Counting List Homomorphisms and Graphs with Bounded Degrees

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Abstract

We discuss a number of variants of the homomorphism problem, where input graphs G are to be mapped homomorphically to a fixed target graph H. We set up a structure unifying the work on finding graph homomorphisms, due to the authors, of counting graph homomorphisms, due to Dyer and Greenhill, and of finding list homomorphisms, due to Feder, Hell, and Huang.

We also consider the effect of restricting the maximum degree of the input graph. We identify a number of interesting problems in this context. Our main result is a complete classification of the complexity (as polynomial time solvable or $\#P$-complete) of the problem of counting list homomorphisms of graphs with bounded degrees.

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1 Introduction

Our graphs are undirected and without multiple edges, but may have loops. A graph without loops is called *irreflexive*, and a graph in which each vertex has a loop is called *reflexive*.

Let G and H be graphs. A *homomorphism* f of G to H is a mapping $f : V(G) \to V(H)$ such that $f(g)f(g')$ is an edge of H whenever gg' is an edge of G. Each fixed graph H gives rise to a decision problem in which one is to decide whether or not a given input graph G admits a homomorphism to H. This basic decision problem has been studied by several authors, and a complete classification of its complexity has been given in [9]. A number of variants of this basic problem has been considered since. In each of the ones discussed in this paper, the graph H is fixed. Instead of deciding the existence of a homomorphism of G to H, we may want to count the total number of such homomorphisms. This problem arises in statistical physics [1], and a complete classification of its complexity has been given in [3]. In another direction, one may ask whether the problem of deciding the existence of a homomorphism of G to H is any easier when G is restricted to have bounded degrees. It turns out that this is so in certain cases, although the full answer is not yet known, [7]. Finally, there is a version in which the input consists of graph G together with lists, $L(g) \subseteq V(H), g \in V(G)$, and the question to decide is whether or not there exists a homomorphism f of G to H in which each $g \in V(G)$ has $f(g) \in L(g)$. The complexity of deciding the existence of a list homomorphism has been completely classified in [4, 5, 6].

Clearly, these variants can be combined in various ways - we may want to count list homomorphisms, or decide the existence of list homomorphisms for graphs with bounded degrees, and so on.

To organize these variants, we introduce the following notation: HOM_H will denote the basic problem (for a fixed graph H) – given an input graph G is there a homomorphism of G to H ?

We will denote the variants by introducing the superscripts $+L$ (to mean we are considering lists), $+C$ (to mean we are interested in counting the number of homomorphisms, or list homomorphisms, rather than just deciding their existence), and $+\Delta$ (to mean the degrees of the input graphs are at most Δ). Moreover, we shall omit the reference to H if we want to discuss the problem in general - over all possible graphs H. Thus, for example, HOM^{+C+L} is the family of problems HOM_H^{+C+L} of counting list homomorphisms to H, over all possible target graphs H.

To be precise, here are the definitions of the individual problems:

- HOM_H

  Instance: A graph G

  Goal: Decide if there is a homomorphism of G to H

- HOM_H^{+C}

  Instance: A graph G

  Goal: Count the number of homomorphisms of G to H

- HOM_H^{+L}

  Instance: A graph G together with lists $L(g) \subseteq V(H), g \in V(G)$

  Goal: Decide if there is a homomorphism of G to H such that $f(g) \in L(g)$ for each $g \in V(G)$

- HOM_H^{+C+L}

  Instance: A graph G together with lists $L(g)$ as above

  Goal: Count the number of homomorphisms of G to H such that each $f(g) \in L(g)$

- $\mathrm{HOM}_H^{+\Delta}$, $\mathrm{HOM}_H^{+C+\Delta}$, $\mathrm{HOM}_H^{+L+\Delta}$, $\mathrm{HOM}_H^{+C+L+\Delta}$, are defined analogously, except the instance graphs G are restricted to have all degrees less than or equal to Δ.

A summary of all the results and open problems is presented in a table at the end of the article.

