I. Schur, C.E. Shannon and Ramsey Numbers, a short story. *

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1 Introduction

Let $\mathcal{G}_{\sqrt{3}}$ be the graph with vertex set $\mathcal{S}$, the unit sphere of the Hilbert Space ℓ_2, with edges between any two points whose distance $> \sqrt{3}$. Clearly, $\mathcal{G}_{\sqrt{3}}$ is a triangle free graph. Does $\mathcal{G}_{\sqrt{3}}$ contain every finite triangle free graph as an induced subgraph?

In July 1974, at the IMC in Vancouver we asked Paul Erdős the following question:

> *Can every triangle-free graph on n vertices be embedded on the unit sphere in a Euclidean space E^d so that vertices connected by an edge will be at a distance $> \sqrt{3}$ apart?*

This question was motivated by an attempt to tackle the Ramsey Number $R(3,3,\ldots,3)$ (the largest number of vertices in a complete graph admiting an m-coloring of its edges with no monochromatic triangle). More specifically, an attempt to solve one of the "prized" Erdős problems by proving that there is a constant A such that $R(3,3,\ldots,3) \leq A^m$.

Erdős was excited by the problem and gave it a large exposure by including it in various "traditional" presentations including his presentation of problems in the 1975 Aberdeen Conference on Combinatorics [8]. Through numerous proddings, questions, ideas and suggestions by Erdős many were led on a wonderful tour of problems and results related to realizing graphs as points in Euclidean spaces with edges determined by distances.

In this note, we wish to trace the tour of this topic as it was expertly guided by Erdős, early origins of Ramsey Theory and the current state of related open problems. An earlier version of this note appeared in [26].

In section 2 we briefly disscuss the long historical investigation of the Ramsey Numbers $R(3,3,\ldots,3)$, their estimation, and some related problems which we trace back to the work of Issai Schur over 80 years ago. In section 3 we show the connection between the problem posed and $R(3,3,\ldots,3)$, in section 4 we trace the rise and fall of the conjecture and describe the current situation. Throughout, Erdős' guiding suggestions and questions will be highlighted.

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2 *The Ramsey Number* $R(3, 3, \ldots, 3)$

For brevity, denote by $R(3; m)$ the Ramsey number $R(3, 3, \ldots, 3)$ (with 3 being iteretad m times). It seems today that the study of these numbers actually predates Ramsey's Theorem by more than a decade. The paper [32] by Issai Schur is very closely related to these numbers and we are going to cover this in some detail. We believe that this is a worthy cause!

The paper appeared in 1916. That year was a transition year for Schur. Since 1911 he has been a successor to Felix Hausdorff at Universität Bonn and in 1916 he accepted a prestigeous position at Universität Berlin where he was later promoted to full professor in 1919. These years were certainly Schur's mathematical prime years (Schur was born in 1875; see [33] with a detailed introduction by A. Brauer). Schur's main field was algebra and number theory. However he was keen to pursue new paths and problems and so it appears that in the twenties and thirties he was a driving force of the development of a theory which later became Ramsey Theory. Not only did he publish the first paper containing a "Ramsey type" statement (of course this may be disputed; other earlier contributions include Hilbert's famous paper [19] and, of course, Mrs. Pigeonhole could make a similar claim), but Schur also identified problems which led to Van der Waerden's theorem and posed questions which inspired his student R. Rado to develop perhaps the most important pre-war contribution to Ramsey Theory [28]. Schur's work was very precise. Even after 80 years his original paper merits a close scrutiny.

He starts with a statement of a theorem proved by L. E. Dickson (Journal für reine und angewandte Mathematik, vol. 135):

> The modular equation $x^m + y^m \equiv z^m \ (mod\, p)$ can be solved by numbers x, y, z which are not divisible by p if p is larger than M which depends on m only.

As another motivation for his work, Schur quotes a result of Hurwitz (Journal für die reine und angewandte Mathematik, vol. 136) about modular forms of Fermat's Theorem. Schur's goal was to provide yet another proof of Dickson's theorem. He remarked that Dickson gave two proofs (which he described as tedious) while *"another although as well not easy proof follows from general studies of Hurwitz"* (with no reference given). Thus Dickson's theorem attracted the attention of several people. Schur's motivation was to provide an easy, elementary proof. He accomplished this by deriving Dickson's theorem as a consequence of a *"very easy Lemma, which belongs more to combinatorics than to number theory"*.

