ASPECTS OF STRUCTURAL COMBINATORICS (Graph Homomorphisms and Their Use)

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Abstract

This paper is based on a course delivered by the author at NCTS, National Chiao Tung University, Taiwan in Febuary 1999. We survey results related to structural aspects of graph homomorphism. Our aim is to demonstrate that this forms today a compact collection of results and methods which perhaps deserve its name : structural combinatorics. Due to space limitations we concentrate on a sample of areas only: representation of algebraic structures by combinatorial ones (graphs), the poset of colour classes and corresponding algorithmic questions which lead to homomorphism dualities, blending algebraic and complexity approaches.

1 Introduction

Graph theory receives its mathematical motivation from the two main areas of mathematics: algebra and geometry (topology) and it is fair to say that graphical notions stood at the birth of algebraic topology. Consequently various operations and comparisions (relations) for graphs stress either its algebraic aspects (e.g. colorings and various products and spaces associated with graphs) or its geometrical aspects (e.g. contraction, subdivision). It is only natural that the key place in modern graph theory is played by (fortunate) mixtures of both approaches as exhibited best by the various modifications of the notion of graph minor. However from the algebraic point of view perhaps the most natural graph notion is that of a homomorphism:

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Given two graphs G and G' a *homomorphism* f of G to G' is any mapping $f : V(G) \to V(G')$ which satisfies the following condition :

$$[x, y] \in E(G) \Rightarrow [f(x), f(y)] \in E(G')$$

.

This condition should be understood as follows: on both sides of the implication one considers the same type of edges (undirected or directed). Analogous definitions give the notions of homomorphism for hypergraphs (set systems) and relational systems (of a given type, to be specified later).

The existence of a homomorphism from G to H is denoted by $G \to H$, in which case we also say that *G is homomorphic to H*, the non-existence of such a homomorphism is denoted by $G \not\to H$ and in such a case we say that *G is not homomorphic to H*. If G is homomorphic to H and also H is homomorphic to G then we say that G and H are *homomorphically equivalent* (or simply *hom-equivalent*) and we denote this by $G \sim H$.

The homomorphism is an algebraic notion which in graph theory finds its way to problems related to products, reconstruction and chromatic polynomials, just to name a few.

A combinatorial approach is motivated usually by the connection of homomorphisms to vertex coloring, expressed by the following observation which holds for every undirected graph G:

$$G \to K_k \text{ if and only if } \chi(G) \leq k.$$

An algebraic approach leads to *groups*, *monoids*, *posets* and *categories*.

In categories, the most general of these concepts, we speak about *objects* (for example, graphs), *morphisms* (for example, homomorphisms) and *composition* (for example, the composition of mappings).

Abstractly, we can think of this situation as an oriented multigraph with labeled arcs together with "composition of arrows".

Fig 1

However this (categorical) setting brings out certain features and gives rise to new perspectives. A spectacular example we want to introduce now:

Given two graphs G, H we denote by $Hom(G,H)$ the set of all homomorphisms from G to H; formally $Hom(G,H) = \{f; f : G \to H\}$ (sometimes the notation $< G,H >$ is used).

We also denote by $h(G,H)$ the cardinality of the set $Hom(G,H)$; formally, we put $h(G,H) = |Hom(G,H)|$. And by $\mathbf{h}(H)$ (sometimes the notation $< H >$ is used) the infinite vector whose coordinates are indexed by finite graphs; we consider non isomorphic graphs only. More formally,

$$\mathbf{h}(H) = (h(G,H); G \text{ finite graph},)$$

.

This formal approach (which typically involves large sets and which in this setting relates to invariants like the Tutte-Grothendieck polynomial and to problems in statistical physics) has been a remarkable success in many respects. In addition to recent applications to problems related to statistical physics, see e.g. [65], [10], [6], we wish to stress purely combinatorial problems. In order to motivate this by the following two results:

Theorem 1.1 (Lovász [42]) *For any two graphs G, H:*

$$\mathbf{h}(G) = \mathbf{h}(H) \textit{ if and only if } G \cong H$$

.