2 Without Degree Constraints

The following is the classification of the **basic problem**:

Theorem 1 [9] HOM_H *is polynomial time solvable when* H *has a loop or is bipartite, and is* NP*-complete otherwise.*

When lists are added, we have the following classification:

Theorem 2 [6] HOM_H^{+L} *is polynomial time solvable when* H *is a bi-arc graph, and is* NP*-complete otherwise.*

Bi-arc graphs are defined as follows: Let C be a fixed circle, with two specified points n and s. A *bi-arc* is an ordered pair of arcs (N, S) on C such that N contains n but not s, and S contains s but not n. A graph H is a *bi-arc graph* if there exists a family of bi-arcs $(N_h, S_h), h \in V(H)$, such that for any $h, h' \in V(H)$ one of the following two alternatives must happen: Either h and h' are adjacent in H, N_h intersects $S_{h'}$, and $N_{h'}$ intersects S_h; or h and h' are not adjacent in H, N_h does not intersect $S_{h'}$, and $N_{h'}$ does not intersect S_h. (In both cases $h = h'$ is possible.) We note, cf. [4, 5], that a reflexive graph is a bi-arc graph if and only if it is an *interval graph*, and an irreflexive graph is a bi-arc graph if and only if it is bipartite and its complement is a *circular arc graph.*

The complexity of **counting** homomorphisms has been classified by M. Dyer and C. Greenhill:

Theorem 3 [3] HOM_H^{+C} *is polynomial time solvable when each component of* H *is either a reflexive complete graph or an irreflexive complete bipartite graph, and is* $\#P$*-complete otherwise.*

We now complete the picture by classifying the complexity of **counting list homomorphisms** (the same result has been independently proved in [2]):

Theorem 4 HOM_H^{+C+L} *is polynomial time solvable when each component of* H *is either a reflexive complete graph or an irreflexive complete bipartite graph, and is* $\#P$*-complete otherwise.*

Proof: When H is not of the kind described, we have seen that even the counting of ordinary homomorphisms is $\#P$-complete. Since homomorphisms are just list homomorphisms for inputs G with each $L(g) = V(H)$, the $\#P$-completeness follows. To count the number of list homomorphisms of G to H, when H is as described above, it will suffice to count the number of list homomorphisms of a connected G to a connected H (which is then either a reflexive complete graph or an irreflexive complete bipartite graph). Indeed, the number of list homomorphisms of a connected G to a disconnected H is the sum of the counts of the list homomorphisms of G to H' all the components H' of H, and the number of list homomorphisms of a disconnected G to H is the product of the counts of list homomorphisms of all components G' of G to H. But counting list homomorphisms of a connected G to such a connected H is easy: If H is a reflexive complete graph, then any mapping conforming to the lists is a homomorphism, thus the count is the product of the sizes of the *reduced* lists. (The reduced list of a vertex consists of all those members of the original list which belong to the component of H under consideration.) If H is an irreflexive bipartite graph, then the count is 0 when G contains an odd cycle of any size, including size one (i.e., a loop). Otherwise G is also a connected irreflexive bipartite graph and the count is the sum of the counts assuming the first part of G maps to the first part of H and the second part of G to the second part of H, or conversely. Each of these counts is easy to evaluate by taking a product of the appropriately reduced lists.

3 With Degree Constraints

In [7, 10] we have investigated the effect of **adding degree constraints to the basic problem** - introducing the family of problems $\mathrm{HOM}^{+\Delta}$. It turns out there are graphs H such that $\mathrm{HOM}_H^{+\Delta}$ is polynomial time solvable, even though without the degree constraints, the corresponding problem HOM_H is NP-complete, i.e., H is nonbipartite. An immediate example of this phenomenon is the problem of three-colouring:

Theorem 5 *Let $H = K_3$:*
If $\Delta = 3$ then $\mathrm{HOM}_H^{+\Delta}$ is polynomially solvable.
If $\Delta \geq 4$ then $\mathrm{HOM}_H^{+\Delta}$ is NP-complete.

(Both these observations are well known, cf. [7]; the first one is due to the theorem of Brooks which implies that a graph with all degrees at most three is three-colourable if and only if it contains no component isomorphic to K_4.)

There are more interesting examples [7]. Let $\Delta \geq 3$. It is shown in [8] that for any connected graph A there exists a graph H such that, for each graph G with degrees at most Δ, G admits a homomorphism to H if and only if A admits a homomorphism to G. Since A is fixed, the existence of a homomorphism of A to G can be tested in polynomial time (in terms of the size of G), and therefore we have a polynomial algorithm for $\mathrm{HOM}_H^{+\Delta}$.

Here is a concrete nonbipartite example of such a construction: Let X be any set. We construct a graph $H(X)$ as follows: The vertices are ordered pairs (x, T), where T is a three-element subset of X, and x is an element of $X \setminus T$. Two vertices $(x, T), (x', T')$ are adjacent in $H(X)$ just if T, T' are disjoint, and $x \in T', x' \in T$. It is proved in [8] that a cubic graph G admits a homomorphism to $H(X)$ if and only if G is triangle-free, as long as the set X is large enough.

