Embedding Graphs in Euclidean Spaces, an exploration guided by Paul Erdős *

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1 Introduction

In July 1974 M.R. met Erdős in the IMC in Vancouver and asked him the following question:

> *Can every triangle-free graph on n vertices be embedded on the unit sphere in a Euclidean space E^d so that vertices connected by an edge will be at a distance $> \sqrt{3}$ apart?*

The motivation for this question was an attempt to tackle the Ramsey Number $R(3, 3, \ldots, 3)$. More specifically, an attempt to solve one of the "prized" Erdős problems by proving that there is a constant A such that $R(3, 3, \ldots, 3) \leq A^n$.

by Erdős, Fan Chung's etc.

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Erdős was excited by the problem and gave it a large exposure by including it in his presentation of problems in the 1975 Aberdeen Conference on Combinatorics [7]. Through numerous proddings, questions and suggestions by Erdős many were led on a wonderful tour of problems and results related to realizing graphs as points in Euclidean spaces with edges determined by distances.

In this note, we wish to trace the tour of this topic as it was expertly guided by Erdős and the current state of related open problems.

In section 2 we briefly disscuss the long history of investigation of numbers $R(3, 3, \ldots, 3)$, their estimation, nad some related problems which start with the work of Issai Schur 80 years ago. In section 3 we show the connection between the problem posed and the Ramsey Number $R(3, 3, \ldots, 3)$, in section 4 we trace the rise and fall of the conjecture and describe the current situation. Throughout, Erdős' guiding suggestions and questions will be highlighted.

2 *The Ramsey Number* $R(3, 3, \ldots, 3)$

For brevity, denote by $R(3; m)$ the Ramsey number $R(3, 3, \ldots, 3)$ (with 3 beeing iteretad m times. It seems today that the study of these numbers actually predates Ramsey theorem by more than a decade. The paper [27] by Issai Schur is very closely related to these numbers and we are going to cover this in a greater detail. We believe that this is worth of doing!

The paper [27] appeared in 1916. That year was a transition year for Schur. Since 1911 he has been a successor of Felix Hausdorf at Universität Bonn and in 1916 he accepted prestigeous position at Universität Berlin where he has been promoted to the full professor in 1919. These years are certainly Schur's prime times as a mathematician. (Schur was born in 1875; see [28] with a detail introduction by A. Brauer.) Schur's main field was algebra and number theory. However he has been keen to pursue new path and problems and so it appears that in twenties and thirties he has been a driving force of the development of theory which later became Ramsey theory. Not only he published the first paper containing a "Ramsey type" statement (ofcourse this is always a disputable statement; other earlier contribution include famous Hilbert's paper [17] and, ofcourse, Mrs. Pidgeonhole). But

Schur also isolated problem which led to Van der Waerden theorem and posed ques! tions which inspired his studen

. Rado to develop perhaps the most important pre-war contribution to Ramsey Theory [29]. Schur's work was very precise. Even after 80 years it is worth to look into his original paper:

Schur starts with a statement of a theorem proved by L. E. Dickson (Journals für reine und angewandte Mathematik, vol 135):

> The modular equation $x^m + y^m \equiv z^m (modp)$ can be solved by numbers x, y, z which are not divisible by p if p is larber than M which depends on m only.

As another motivation of his work Schur quotes a result of Hurwitz (Journal fúr die reine und angewandte Mathematik, vol. 136) about modular forms of Fermat's Theorem. The goal of Schur is to provide yet another proof of Dickson's theorem. It is remarked that Dickson gave two proofs (which are described as tedious) while "another although aswell not easy proof follows from general studies of Hurwitz" (and no reference is given). Thus Dickson's theorem was subject of an attention of several people. The motivation of Schur was to provide an easy solution. To do so he derives Dickson theorem as a consequence of "very easy Lemma, which belongs more to combinatorics than to number theory":

Lemma 2.1 *If one divides arbitrarily numbers* $1, 2, \ldots, N$ *in* m *rows then, providing that* $N > m!e$*, one of the rows contains two numbers together with their difference.*

Schur proceeds by a textbook derivation of Dickson's theorem using his Lemma. He then proves Lemma. By today methods this is easy to prove via Ramsey Theorem. Schur gave a proof whose mild formalism is perhaps worth to recall (we try to keep as much of the original notation as possible):

Schur proceeds by assuming contradiction: there exists $N > m!$e such that the numbers $1, 2, \ldots, N$ can be divided in m rows so that no row contains two numbers together with their difference(in later terminology such sets are called sum free sets). He choses the row Z_1 wich contains most of the numbers $x_1 < \ldots < x_{n_1}$. It is then $N \leq n_1 m$. He then proceeds by considering differences

(2) $x_2 - x_1, x_3 - x_1, \ldots, x_{n_1} - x_1$.