Lemma 2.1 *If one divides arbitrarily the numbers* $1, 2, \ldots, N$ *in* m *rows then, provided that* $N > m!e$, *one of the rows contains two numbers together with their difference.*

In his paper, Schur first derived Dickson's theorem using his Lemma in a "textbook" fashion. He then proved the Lemma. By today's methods it is easy to prove the Lemma via Ramsey's Theorem. Schur gave a proof whose mild formalism is perhaps worth recalling (we try to keep as much of the original notation as possible):

We proceed by contradiction: assume that there exists an $N > m!$e such that the numbers $1, 2, \ldots, N$ can be divided into m rows so that no row contains two numbers together with their difference (in today's terminology such sets are called *sum free sets*). We choose the largest row, say Z_1 which contains the integers $x_1 < \ldots < x_{n_1}$. Clearly:

$$N \leq n_1 m \tag{1}$$

We then continue by considering the differences:

$$x_2 - x_1 < x_3 - x_1 < \ldots < x_{n_1} - x_1 \tag{2}$$

These $n_1 - 1$ differences do not belong to Z_1 and thus they are distributed among the remaining $m-1$ rows. Again, let Z_2 be the largest row containing say n_2 of the differences 2:

$$Z_2 = \{x_\alpha - x_1 < x_\beta - x_1 < x_\gamma - x_1 < \ldots\} \tag{3}$$

As before we have:

$$n_1 - 1 \leq n_2(m-1) \tag{4}$$

If we subtract $x_\alpha - x_1$ from the rest of the numbers in 3 we get the $n_2 - 1$ differences:

$$x_\beta - x_\alpha < x_\gamma - x_\alpha < \ldots \tag{5}$$

These differences are distributed among the remaining $m-2$ rows. Continuing this process we get after $m-1$ repetitions the sequence: $n_1, n_2, \ldots, n_{m-1}$ which satisfies the inequalities:

$$n_\mu - 1 \leq n_{\mu\ +1}(m-\mu) \tag{6}$$

From (6) we get:

$$\frac{n_1}{(m-1)!} \leq \frac{1}{(m-1)!} + \frac{1}{(m-2)!} + \ldots + \frac{1}{(1)!} < \mathrm{e}. \tag{7}$$

(Alas, here is a single error we found in [32]: the three dots ... were missing!) Thus:

$$N \leq mn_1 < m!\mathrm{e} \tag{8}$$

contrary to the assumption.

Closing remarks in [32] are quite interesting. Schur first observes that Dickson proved that M can be chosen as $M = m^4 - 6m^3 + 13m^2 - 6m + 1$. He then continues to remark that such a bound cannot be obtained by his approach. Towards this end he defines N_m as the largest N so that $1, 2, \ldots, N$ can be partitioned into m sum free sets. He then shows that the numbers N_m satisfy the recursion $N_{m+1} \geq 3N_m + 1$ and thus also $N_m \geq \frac{3^m - 1}{2}$. These are now standard arguments, repeated many times, see e.g. [17]. Schur also remarks that the actual value of N_m is equal to the lower bound for $m \leq 3$ (and this does not hold for any other value of m as we now know). What a wealth of material in such a short note, merely 2 and half pages long!

Schur's Lemma was generalized in various directions, together with Van der Waerden's theorem (also conjectured by Schur) they are key number theoretical Ramsey-type results (see [18], [25]).

Returning to Ramsey's Theorem, denote by S_m the smallest number which satisfies Schur's lemma (these numbers are called Schur numbers). It is now clear that $S_m = N_m + 1$ and that $R(3; m) \geq S_m$ for we can define a coloring of the edges of $K_{N_m + 1}$ as follows: the edge xy gets the color i if $|x - y|$ belongs to the i-th row of the partition of $1, 2, \ldots, N_m$ into sum free sets. Combining our bounds we have $\frac{3^m + 1}{2} \leq R(3; m) \leq m!\mathrm{e}$. These bounds are basically the best currently known. The best asymptotical bounds are $R(3; m) \leq m!(\mathrm{e} - \frac{1}{24})$ (see [7]) and $R(3; m) \geq c(321)^{k/5} > c(3.17176)^k$ for some positive constant c (see [13]). Finding the limit value of $R(3; m)^{1/m}$ (known to exist) is one of P. Erdős favorite problems. It still carries an "Erdős bounty" of \$100.