Theorem 6 [8] *Let $H = H(X)$, where $|X| \geq 22$, and let $\Delta = 3$. Then $\mathrm{HOM}_H^{+\Delta}$ is polynomial time solvable.*

It is easy to see that $H = H(X)$ is not bipartite (in fact, it has a fairly big chromatic number [7]), thus HOM_H is NP-complete.

There are, on the other hand, many nonbipartite graphs H for which $\mathrm{HOM}_H^{+\Delta}$ remains NP-complete, even for $\Delta = 3$:

Theorem 7 [7] *Let $H = C_{2k+1}, k \geq 2$, and $\Delta \geq 3$. Then $\mathrm{HOM}_H^{+\Delta}$ is NP-complete.*

In fact, the same conclusion ($\mathrm{HOM}_H^{+\Delta}$ NP-complete) applies [7] whenever H is a triangle free graph in which each vertex belongs to a pentagon and in which no two pentagons share more than one edge. (This does not include the Petersen graph, and we do not know the complexity of $\mathrm{HOM}_H^{+\Delta}$ when H is the Petersen graph and $\Delta = 3$.) We also know [7] there exist graphs H of arbitrarily high girth and chromatic number, for which $\mathrm{HOM}_H^{+\Delta}$ is NP-complete whenever $\Delta \geq 3$.

Based on the types of reductions used in proving completeness results, T. Feder made the following (meta-)conjecture (private communication):

Conjecture 1 *Any homomorphism type problem (including list homomorphisms, and homomorphisms of more general structures) which is NP-complete without degree constraints is also NP-complete with degree constraints, provided the degree bound is high enough.*

According to Feder's conjecture, each nonbipartite graph H admits a minimum value of Δ such that $\mathrm{HOM}_H^{+\Delta}$ is NP-complete. Determining this value of Δ (i.e., classifying the complexity of all problems in $\mathrm{HOM}^{+\Delta}$) is an interesting open problem.

M. Dyer and C. Greenhill [3] have also considered the effect on the complexity of **counting the number of homomorphisms of graphs with bounded degrees** - yielding the family of problems $\mathrm{HOM}^{+C+\Delta}$. They made the following conjecture, asserting that restricting the degrees of input graphs does not change the classification of **any** basic counting problem:

Conjecture 2 *If HOM_H^{+C} is $\#P$-complete (i.e. if H has a component that is not a reflexive complete graph or an irreflexive complete bipartite graph), and $\Delta \geq 3$, then $HOM_H^{+C+\Delta}$ is also $\#P$-complete.*

They have verified their conjecture in the following special case:

Theorem 8 [3] *Suppose H is a graph whose adjacency matrix is non-singular and which contains a component that is not a reflexive complete graph or an irreflexive complete bipartite graph, and let $\Delta \geq 3$. Then $\mathrm{HOM}_H^{+C+\Delta}$ is $\#P$-complete.*

It follows (as shown in [3]) that for each graph H for which HOM_H^{+C} is $\#P$-complete there exists a Δ such that also the problem $\mathrm{HOM}_H^{+C+\Delta}$ is $\#P$-complete. This lends further support for Conjecture 1.

To complete our catalogue of the variants of the homomorphism problems, we only have to deal with **lists in conjunction with degree bounds**, i.e., with the problem families $\mathrm{HOM}^{+L+\Delta}$ and $\mathrm{HOM}^{+C+L+\Delta}$.

Theorem 9 *If H is a bi-arc graph, then $\mathrm{HOM}_H^{+L+\Delta}$ is polynomial time solvable.*

If H is any irreflexive odd cycle and $\Delta \geq 3$, then $\mathrm{HOM}_H^{+L+\Delta}$ is NP-complete.

Proof: Since $\mathrm{HOM}_H^{+L+\Delta}$ is a subproblem of HOM_H^{+L}, the polynomial time solvability follows from Theorem 2. On the other hand, $\mathrm{HOM}_H^{+\Delta}$ is a subproblem of $\mathrm{HOM}_H^{+L+\Delta}$, thus the NP-completeness of $\mathrm{HOM}_H^{+L+\Delta}$ for any odd cycle H of length at least five follows from Theorem 7. Only when H is the triangle K_3 and $\Delta = 3$, is $\mathrm{HOM}_H^{+\Delta}$ polynomial time solvable, cf. Theorem 5; in this special case, $\mathrm{HOM}_H^{+L+\Delta}$ still turns out to be NP-complete, as has been verified by T. Feder (personal communication).

