is one of the underlying ideas of our approach.

There is another drawback of the above paradigm. In all of these criteria (with exception of angular resolution, where we want to maximize, and symmetry which is a structural property) we are aiming for a minimization (for example we want to minimize the total length of our drawing). That of course means that we have to optimize these criteria (as they are sometimes mutually pointed against each other) and we have to add to our paradigm preferences among them. However as optimization problems these criteria are computationally hard (see [5]).

Our approach is different and we believe it could add a new aspect to visualization and analysis of visual data.

Let us first specify objects which are relevant to our method:

A *drawing* D is a finite set of curves in Euclidean plane. (As our drawings are man made we assume that the set is finite.) A curve is a continuous image of unit interval in plane (we mean a "nice" image; this is not place for technicalities).

A *painting* can be any image, including photographs. It typically consists of differently colored areas.

An *engraving* is a very special kind of object, since it can be included in the previous two classes. However seeing it as a drawing gives a lot more information.

For an infinite line L and a drawinf D denote by $i(L, D)$ the number of intersections of the line L and the drawing D. We define the *Combinatorial Entropy* $H_c(D)$ of a drawing D as the expected value of $i(L, D)$ where expectation relates to the random selection of line L. Combinatorial Entropy is also called *Fractional Length* from reasons to be mentioned later, see [1, 2].

By virtue of this definition we note the following:

i. The combinatorial entropy $H_c(D)$ is easily evaluated by a random generation of lines;

ii. $H_c(D)$ is invariant under transposition and rotation;

iii. $H_c(D)$ is invariant under scaling (i.e. "blowing up").

The role of randomness in the art has been discussed e.g. in [25], [26]. These facts make it possible to evaluate (or very accurately to estimate) combinatorial entropy of many drawing schemata, drawings of artists (we systematically tested some of the early works of Picasso [32] and Kandinsky [18] as well some drawings from [31]). We also used the analytical description of tertiary structure of some of the proteins (provided by P. Pančoška, compare [17]). The software developed in [1] uses standard tools of digital image processing, see [3], [34] and allows to handle a very broad spectrum of examples. Most difficulties are with paintings, because they must be transformed into drawings in order to count number of intersections (several filters are applied on the picture to find its contours, the resulting picture is drawing as we defined it before).

Let us state some specific examples: the above Janáček score had combinatorial entropy 18.55 which is quite similar to the combinatorial entropy 17.87 of the above drawing of Načeradský and Nešetřil taken from [31]. The three graph drawings depicted above have combinatorial entropy (from left to right) 5.70, 5.37, 6.96.

The method is flexible enough to handle complex drawings and pictures. For example classical prints from J. Verne novels when considered as a dense network of individual lines (no blurring). For example the following picture (due to Roux) [38] which serves as the front illustration of [22] has combinatorial entropy 136.6.

As expected (both by the intuition and a few trials) the $n \times n$ grid has approximate combinatorial entropy n. (As lattices are given analytically we can perform this experiment for very large n). Many more examples are contained in [1, 2].

What do these numbers mean? What is a significance of the combinatorial entropy of a drawing? The definition of the combinatorial entropy is motivated by the research done by prof. Michel Mendes France in a series of papers devoted to the analysis of curves, [25]. He defines the temperature $T(D)$ of a curve D by the following formula

$$T(D) = (\log \frac{E(i(L,D))}{E(i(L,D)) - 1})^{-1}$$

where expectation relates to the random selection of the line L.

He related this parameter to *entropy, dimension* and other parameters which he defined in an analogy to *fractal theory, statistical physics* and *geometric probability* and *integral geometry.* These definitions rest on classical theorems due to Steinhaus [37] (which in turn goes to Poincaré). One can show that the $\log H_c(D)$ is a good approximation of the entropy of a curve. Thus the name *Combinatorial Entropy* for our invariant.

By viewing a drawing as a set of curves and thinking of *eulerian trail* [22](in each component of the drawing) as a new curve of double length (as we traverse every segment of a drawing twice) we can define the the temperature by the same formula for a more general class of pictures (drawings). For large number of intersections (of a drawing D with a line L) the temperature $T(D)$ is approximately equal to the average value of $i(D,L) - 1 = H_c(D) - 1$ while the entropy $H(D)$ (in the classical sense) is approximated by the logarithm of the combinatorial entropy $H_c(D)$. It follows (and this is Steinhaus' theorem) that the combinatorial entropy $H_c(D)$ of a drawing D is approximated by the ratio

$$2\ell(D)/c(D)$$

where $\ell(D)$ denotes the total length of the drawing D and $c(D)$ denotes the length of circumference of the (convex closure) of D. (This suggests the alternative name fractional length, see [30] for details.)