It has been shown by Lovász [43] that Theorem 1 holds in most "combinatorial" categories covering particularly relational systems of arbitrary type.

Theorem 1.2 (Müller [48]) *If G is a graph with n vertices and $m > nlogn$ edges then G is edge reconstructible.*

Concerning Theorem 1.2 let us first recall the famous edge reconstruction conjecture (see e.g. survey by Bondy [5]):

Conjecture 1 *For undirected graphs G, H with at least 4 edges the following two statements are equivalent*

(i) *G and H are isomorphic;*

(ii) *There exists a bijection* $\iota : E(G) \to E(H)$ *such that* $G - e \cong H - \iota(e)$ *for every* $e \in E(G)$.

This conjecture fails to be true for several (in fact four) small graphs, hence the assumption that G and H have as least 4 edges. Obviously (i) implies (ii) and thus the validity of the opposite implication is the core of the conjecture.

Now it is well known that the above condition (ii) is equivalent to the following condition (ii'):

There exists a bijection

$$\iota : \{A; A \subset E(G)\} \longrightarrow \{B; B \subset E(H)\}$$

such that $(V(G), A)$ is isomorphic to $(V(H), \iota(A))$ for every $A \subset E(G)$.

The edge reconstruction conjecture is related to Ulam's vertex reconstruction conjecture and Müller's result is one of the strongest results supporting its validity. As an illustration of the techniques we prove the following (less technical) results of Lovász [43] which motivated Müller's result.

Theorem 1.3 ([43]) *Let* G *be a graph with* n *vertices and* m *edges, let* $m > \binom{n}{2}/2$. *Then* G *is edge-reconstructible.*

Proof. Let $G = (V, E)$, $H = (V, E')$ be graphs such that the condition (ii') holds:

Let $\iota : \{A, A \subset E\} \to \{B; B \subset E'\}$

be a bijection such that

$$(V, A) \cong (V, \iota(A)).$$

We shall need two more definition:

Given two graphs G_1, G_2 we denote by $i(G_1, G_2)$ the number of *injective* homomorphisms $f : G_1 \longrightarrow G_2$. $\overline{G}$ denotes the complement of graph G: $V(\overline{G}) = V(G)$, $E(\overline{G}) = \binom{V(G)}{2} - E(G)$.

Using Inclusion-Exclusion Principle we can express the number of injective homomorphisms as follows:

$$i(G, \overline{H}) = i((V, \emptyset), (V, \emptyset)) - \sum_{e_1 \in E} i((V, \{e_1\}), H) + \sum_{e_1 \neq e_2 \in E} i((V, \{e_1\, e_2), H) - \ldots + (-1)^{|E|} i(G, H). \tag{1}$$

Similarly

$$(2) \qquad i(H,\overline{H}) = i((V,\emptyset),(V,\emptyset)) - \sum_{e_1\in E'} i((V,\{e_1\}),H) + \sum_{e_1\neq e_2\in E'} i((V,\{e_1\,e_2\}),H) - \ldots + (-1)^{|E'|} i(H,H).$$

Obviously $|E| = |E'|$ but also, according to our assumption (ii'), for every $1 \leq k < |E|$ the bijection ι associates to any set $\{e_1,\ldots,e_k\}$ a set $\iota(\{e_1,\ldots,e_k\}) = \{e'_1,\ldots,e'_k\}$ such that $(V,\{e_1,\ldots,e_k\}) \cong (V,\{e'_1,\ldots,e'_k\})$.

Thus the terms on the right side of expressions (2) and (3) are pairwise the same. Thus

$$(3) \qquad i(G,\overline{H}) - i(H,\overline{H}) = (-1)^{|E|}(i(G,H) - i(H,H)).$$

However the left side of (3) is zero as both G and H have too many edges. On the other side $i(H,H) > 0$ and thus $i(G,H) \neq 0$ too.

Let us sumarize : we proved that there exist injective homomorphism $G \longrightarrow H$.

Now we can repeat the same proof for pairs $i(G,\overline{G})$ and $i(H,\overline{G})$ and we obtain similarly $i(H,G) \neq 0$.