It seems that each graph G which is not a bi-arc graph will have a minimum Δ for which $\mathrm{HOM}_H^{+L+\Delta}$ is NP-complete (see Conjecture 1). We do not know what these minimum values are, aside from the odd cycles as cited. To classify the complexity of problems $\mathrm{HOM}_H^{+L+\Delta}$ appears to be an interesting open problem.

We close our discussion with the following complete classification of the complexity of problems in the family $\mathrm{HOM}^{+C+L+\Delta}$:

Theorem 10 *If each component of H is a reflexive complete graph or an irreflexive complete bipartite graph, then $\mathrm{HOM}_H^{+C+L+\Delta}$ is polynomial time solvable. Otherwise $\mathrm{HOM}_H^{+C+L+\Delta}$ is #P-complete.*

Proof: In the former case, we argue that $\mathrm{HOM}_H^{+C+L+\Delta}$ is a subproblem of HOM_H^{+C+L}, which is solvable in polynomial time by Theorem 4. In the latter case H contains a component G which

1. contains both a vertex with a loop and a vertex without a loop, or
2. is reflexive but not complete, or
3. is irreflexive but not complete bipartite.

Below we give the adjacency matrices of four small graphs A, B, C, D (for simplicity we shall denote the graphs and their adjacency matrices by the same symbol). The graph A consists of two adjacent vertices exactly one of which has a loop. Any component G satisfying condition 1 above must contain A as an induced subgraph. The graph B is the reflexive path with three vertices. Any component G satisfying condition 2 above must contain B as an induced subgraph. The graph C is the irreflexive triangle K_3, and the graph D is the irreflexive path with four vertices. Any component G

satisfying condition 3 is either not bipartite, and hence contains C or D as an induced subgraph, or is bipartite but not complete bipartite, and hence contains D as an induced subgraph.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus H contains as an induced subgraph one of A, B, C, D. It is easy to verify that each of the adjacency matrices A, B, C, D is non-singular. Thus the problems $\mathrm{HOM}_H^{+C+\Delta}$ for $H = A, B, C, D$ are $\#P$-complete, according to Theorem 8. Since H contains one of these as an induced subgraph, $\mathrm{HOM}_H^{+C+L+\Delta}$ must also be NP-complete. Indeed, if, say, A is an induced subgraph of H, then any instance of $\mathrm{HOM}_A^{+C+\Delta}$ may be viewed as an instance of $\mathrm{HOM}_H^{+C+L+\Delta}$ (where each list is either $V(H)$ or $V(A)$).

Our result shows that, at least for counting list homomorphisms, adding the degree constraints did not change the classification of any graph H - thus lending further support to Conjecture 2.

Finally, we remark that our frequent assumption that $\Delta \geq 3$ is justified:

Theorem 11 *If $\Delta \leq 2$, then all problems in* HOM^{Δ}*,* $\mathrm{HOM}^{+C+\Delta}$*,* $\mathrm{HOM}^{+L+\Delta}$*, and* $\mathrm{HOM}^{+C+L+\Delta}$ *are polynomial time solvable.*

Proof: Clearly, it suffices to treat $\mathrm{HOM}_H^{+C+L+\Delta}$, since it contains all the other problems. If G is *any graph of bounded treewidth* (and graphs with maximum degree at most two have treewidth at most two), one can count the number of list homomorphisms (to any H) by the standard techniques dealing with nice tree decompositions, cf., e.g., [2].

SUMMARY TABLE

count?	lists?	degrees?	name	source	theorem	easy graphs
0	0	0	HOM	[9]	1	*-bipartite
0	1	0	HOM^{+L}	[6]	2	bi-arc
1	0	0	HOM^{+C}	[3]	3	*-complete
1	1	0	HOM^{+C+L}	here	4	*-complete
0	0	1	$\text{HOM}^{+\Delta}$	[7]	5,6,7	OPEN
0	1	1	$\text{HOM}^{+L+\Delta}$	here	9	OPEN
1	0	1	$\text{HOM}^{+C+\Delta}$	[3]	8	OPEN
1	1	1	$\text{HOM}^{+C+L+\Delta}$	here	10	*-complete

(A graph is **-bipartite* if it is bipartite or has a loop, and is **-complete* if each component is either a reflexive complete graph or an irreflexive complete bipartite graph.)

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