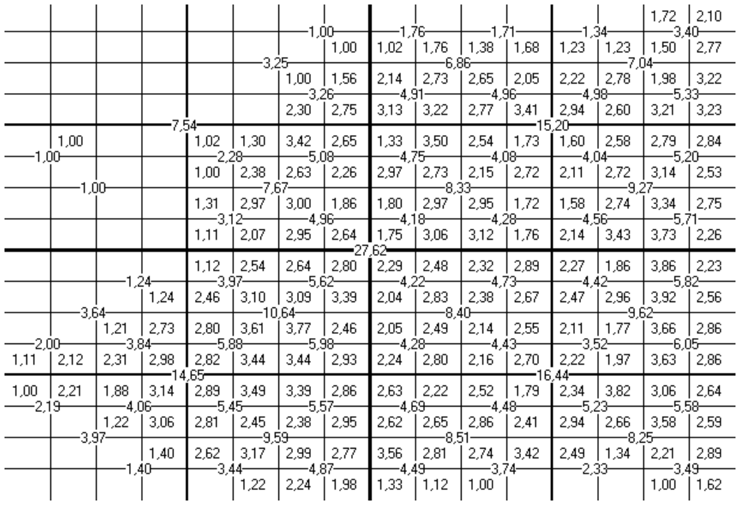
Based on these interpretations $H_c(D)$ measures the information content and the amount of work which the artist (explicitly) put into his drawing. On the other hand, by comparing various drawings of the same object (or theme) the lower $H_c(D)$ indicates the elegance and simplicity of the output. However as such the combinatorial entropy captures only the global properties of the drawing (expressed by the total length of the drawing and the circumference). Still, via the average number of intersections it captures, implicitly, many properties of the drawing. In [30] we also proposed a technique (based on the algebraical structure *cogroup*) to generate harmonious objects. In our setting this can be formulated as follows:

Hereditary Combinatorial Entropy Thesis

A *harmonious* (or aesthetically pleasing) drawing or design has combinatorial entropy in each of its (meaningful) parts proportional to the global combinatorial entropy. Here a meaningful part is a part which reflects the properties of D.

For the following (author) drawing (of our institute in Prague)

we found the following distribution of combinatorial entropies (this is based on the regular partition of the square into smaller squares; the combinatorial entropy of the drawing restricted to a particular subsquare is listed in the middle of the subquare):



7 Constructible World

The spiritual contents of art is growing. More and more an artistic object is an act, quoting Damisch, a move of the mind. A painter thinks and he/she does so through and in his/her painting, just in the same way as a mathematician thinks and he/she does so through his/her results: definitions, theorems, proofs. We do not have here in mind only conceptual art and the like, we think of art generally. And this is not a revolution. Only the dust of time, misinterpretations of history and the pride of contemporaries covered all the bold moves (how fitting a word in this context) of old masters. Let us summarize:

Our intention in this short paper has been to draw some parallells between art and mathematics mainly from the creative point of view. We have tried to stress some similarities. [28, 29, 30] contains some more and perhaps more complete material. The interested reader probably knows many other examples of such similarities from his/her experience.

Is the existence of such similarities just a coincidence? We do not know. Let us end this note in a highly speculative way:

In mathematics there is a principle called 'Church thesis' which in essence claims that every (intuitive) algorithmic procedure takes a particular form, a form expressed by (any) universal computing machine. Maybe something like Church's thesis holds in general for human activities. Let us try to formulate this thesis as follows:

Creative Thesis([29])
All sufficiently deep human activities, all sufficiently deep understandings, have profound similarities. This is exhibited in the way the work (knowledge) is organized, in the way it is revealed and in the way it interacts with other activities.

Admittedly this is too vague. But this one has to expect at such a level of generality.

Perhaps one should interpret the thesis positively. How often are stressed only the conflicting features of artistic and scientific communities, say, along the lines of feelings and sensibilities as opposed to mind and brain activities. We feel that these views are often superficial and isolationist, and sometimes simply an overreaction. On the other hand there is a numerous supporting evidence and two such examples were considered in [29]: sketches or sketching and minimalistic trends.

By the same token it is difficult to find a counterexample to the thesis. Mathematics, in particular, is a rich generator of complex patterns and as such it is often used (above we mentioned some examples). So it will be difficult to break the mathematical trap - everything goes, everything is possible. There is no doubt that the new means of technology and guided by math- and computer-devices will lead to new art form. But it is a conviction of this author that to break the trap of uniformity and unbelievable combinatorial variety of technology will require some profound and probably not yet discovered ideas. And this is probably the realm of scientists and artists, maybe for the first time since the Renaissance. We do not know yet.

But we have to try...

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