These differences do not belong to Z_1 and thus they are distributed to the remaining $m-1$ rows. Let Z_2 be the row containing most, say n_2 of the differences (2): (3) $x_\alpha - x_1, x_\beta - x_1, x_\gamma - x_1, \ldots$. It is then $n_1 - 1 \leq n_2(m-1)$. If we subtract the first number of (3) from the rest then we get the differences (4) $x_\beta - x_\alpha, x_\gamma - x_\alpha, \ldots$ which are distributed among the remaining $m-2$ rows. Again one considers that row which contains most, say n_3 of numbers (4). It is $n_2 - 1 \leq n_3(m-2)$. Continuing this way we get certain $m' \leq m$ and numbers $n_1, n_2, \ldots, n_{m'}$ which satisfy the inequalities (5) $n_\mu - 1 \leq n_{\mu\ +1}(m-\mu)$ and $n_{m'} = 1$. From (5) we get $\frac{n_1}{(m-1)!} \leq \frac{1}{(m-1)!} + \frac{1}{(m-2)!} + \ldots + \frac{1}{(m-m')!} <$ e. (Alas, here is a single mistake we found: the three dots $\ldots$ are missing!) Thus $N \leq mn_1 < m!$e a contrary to our (i.e. Schur's) assumption.

Closing remarks of [27] are quite interesting. Schur first observes that Dickson proved that M can be chosen of the form $M = m^4 - 6m^3 + 13m^2 - 6m + 1$. He then rontinues to remark that such a bound cannot be obtained by his approach. Towards this end he defines N_m as the largest N so that $1, 2, \ldots, n$ can be partitioned into sum free sets. He then shows that the numbers N^m satisfy the recursion $N_{m+1} \geq 3N_m + 1$ and thus also $N_m \geq \frac{3^m - 1}{2}$. These are usual arguments which were repeated many times, see e.g. [15]. Schur also remarks that the actual value of N_m is equal to the lower bound for $m \leq 3$ (and this does not hold for any other value of m as we know by now). What a wealth of material contains this short note of 2 and half pages!

Schur theorem has been generalized in various directions and together with Van der Waerden's theorem (also conjectured by Schur) it is the key number theoretical Ramsey-type result (see [16], [22]). Returning to the Ramsey theorem denote by S_m the smallest number which satisfies Schur's lemma (these numbers are called Schur numbers). It is now clear that $S_m = N_m + 1$ and that $R(3; m) \geq S_m = 1$ for we can define a coloring of edges of K_{N_m+1} as follows: the edge xy gets i-th color if $|x - y|$ belongs to the i-th row of the partition of $1, 2, \ldots, N_m$ into sum free sets. Combining our bounds we have $\frac{3^m+3}{2} \leq R(3; m) \leq m!$e. These bounds are basically the best possible. The best asymptotical bounds are $R(3; m) \leq m!(\mathrm{e} - \frac{1}{24})$ (see [6]) and $R(3; m) \geq c315^{m/5} = 3.16^m$ (see [13]). It is an outstanding problem (due to Erdős to find the limit value of $R(3; m)^{1/m}$. This is a \$100 problem of Erdős and one of his favourite one. The limit is known to exist.

3 *Geometric graphs and Ramsey Numbers* $R(3, 3, \ldots, 3)$

Graphs defined by means of point sets in metric spaces and their distances have a long history. In Ramsey theory they were used mostly by P.Erdős to get bounds on Ramsey numbers, $R(3, m)$ in particular (please note the difference between $R(3, m)$ and $R(3; m)$!) Alternatively the later problem relates to the size of a maximal independent set $\alpha(G)$ in a triangle-free graph G. In yet another setting this is related to small triangle free graphs G with a given chromatic number. The following is then seminal Erdős construction. He constructs a graph G as follows: vertices of G are points of an n-dimensional unit sphere with two points beeing joined if their Euclidean distance exceeds $\sqrt{3}$, see e.g. [10]. Also the recent best constructive lower bound $R(3, m) \geq c.n^{\frac{3}{2}}$ ([1] is of the geometrical nature (although this graph is closer to [21] than to distance graphs). Although weaker than the bounds obtained by probabilistic methods the distance graphs still proved to be a standing source of an inspiration in discrete mathematics and geometry, see e.g. [23], [11], [12], [4].