We conclude this section with the following anecdote on the early interaction between the two great mathematicians, I. Schur and P. Erdős. In his first paper as a first year undergraduate, P. Erdős' gave a simple, elementary proof of Bertrand's postulate that for every positive integer n there is a prime p between n and $2n$. Later, he gave an elementary proof to Breusch's theorem and extended it. Breusch was a student of Schur. These results constituted P. Erdős doctoral dissertation which he wrote as a second-year undergraduate. When a little later, Erdős solved other problems of Schur he dubbed him *"der Zauberer von Budapest"* ("the magician of Budapest"). For more details see the fascinating article by B.Bollobás [5].

3 $R(3,3,\ldots,3)$ *and geometric graphs*

Graphs defined by means of point sets in metric spaces and their distances have a long history. In Ramsey Theory they were used frequently by P. Erdős to get bounds for Ramsey numbers in general and $r(3,m)$ in particular (please note the difference between $r(3,m)$ and $R(3;m)$, $r(3,m)$ relates to the largest size of a triangle free graph G with independence number $\alpha(G)=m$). In yet another setting this is related to small triangle free graphs G with a given chromatic number. The following is a seminal Erdős construction: he constructs a graph G as follows: vertices of G are points of an n-dimensional unit sphere with two points being joined if their Euclidean distance exceeds $\sqrt{3}$, see e.g. [11]. Also the recent best constructive lower bound $r(3,m)\geq c.m^{\frac{3}{2}}$ [1], is of the geometric nature although this graph is closer to Kneser graphs than to distance graphs. Although weaker than the bounds obtained by probabilistic methods the distance graphs still proved to be an outstanding rich source of inspiring tools and challenging problems in discrete mathematics and geometry, see e.g. [27], [12], [14], [4]. A different construction of the lower bound $r(3,m)\geq cm^{\frac{3}{2}}$ was given recently by N. Alon and P. Pudlák.

We continue this section by establishing the connection between the embedding problem and $R(3,3,\ldots,3)$. We trace the start of this connection to an observation by Erdős, McEliece and Taylor, [9] and by Erdős, Chvátal and Hedrlín, [6] that connects $R(3,3,\ldots,3)$ with the strong product of graphs defined as follows:

Definition 1 *The strong product of two graphs G_1,G_2, denoted by $G_1\boxtimes G_2$ is defined by:*

1. $V(G_1\boxtimes G_2)=V(G_1)\times V(G_2)$
2. $E(G_1\boxtimes G_2)=\{(gh,g'h')\mid (g,g')\in= E(G_1),(h,h')\in= G_2\}$ ($\in=$ *means "in" or* $h=h'$)

More specifically, it is proved in [9] and [6] that given n and k, there exists a graph G with independence number $\alpha(G)=k-1$ such that $\alpha(G^n)=R(k,k,\ldots,k)=R(k;n)$. (Here G^n denotes the strong product of n copies of the graph G).) In particular, if $k=2$ there exist a graph G with independence number 2 such that $\alpha(G^n)=R(3,3,\ldots,3)=R(3;n)$. For completenes, we include this connection.

Let $G_1,\ldots,G_n$ be n graphs. Let A be an independent set in the product $G_1\boxtimes G_2\boxtimes\ldots\boxtimes G_n$. We consider the vertices of A as the vertices of a complete graph. For any pair of vertices, $(g_1,g_2,\ldots,g_n)$ and $(h_1,h_2,\ldots,h_n)$ there is a first index j such that $g_j\neq h_j$ and $(g_j,h_j)\notin E(G_j)$. We color the edge connecting these two vertices in our complete graph by the color j. Clearly, a j-monochromatic complete subgraph corresponds to an independent set in G_j hence its size is atmost $\alpha(G_j)=\alpha_j$. Hence $|A|\leq R(\alpha_1+1,\alpha_2+1,\ldots,\alpha_n+1)$.
Conversely, we wish to show how graphs G_i satisfying $\alpha(G_i)=\alpha_i$ and $\alpha(G_1\boxtimes G_2\boxtimes\ldots\boxtimes G_n)=R(\alpha_1+1,\alpha_2+1,\ldots,\alpha_n+1)$ can be constructed. Given a coloring χ of $E(K_r)$ in n colors avoiding a monochromatic j colored complete subgraph of order α_j+1 we can construct graphs G_i, $1\leq i\leq n$, as follows: $V(G_i)=V(K_r)$, $(x,y)\in E(G_i)$ iff $\chi(x,y)\neq i$. Thus an independent set of vertices in G_i is an i colored clique in K_r and therefore $\alpha(G_i)\leq\alpha_i$. It is easy to see that the "diagonal" set $\{(x,\ldots,x)\mid x\in K_r\}$ is an independent set in $G_1\boxtimes G_2\boxtimes\ldots\boxtimes G_n$ of size r.
It follows that $R(n;m)=\max\{\alpha(G^m)\}$ where the maximum is taken over all graphs G with $\alpha(G)=n-1$. This is easily done by replacing each graph G_i in the previous derivation by the graph consisting of the disjoint union of the G_i's with edges added between any two vertices belonging to different G_i's.