But then, as our graphs are finite, we have $G \cong H$. (Alternatively, it suffices to prove $i(G,H) \neq 0$ as we know that both G and H have the same number of edges. Thus any injective homomorphism, i.e. subgraph, is necessary isomorphism.) ♡

Perhaps this example of use of *combinatorics of maps* provides a good motivation for this paper where we want to introduce more examples. Due to the space limitations we have to concentrate on a few sample areas only.

In Chapter 2 we deal with algebraic aspects of graph homomorphisms from the point of view of categories (of graphs and their homomorphisms) and posets (induced by the existence of homomorphism). Particularly we review the recent development related to the notion of density (and we give three proofs of the fundamental result for undirected graphs).

In Chapter 3 we survey complexity questions, both hard and polynomial instances of the basic decision problem (the existence of an H-colloring). We close with a characterization of finitary dualities which as an analogy for colorings of the Robertson-Seymour-Thomas program. We close the paper with yet another view relating this paper to fundamental results (and insights) of P. Erdös.

The paper is organized as follows:

Chapter 2: Ordering by Homomorphism (Structure of Color Classes)

2 Ordering by Homomorphisms (Structure of Color Classes)

2.1 Categories and representations

Consider *all* finite graphs together with all homomorphisms between them. What we can say about a structure of such a situation?

This is certainly a complicated situation, on the first glance undescribably complicated. But there is a simple basic structure which underlies this situation and in fact this is a common structure to many situations. This underlying structure is called a *category*.

In order to define a category K we always specify *objects* (we denote them by capital letters A, B, C, ...) and *morphisms*.

Morphisms are labeled arrows denoted by $A \xrightarrow{f} B$ or $f : A \longrightarrow B$. What this means is that each morphism $f : A \longrightarrow B$ has specified two objects $d(f) = A$ and $r(f) = B$ (the *domain* and the *range* of f). Denote by $< A, B >$, or $Hom(A, B)$, the set of all morphisms f satisfying $d(f) = A$, $r(f) = B$ (more precisely we should write $< A, B >_{\mathcal{K}}$). As we work exclusively with finite objects (like finite graphs), we assume that for any pair of objects A, B the set $< A, B >$ is *finite* (and of course it may be empty).

Two more features describe our situation:
For every triple A, B, C of objects we have a mapping

$$\circ :< B, C > \times < A, B > \longrightarrow < A, C >$$

which assigns to morphisms f, g their *composition* $f \circ g$ (this composition $f \circ g$ need not be a composition of maps as even f, g need not be mappings). We further assume that the operation $\circ$ (it is in fact a partial operation) is *associative*: $(f \circ g) \circ h = f \circ (g \circ h)$ whenever one of the sides of the equality is defined.

And finally we are given to very object A a morphism $\mathbf{1}_A$ which satisfies

$$\mathbf{1}_A \circ f = f \text{ and } g \circ \mathbf{1}_A = g$$

whenever the left hand side makes sense. $\mathbf{1}_A$ is called the *identity* on A.

This completes the description of our situation. If objects, morphisms, composition and unit objects are specified and the above minimal requirements ($< A, B >$ finite set, identity and associativity of $\circ$) then we say that we have an instance of a *category*. (We specified the notion of category in finite set theory, we make no attempts to generalize to infinity; so we have countably many objects and morphisms but between any two objects only finitely many morphisms. This is a paper on finite combinatorics.)

Categories are abundand and so is the literature about them (we want to single out three books : MacLane classical modern [45], very elementary but rigorous [41] and [61] which is closest to our combinatorial setting).

Here are some examples:
SET = category of all finite sets and all mappings between them;
ORD = category of all finite linearly ordered sets and all monotone mappings between them (this is also called *simplicial category*);
GRA = category of all finite graphs and all their homomorphisms;
$(X, \leq)$ = the category induced by any partially ordered set : $x \longrightarrow y$ iff $x \leq y$; in this category $< x, y >$ consists from at most one morphism - such a category is called *thin*;
Any group $(X, \cdot, e)$ can be considered as a category with one object X only, morphisms $X \longrightarrow X$ are labeled by elements of group with composition defined as multiplication;
Any monoid $(X, \cdot, 1)$ can be treated similarly as a category with one object, (*monoid* is a semigroup with unite element).