We continue this section by establishing the connection between the embedding problem and $R(3, 3, \ldots, 3)$. We trace the start of this connection to an observation by Erdős, McEliece and Taylor, [8] and by Erdős, Chvátal and Hedrlín, [5] that connects $R(3, 3, \ldots, 3)$ with the strong product of graphs. More specifically, they proved that given n and k there exists a graph G with independence number $\alpha(G) = k$ such that $\alpha(G^n) = R(k+1, k+1, \ldots, k+1)$. In particular, if $k = 2$ there is a graph G with independence number 2 such that $\alpha(G^n) = R(3, 3, \ldots, 3)$. On the other hand, the Shannon Capacity of a graph [21], provides a tool for bounding the independence numbers $\alpha(G^n)$ for all integers n. Specifically, if $\theta(G)$ denotes the Shannon Capacity of G then $\alpha(G^n) \leq \theta^n(G)$. Hence any *"good"* upper bound on $\theta(G)$ may provide the desired bound for the Ramsey Number $R(3, 3, \ldots, 3)$. Our hope was that embeddability will provide the *"good"* upper bound.

Definition 1 *A graph G is called α-embeddable in the Euclidean space E^d if there is a bijection $f : V(G) \to S^{d-1}$ such that $(g, g') \in E(G) \Longleftrightarrow \|f(g) - f(g')\| > \alpha$.*

Definition 2 *We say that a graph G has an α-representation in E^d if there is a bijection $f : V(G) \to S^{d-1}$ such that $\|f(g) - f(g') \leq \alpha\|$ with equality iff $(g, g') \in E(G)$.*

We note first that every graph of order n has an *α-representation* in some space E^d of dimension $d \leq n-1$. To see this, let A be an $n \times n$ matrix with $A_{i,i} = 1$, $A_{i,j} = \beta$ if $(g_i, g_j) \notin E(G)$ and $A_{i,j} = -\beta$ if $(g_i, g_j) \in E(G)$. Clearly, if β is small enough A will be positive semi-definite and hence $A = M \times M^{tr}$. The mapping $f(g_k) \rightarrow M_k$ where M_k is the k-th row of M is an *α-representation* of G with $\alpha = \sqrt{2+2\beta}$. Obviously, if G has an *α-representation* then G is *α'-embeddable* for some $\sqrt{2} < \alpha' \leq \alpha$. The following theorem shows that these notions can be tightly coupled for some families of graphs.

Theorem 1 *Let $\mathcal{G}$ be a family of graphs such that $G \in \mathcal{G} \Longrightarrow G' \in \mathcal{G}$ for every subgraph G' of G. Assume that every graph $G \in \mathcal{G}$ is α- embeddable then every $G \in \mathcal{G}$ has an α-representation.*

Proof To be supplied later.

□

The powerful notions of *orthogonal representation* and *stem vectors*, defined below, were introduced by Lovász in his seminal paper [21] on the Shannon Capacity.

Definition 3 *An orthogonal representation of a graph G is a $\sqrt{2}$ representation of $\overline{G}$ (the complement of G).*

Definition 4 *A stem vector for an em orthogonal representation of a graph G is a vector b such that $< b, v > \geq 1 \; \forall v \in V(G)$.*

In [21] Lovász proved that if b is a stem vector for an orthogonal representation of G then $\theta(G) \leq ||b||^2$. By a simple lifting process we can connect the *α-representation* of a graph G and it's Shannon Capacity as shown by the next theorem:

Theorem 2 *If the graph G has an α-*representation*, $\alpha > \sqrt{2}$ then G has an orthogonal representation with a stem vector b satisfying $||b||^2 = \frac{\alpha^2}{\alpha^2-2}$.*

Proof A short Proof is coming shortly... $\square$

We are now ready to tie our initial problem with the attempted assault on the Ramsey Number $R(3,3,\ldots,3)$. Assume that indeed all triangle-free graphs are *α-embeddable* for some fixed $\alpha > \sqrt{2}$. Since subgraphs of a triangle-free graph are also triangle-free, by 1 they also have an *α-representation*. Clearly, this implies that every graph G with independence number $\alpha(G) = 2$ has a representation such that $(g, g') \notin E(G) \Longrightarrow ||g - g'|| = \alpha$. From 2 we could then deduce that for every such graph, it's Shannon Capacity will satisfy $\theta(G) \leq \frac{\alpha^2}{\alpha^2 - 2}$ and since $\alpha(G^n) \leq \theta^n(G)$ using [7] we would get that $R(3,3,\ldots,3) \leq (\frac{\alpha^2}{\alpha^2-2})^n$.