The Shannon Capacity of a graph [24], provides a tool for bounding the independence numbers $\alpha(G^n)$ for all integers n. Recall, that for a given graph G we define the *Shannon Capacity* $\theta(G)$ of G by the formula $\theta(G)=\sup\alpha(G^n)^{1/n}$. It follows that $\alpha(G^n)\leq\theta^n(G)$. Hence any upper bound on $\theta(G)$ provides a desired bound for the Ramsey Number $R(3,3,\ldots,3)$ (when $\alpha(G)=2$). Our hope was that embeddability will provide the *"good"* upper bound.

Definition 2 *We say that a graph G is α-embeddable in E^d if there is a bijection $f: V(G)\to S^{d-1}$ such that $(g,g')\in E(G)\Longrightarrow \|f(g)-f(g')\|>\alpha$. The embedding is* faithful *if $\|f(g)-f(g')\|<\alpha$ when $(g,g')\notin E(G)$.*

Definition 3 *We say that a graph G has an α-representation in E^d if there is a bijection $f: V(G)\to S^{d-1}$ such that $(g,g')\in E(G)\Longleftrightarrow \|f(g)-f(g')\|=\alpha$.*

We note first that every graph of order n has an *α-representation* in some space E^d of dimension $d\le n-1$. To see this, let A be an $n\times n$ matrix with $A_{i,i}=1$, $A_{i,j}=\beta$ if $(g_i,g_j)\notin E(G)$ and $A_{i,j}=-\beta$ if $(g_i,g_j)\in E(G)$. Clearly, if β is small enough A will be positive semi-definite and hence $A=M\times M^T$. The mapping $f(g_k)\to M_k$ where M_k is the k-th row of M is an *α-representation* of G with $\alpha=\sqrt{2+2\beta}$. Thus every graph G has an *α-representation* (*α-embedding*) for some $\alpha>\sqrt{2}$. A natural question then is to determine for a given graph G, the largest α for which G admits an α-representation (embedding). The following theorem shows that these notions are tightly coupled for some families of graphs.

Theorem 1 *Let $\mathcal{G}$ be a hereditary family of graphs (i.e. $G'\in\mathcal{G}$ for every subgraph G' of $G\in\mathcal{G}$). Assume that every graph $G\in\mathcal{G}$ is faithfully α-embeddable. Then every graph $G\in\mathcal{G}$ has an α-representation.*

Proof. We first note that if a graph G is *α-embeddable* in E^d then we can embed it in E^{d+1} so that vertices coonected by an edge will be at a distance $\ge\alpha$. To see this, map the vertex $(x_1,\ldots,x_d)$ to the vertex $(\beta x_1,\ldots,\beta x_n,\sqrt{1-\beta^2})$. For a properly chosen β, some of the edges will have length α. We call this process *lifting*.

Assume that the theorem is false. There are then graphs $G\in\mathcal{G}$ which do not admit an *α-representation*. Among these, we select those graphs that have the smallest size, that is the smallest number of edges. From these graphs, let G be a graph such that among all possible embeddings that allow edges of length α, the number of edges of length $>\alpha$ is minimized.

Let $f(i)$ be the embedding and let (i',j') be an edge of G that has length $>\alpha$ and furtheremore, it has the shortest length among all edges with length $>\alpha$. The graph G' obtained from G by removing the edge (i',j') is a subgraph of G and by our assumptions $G'\in\mathcal{G}$. Since it's size is smaller than the size of G, it has an embedding $g: V(G')\to E^d$ such that $\|g(i)-g(j)\|\le\alpha$ with equality iff $(i,j)\in E(G')$.