As indicated by these examples, category theory is a *minimal calculus* common to most mathematical theories. It is a (rather schematic) world in which most mathematics (and mathematicians) live.

We mostly use categories and category theory to motivate and to formulate easily general things which otherwise would be hard to describe. We usually do not use calculus of categories to prove a particular statement. But there are exceptions. Even in combinatorics there are exceptions and some of them we shall describe in this chapter.

We need to compare categories. This is straightforward (by now; however it took some time before proper concepts were isolated):

Let $\mathcal{K}$, $\mathcal{L}$ be categories. A mapping F which maps

$$F : OBJECTS(\mathcal{K}) \longrightarrow OBJECTS(\mathcal{L})$$

$$F : MORPHISMS(\mathcal{K}) \longrightarrow MORPHISMS(\mathcal{L})$$

is called *functor* providing the following holds:

(i) $r(F(f)) = F(r(f))$

(ii) $d(F(f)) = F(d(f))$;

(iii) $F(f) \circ F(f) = F(f \circ g)$; (more exactly we should write $E(f) \circ_{\mathcal{L}} F(g) = F(f \circ_{\mathcal{K}} g)$)

(iv) $F(\mathbf{1}_A) = \mathbf{1}_{F(A)}$

for all morphisms f, g (providing that right hand side in (iii) is defined).
We write $F : \mathcal{K} \to \mathcal{L}$.

We say that a functor F is *faithful* providing it is 1-1 on every set $< A, B >$ of morphisms in $\mathcal{K}$. We say that a functor F is *embedding* providing F is 1-1 (both on $OBJECTS(\mathcal{K})$ and $MORPHISMS(\mathcal{K})$) and

$$\{F(f); f \in < A, B >\} = < F(A), F(B) >$$

Finally, we say that functor
$F : \mathcal{K} \longrightarrow \mathcal{L}$ is an *embedding of* $\mathcal{K}$ *into* $\mathcal{L}$ or that F is a *representation of* $\mathcal{K}$ *in* $\mathcal{L}$.

The notion of embedding or representation should capture the following results:

Theorem 2.1 ([14]) *Every group is isomorphic to the group of automorphisms of a graph.*

Theorem 2.2 ([20]) *Every monoid is isomorphic to the monoid endomorphisms of a graph.*

Theorem 2.3 ([25]) *Every finite category can be represented by balanced directed graphs.*

(An oriented graph is *balanced* if any cycle contains the same number of forwarding and backwarding arcs.)

All these theorems are consequences of much more general approach which we shall now outline.

2.2 Concreteness - a combinatorial obstacle

The above results indicate that somehow everything is true. There are embeddings of categories in all possible directions. It seems that we can represent every category $\mathcal{K}$ by any other $\mathcal{L}$ if only $\mathcal{L}$ is sufficiently "non-trivial" (however there are exceptions, bounded degree, bounded genus, orientation, see [1], [2], [26]. The situation seems to be reminiscent to theory of NP-completeness where the initial joy (of being able to "narrow" $P - NP$ gap) turned to realistic scepticism (that virtualy "everything" seems to be NP-complete).

But for representation of categories there is one striking difference. There is a non-trivial necessary condition when a category is representable, say, by the category GRA of all finite graphs and all their homomorphisms. This is related to the following notion:

Definition 1 *A category $\mathcal{K}$ is said to be* concrete *if there is a faithful functor $F : \mathcal{K} \longrightarrow SET$.*

Most categories ("from real life") are concrete as the morphism between their objects represent (special) mappings and one mapping usually does not correspond to two different morphisms(which corresponds to the faithfullness of the functor).