4 *The rise and fall of embeddability of triangle-free graphs*

In the summer of 1976 M.R met Erdős in Vancouver again. His interest in the $\sqrt{3}$- embeddability was still very active. We noted that all graphs that are $\sqrt{3}$-embeddable in R^d are $(d+1)$-colorable but suspected that even bipartite graphs may require high dimension if they are $\sqrt{3}$-embeddable. Erdős suggested to explore the $\sqrt{3}$-embeddability of the bipartite graph $G(n, 2^n)$ defined as follows: it's first partition is a set A with n vertices and the second partition is a set B with 2^n vertices. With every subset $A' \subset A$ we associate a unique vertex in B and connect this vertex with all vertices in A'. B. Alspach and M.R. proved [3] that indeed all bipartite graphs are $\sqrt{3}$-embeddable and that indeed Erdős' initial hunch, that the dimension cannot be fixed, was true. In [3] it was also shown that the smallest dimension in which $G(n, 2^n)$ can be $\sqrt{3}$-embedded is $n - 1$.

Sometime later in that summer, M.R. met David Larman and told him about the $\sqrt{3}$-embeddability problem. Larman [20] constructed the first examples of triangle free graphs that are not $\sqrt{3}$-embeddable in any Euclidean space R^d. Larman actually constructed triangle free graphs that are not even $\sqrt{\frac{8}{3}}$ embeddable. While this result did not destroy the hope of proving that $R(3,3,\cdots,3) \leq A^n$ for some constant A it did cast a doubt whether this inequality can be established via the embeddability question. Indeed in his paper Larman conjectured that for every $\epsilon > 0$ one can find triangle free graphs that are not $\sqrt{2+\epsilon}$-*embeddable* in any Euclidean space R^d.

In 1981 László Lovász called the attention of M.R to a beautiful paper by Konyagin [19] and suggested that it might help prove Larman's conjecture. In this paper, in response to a problem posed by Lovász , Konyagin constructed n unit vectors $u_1, \cdots, u_n$ such that any 3 distinct vectors contain at least one orthogonal pair and $\| \sum_{j=1}^{n} u_j \| > cn^{0.54}$. Using this result M.R .[25] proved Larman's conjecture. Shortly afterwards V. Rödl [24] found, for every $\epsilon > 0$, another construction of triangle free graphs that are not $\sqrt{2+\epsilon}$-*embeddable* in any Euclidean space R^d. While the last two results "slammed the door shut" on the attempted assault on $R(3,3,\cdots,3)$ via the embeddability, our tour of the topic was not finished.

In 1988 M.R. gave a talk in Noga Alon's seminar in Tel Aviv. In this talk, attended by Erdős, Konyagin's result and a stronger result proved later by Kashin and Konyagin, [18] was discussed. Erdős asked: "what is the maximum number of vectors in the Euclidean space R^d such that among any 3 distinct vectors there is at least one pair of orthogonal vectors?" We called such sets *almost orthogonal*. Clearly, if we take two disjoint sets, each containing d mutually orthogonal vectors, we get a set of $2d$ *almost orthogonal* vectors in R^d. We believed that this is not the correct upper bound. In the summer of 1989 we met again in a conference in Leibnitz Austria, M.R. suggested an approach that may lead to a construction of more than $2d$ *almost orthogonal* vectors in R^d. Erdős interrupted and said: "I do not see how to construct even $2d+1$ *almost orthogonal* vectors in R^d!". Once again, his hindsight proved to be correct. In 1991 M.R [26] proved that indeed the maximum number of *almost orthogonal* vectors in R^d is $2d$. The proof uses the number 3 in a very strong way that does not lend it to be used for larger numbers. Erdős immediately asked what happens if we replace the number 3 by 4 or more generally by k. The same question was also asked later by Z. Füredi and R. Stanley [14]. Recently, Noga Alon and Mario Szegedy proved that if k and d are large enough then there are exponentially many vectors in R^d such that among any k distinct vectors there is at least one orthogonal pair. The question whether R^d can contain more than $3d$ vectors such that among any 4 distinct vectors there is an orthogonal pair or the smallest k and d for which more there are more than $(k-1)d$ vectors in R^d such that among any k distinct vectors there is a least one orthogonal pair are still waiting to be resolved.

We are sure that Erdős knows now the answers to these questions , but as a trong believer in the SF he'll have us work hard to figure the answers.

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