Consider now the matrices $A_{i,j}=<f(i),f(j)>$ and $B_{i,j}=<g(i),g(j)>$. Both matrices A and B are positive semi-definite Gram matrices. Hence for every $0\le t\le 1$ the matrix $C=tA+(1-t)B$ is also positive semi-definite. Furtheremore, if $(i,j)\in E(G)\cap E(G')$ then $A_{i,j}=B_{i,j}=-\beta\Longrightarrow C_{i,j}=-\beta$. Since $(i',j')\notin E(G')\Longrightarrow B_{i,j}>-\beta$. Since $A_{i',j'}<-\beta$ clearly we can choose $0<t<1$ so that $C_{i',j'}=tA_{i,j}+(1-t)B_{i,j}=-\beta$. Now C is positive semi-definite hence $C=M\times M^T$. It is easy to see that the map: $h(i)=M_i$, where M_i is the i^{th} row of M is an embedding of G in E^d such that $(i,j)\in E(G)\cap E(G')\Longrightarrow\|h(i)-h(j)\|=\alpha$ and also $\|h(i')-h(j')\|=\alpha$. This contradicts our assumption that the map f minimizes the number of edges of length $>\alpha$. $\diamondsuit$

The powerful notions of *orthogonal representation* and *stem vectors*, defined below, were introduced by Lovász in his seminal paper [24] on the Shannon Capacity. They provide the best known general tools for bounding $\theta(G)$ from above.

Definition 4 *An orthogonal representation of a graph G is a $\sqrt{2}$ representation of $\overline{G}$ (the complement of G).*

Definition 5 *A stem vector for an orthogonal representation $f : V(G) \to S^d$ of a graph G is a vector* $\mathbf{b}$ *such that* $< \mathbf{b}, f(v) > \geq 1$ *for every* $v \in V(G)$.

In [24] Lovász proved that if $\mathbf{b}$ is a stem vector for an orthogonal representation of G then $\theta(G) \leq ||\mathbf{b}||^2$. This theorem enables us to connect the *α-representation* of a graph G and it's Shannon Capacity by a simple lifting process as shown by the next theorem:

Theorem 2 *If the graph $\overline{G}$ has an α*-representation, *$\alpha > \sqrt{2}$ then G has an orthogonal representation with a stem vector* $\mathbf{b}$ *satisfying* $||\mathbf{b}||^2 = \frac{\alpha^2}{\alpha^2 - 2}$.

Proof. It is easy to see that the lifting $(\sqrt{1-\beta^2}x_1, \ldots, \sqrt{1-\beta^2}x_n, \beta)$ of the *α-representation* of $\overline{G}$ by $\beta = \sqrt{\frac{\alpha^2-2}{\alpha^2}}$ yields an orthogonal representation with stem vector $b = (0, \ldots, 0, \frac{1}{\beta})$ as claimed. ♢

We are now ready to tie our initial problem with the attempted assault on the Ramsey Number $R(3,3,\ldots,3)$. Assume that indeed all triangle-free graphs are *α-embeddable* for some fixed $\alpha > \sqrt{2}$. Since subgraphs of a triangle-free graph are also triangle-free, by Theorem 1 they also have an *α-representation*. Clearly, this implies that every graph G with independence number $\alpha(G) = 2$ has a representation such that $(g, g') \notin E(G) \Longrightarrow ||g - g'|| = \alpha$. From Theorem 2 we could then deduce that for every such graph, it's Shannon Capacity will satisfy $\theta(G) \leq \frac{\alpha^2}{\alpha^2-2}$ and since $\alpha(G^n) \leq \theta^n(G)$ using [8] we would get that $R(3,3,\ldots,3) \leq (\frac{\alpha^2}{\alpha^2-2})^n$. So as long as there is a fixed $\alpha > \sqrt{2}$ we will be able to deduce that: $R(3,3,\ldots,3) \leq A^n$.