To be more precise, for example, the following functor $F : GRA \rightarrow SET$ is faithfull and thus GRA is a concrete category:

$$F(G) = V(G), F(f) = f.$$

But the situation is different and less easy if the morphisms in a category $\mathcal{K}$ are "abstract arrows". Then we have to find sets to represent objects and assign to morphisms mappings between corresponding sets in such a way that the composition in K becomes a simple composition of mappings. Consider for example a poset $(X, \leq)$. If we view $(X, \leq)$ as a category (as we did above; it was a thin category) then what we want to do is to replace each $x \in X$ by a set M_x and each pair $x \leq y$ by a mapping $f_{xy} : M_x \longrightarrow M_y$ so that composition holds : $f_{yz} \circ f_{xy} = f_{xz}$.

Having said that it is easy to guess such a representation. We can put

$$M_x = \{y; y \leq x\}$$

and $f_{xy}(z) = z$ for $z \leq x$ (or,equivalently, $z \in M_x$).

Similarily, to prove that a given group (or a given monoid), if considered as a (single object) category, is concrete, ammounts to find a representation of a group (or monoid) by mappings. This is well known and for example left translations turn every group (or monoid) to *isomorphic* permutation group (or monoid of mappings) and that is what we wanted to prove. After realizing this perhaps the following question is justified:

Problem 1 *Is every category concrete?*

This nice problem took a full decade to solve. A nice combinatorics is involved and curiously enough this result misses most monographies on theory of categories. After all this results perhaps belongs to more combinatorial context.

The solution (and indeed problem itself) started with John Isbell [34] when he discovered that the answer to the above problem is negative:

If a category $\mathcal{K}$ is concrete then it has to satisfy the following *Isbell's condition* which we are going to explain now:

For every two objects A, B of $\mathcal{K}$ denote by $L(A, B)$ the set of all pairs (a, b) of morphisms of $\mathcal{K}$ which satisfy: $d(a) = d(b)$ and $r(a) = A$, $r(b) = B$, see Fig.2.

On the set $L(A, B)$ define an equivalence $\sim$ as follows :

$(a, b) \sim (a', b')$ iff for every pair (f, g) of morphisms $d(f) = A$, $d(g) = B$, $r(f) = r(g)$ holds: $f \circ a = g \circ b \Leftrightarrow f \circ a' = g \circ b'$

In the other words $(a, b) \sim (a', b')$ if no pair of outcoming morphisms (f, g) distinguishes (a, b) from (a', b'); see Fig.3.

(It is easy to see that $\sim$ is an equivalence.)

Now we have

Theorem 2.4 (Isbell [34]) *If $\mathcal{K}$ is a concrete a category then for any pair A, B of objects of $\mathcal{K}$ the equivalence $\sim$ has only finitely many classes.*

Proof.

Let $F : \mathcal{K} \to SET$ be a faithfull functor. So let $\mathcal{K}$ be a concrete category. Then we can identify $\mathcal{K}$ with its F- image in SET and thus we may assume without loss of generality that $\mathcal{K}$ is a category of (some) sets and of (some) mapping between them.

To any pair (a, b) of mappings with $d(a) = d(b) = C$, $r(a) = A$, $r(b) = B$ we associate the following relation $R(a, b)$ on $A \cup B$ (A, B are sets now):

$$R(a, b) = \{(a(u), b(u)); u \in C\}$$

.

Assume now that $R(a, b) = R(a', b')$ and that $f \circ a = g \circ b$. Then obviously also $f \circ a' = g \circ b'$ (as if $f(a(u)) = g(b(u))$ then also $f(a'(u')) = g(b'(u'))$ for some u' satisfying $(a(u), b(u)) = (a'(u'), b'(u'))$). Thus we have that $R(a, b) = R(a', b')$ implies $(a, b) \sim (a', b')$.

But then $R(a, b) \subseteq A \times B$ and thus the number of possible equivalence classes of $\sim$ on $L(A, B)$ is $\leq 2^{|A||B|}$ – a large yet finite number.

$\heartsuit$

Thus there are additional structural combinatorial conditions for concrete categories. This we believe is a surprising fact. Even if it is not surprising then it certainly is the only such additional conditions. However it took a long time (a full decade) before coresponding theorems were proved. These results we shall introduce now. First we complete the solution of Problem 1:

Theorem 2.5. *For a category $\mathcal{K}$ are the following two statements equivalent*

(1) *$\mathcal{K}$ is concrete*

(2) *$\mathcal{K}$ satisfies Isbell's Condition for every pair of its objects.*

This theorem was proved by J. Vinárek [63] (extending an earlier result of P. Freyd [13]) . We proved (1) $\Rightarrow$ (2) only. (2) $\Rightarrow$ (1) is a bit harder. In a sense it may be viewed as an on-line version of the proof of the following special case. In this proof we also introduce an important construction.