4 *The rise and fall of $\sqrt{2+\epsilon}$-embeddability of triangle-free graphs*

In the summer of 1976 we met Erdős in Vancouver again. His interest in the $\sqrt{3}$- embeddability was still very active. We noted that all graphs that are *$\sqrt{3}$-embeddable* in R^d are $(d+1)$-colorable but suspected that even bipartite graphs may require high dimension if they are *$\sqrt{3}$-embeddable*. Erdős suggested to explore the *$\sqrt{3}$-embeddability* of the bipartite graph $G(n, 2^n)$ defined as follows: it's first partition is a set A with n vertices and the second partition is a set B with 2^n vertices. With every subset $A' \subset A$ we associate a unique vertex in B and connect this vertex with all vertices in A'. Indeed, Erdős' idea was right on the mark. B. Alspach and M.R. proved [3] that all bipartite graphs are *$\sqrt{3}$-embeddable* and that indeed Erdős' initial hunch, that the dimension cannot be fixed, was true. In [3] it was also shown that the smallest dimension in which $G(n, 2^n)$ can be $\sqrt{3}$-embedded is $n-1$.

Sometime later in that summer, we met David Larman and told him about the $\sqrt{3}$-embeddability problem. Larman [23] constructed the first examples of triangle free graphs that are not *$\sqrt{3}$-embeddable* in any Euclidean space R^d. Larman actually constructed triangle free graphs that are not even *$\sqrt{\frac{8}{3}}$-embeddable*. While this result did not destroy the hope of proving that $R(3,3,\cdots,3) \leq A^n$ for some constant A, it did cast a doubt whether this inequality can be established via the embeddability question. Indeed in his paper Larman conjectured that for every $\epsilon > 0$ one can find triangle free graphs that are not *$\sqrt{2+\epsilon}$-embeddable* in any Euclidean space R^d.

In 1981 László Lovász called our attention to a beautiful paper by Konyagin [22] and suggested that it might help prove Larman's conjecture. In this paper, in response to a problem posed by Lovász , Konyagin constructed n unit vectors $u_1, \cdots, u_n$ such that any 3 distinct vectors contain at least one orthogonal pair and $||\sum_{j=1}^n u_j|| > cn^{0.54}$. Using this result we proved Larman's conjecture [30]. Shortly afterwards V. Rödl [29] found, for every $\epsilon > 0$, another construction of triangle free graphs that are not $\sqrt{2+\epsilon}$*-embeddable* in any Euclidean space R^d. While the last two results "slammed the door shut" on the attempted assault on $R(3,3,\cdots,3)$ via the embeddability, our tour of the topic was not finished yet.

In 1988 in Noga Alon's seminar in Tel Aviv, attended by Erdős, Konyagin's result and a stronger result proved later by Kashin and Konyagin, [21] was discussed. Erdős asked:

"What is the maximum number of vectors in the Euclidean space R^d such that among any 3 distinct vectors there is at least one pair of orthogonal vectors".

We called such sets *almost orthogonal*. Clearly, if we take two disjoint sets, each containing d mutually orthogonal vectors, we get a set of $2d$ *almost orthogonal* vectors in R^d. We believed that this is not the correct upper bound. In the summer of 1989 we met again in a conference in Leibnitz (Austria). We suggested an approach that may lead to a construction of more than $2d$ *almost orthogonal* vectors in R^d. Erdős interrupted and said: *"I do not see how to construct even $2d+1$ almost orthogonal vectors in R^d".* Once again, his hindsight proved to be correct. In 1991 it was proved that indeed the maximum number of *almost orthogonal* vectors in R^d is $2d$ [31]. The proof uses the number 3 in a very strong way that does not lend it to be used for larger numbers. Erdős immediately asked what happens if we replace the number 3 by 4 or more generally by k. The same question was also asked later by Z. Füredi and R. Stanley. Recently, Noga Alon and Mario Szegedy [2] proved that if k and d are large enough then there are exponentially many vectors in R^d such that among any k distinct vectors there is at least one orthogonal pair. The question whether R^d can contain more than $3d$ vectors such that among any 4 distinct vectors there is an orthogonal pair or the smallest k and d for which there are more than $(k-1)d$ vectors in R^d such that among any k distinct vectors there is at least one orthogonal pair, or more generally m mutually orthogonal vectors are still waiting to be resolved.

The search for bounds for the independence number of G^n when $\alpha(G) = 2$ is quite intriguing even in simple cases. For instance, it is not known whether $\alpha(\overline{C}^n_{2k+1}) > 2^n$ if n is large enough. It can be shown that $\alpha(\overline{C}^k_{2^k+1}) = 2^k$. Thus, if it turns out that $\theta(\overline{C}_{2k+1}) > 2$ we shall have for every integer m, graphs G such that $\alpha(G^k) = 2^k$ for $k \leq m$ and $\alpha(G^{m+1}) > 2^{m+1}$.

Lováz' celebrated computable function $\vartheta(G)$ which gives an upper bound for the Shannon capacity (see [24]) can be used to try and improve known bounds for Ramsey Numbers. For example, the Hoffman-Singleton graph HS (see [20]) may provide an improved lower bound for the Ramsey Numbers $R(2;k)$. This graph has 50 vertices, 7-regular and has girth 5. Let $G = \overline{HS}$ (the complement of the Hoffman-singleton graph). The smallest and largest eigenvalues of G are $-3, 42$ respectively. Hence $\vartheta(\overline{HS}) \leq \frac{10}{3}$ and thus $\alpha(G^n) \leq (\frac{10}{3})^n$ (see [24]). If the Shannon capacity of this graph turns out to be $\frac{10}{3}$ it will yield the improved lower bound $R(2;k) \geq (\frac{10}{3})^k$ (for large k). We conclude with another example of a Ramsey type result where use of the Shannon Capacity and associated tools can produce stronger results than plain counting.

Lemma 4.1 *Let $\mathbf{A(n)} = \{(a_1,\ldots,a_{2n}) | a_i \in 0,1,2,3,4,\ a_{2i-1} < a_{2i}\ 1 \geq i \geq n\}$. Let $\mathbf{B} \subset \mathbf{A(n)}$, $|\mathbf{B}| > 4^n$. There are two distinct vectors $\{b,c\} \subset \mathbf{B}$ and a corresponding index k such that $\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\} = \emptyset$.*

Proof. For each index $1 \leq k \leq n$ we count the number r_k, of distinct pairs $\{a_{2k-1}, a_{2k}\}$ where a ranges over all vectors $a \in \mathbf{B}$. Since $|\mathbf{B}| > \mathbf{4^n}$ there is an index k for which $r_k \geq 5$. But if we take 5 distinct pairs from $\binom{\{1,\ldots,5\}}{2}$ then they must include two disjoint pairs. $\diamondsuit$

A stronger result may be obtained as follows:

Lemma 4.2 *Let* $\mathbf{A(n)} = \{(a_1, \ldots, a_{2n}) | a_i \in 0, 1, 2, 3, 4,\ a_{2i-1} < a_{2i}\ 1 \geq i \geq n\}$. *Let* $\mathbf{B} \subset \mathbf{A(n)}$, $|\mathbf{B}| > 4^n$. *There are two distinct vectors* $\{b, c\} \subset \mathbf{B}$ *such that for every index* $1 \leq k \leq n\ \ |\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\}| = 0 \pmod 2$.

Proof. We first note that $\mathbf{A(1)}$ is the Petersen Graph P and $\mathbf{A(n)} = P^n$. It is well known (see [24]) that the eigenvalues of P are $\{3, 1, -2\}$. And thus the Shannon Cpacity satisfies: $\theta(P) = \alpha(P) = 4$. In particular, $\alpha(P^n) = 4^n$. Since $\mathbf{B} \subset P^n$ has more than 4^n vertices, it cannot be independent. Hence $\mathbf{B}$ contains a pair of vertices $(b, c) \in E(P^n)$. But this means that $|\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\}| = 0 \pmod 2$. $\diamondsuit$

Applying the same argument to $\overline{P}$, the complement of Petersen's graph, we get:

Lemma 4.3 *Let* $\mathbf{A(n)} = \{(a_1, \ldots, a_{2n}) | a_i \in 0, 1, 2, 3, 4,\ a_{2i-1} < a_{2i}\ 1 \geq i \geq n\}$. *Let* $\mathbf{B} \subset \mathbf{A(n)}$, $|\mathbf{B}| > (\frac{5}{2})^n$. *There are two distinct vectors* $\{b, c\} \subset \mathbf{B}$ *such that for every index* $1 \leq k \leq n\ |\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\}| \leq 1$.

Proof. The proof is almost identical to 4.3, except that in this case $\theta(\overline{P}) \leq \frac{5}{2}$ and $1 \leq k \leq n\ |\{b_{2k-1}, b_{2k}\} \cap \{c_{2k-1}, c_{2k}\}| \leq 1$ means that $(b, c) \in E(\overline{P}^n)$. Whether $\frac{5}{2}$ is best possible is not known.

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