Theorem 2.6. *Every finite category K is concrete.*

Proof. We are given finitely many objects $A_1, \ldots, A_n$ together with finitely many morphisms $f : A_i \longrightarrow A_j, 1 \leq i, j \leq n$. Let us define functor which we will denote by symbol $<>$:

For object A_i we put $< A_i >= \bigcup_{k=1}^{n} < A_k, A_i >$;

for morphism $f : A_i \longrightarrow A_j$ we put $< f > (\varphi) = f \circ \varphi$ (for any morphism φ with $Rg(\varphi) = A_i$).

This perhaps needs some explanation which is contained in the following diagram: Fig.4

It holds $< \mathbf{1}_A >= \mathbf{1}_{<A>}$ and $< f \circ_{\mathcal{K}} g >=< f > \circ < g >$.

Moreover if f and g are different morphisms $g, f \in< A_i, A_j >$ of $\mathcal{K}$ then $f \circ \mathbf{1}_{A_i} = f \neq g = g \circ \mathbf{1}_{A_i}$ and thus $< f > \neq < g >$ and $<>$ is a faithfull functor into SET. Thus $\mathcal{K}$ is concrete. ♡

The functor $<>$ introduced in the proof of previous theorem is actually 1-1 and it is called *Cayley-MacLane Representation* from reason which will be evident in the next section.

Remark 1 *Observe how close is the functor $<>$ to the Lovász vector $< A >$. To support (and not to confuse) this connection we decided to use the same symbol in two different (yet related) meanings (in two different parts of this paper).*

2.3 Representations

Thus we know by Theorem 2.6 (and 2.5) that all finite categories are concrete. Now we shall generalize this result (which gives existence of a faithfull functor) to the following results (which give embedding):

Theorem 2.7. *Any finite category is representable by graphs.*
Explicitely: For every finite category $\mathcal{K}$ there exists on embedding $F : \mathcal{K} \longrightarrow GRA$.

This theorem was proved in [21], see e.g. [61],[25] and it extends representations of groups and monoids which were stated earlier as Theorems 2.1 and 2.2.

The proof is a combination of two (by now) standard techniques: we first reduce the problem to relational systems (i.e. colored graphs) and then use a replacement trick to reduce colored graphs to graphs.

2.3.1 Relational Systems Instead of Graphs

Definition 2 *An m-relational system S of order r is a pair $(X; (R_i; i = 1, \ldots, m))$ where $R_i \subseteq X \times X$.*

(Alternatively, an m-relational system is a directed graph with arc colored by m distinct colors.)

Given relational systems $S = (X, (R_i; i = 1, \ldots, m))$ and $S' = (X', ((R'_i; i = 1, \ldots, m))$ a *homomorphism* $f : S \longrightarrow S'$ is a mapping $f : X \longrightarrow X'$ which satisfies for every $i = 1, \ldots, m$:

$$(x, y) \in R_i \Longrightarrow (f(x), f(y)) \in R'_i$$

We shall denote by $REL(m)$ the category of all finite m-relational systems and all homomorphisms between them.

Somehow it is easier to represent categories by relational systems. For example we have the following:

Theorem 2.8. *Every finite category can be represented for some m by $REL(m)$.*

Explicitly : For every finite category $\mathcal{K}$ there exists m and an embedding $F : \mathcal{K} \longrightarrow REL(m)$.

Proof. Let $\mathcal{K}$ has objects $a = \{A_1, \ldots, A_n\}$ and morphisms $m = \{f_1, \ldots, f_m\}$. Let us define functor $F : \mathcal{K} \longrightarrow SET$ as follow

$$F(A) =< A >$$

$$F(f)(\varphi) = f \circ \varphi$$

(This is again *Cayley-Maclane functor*, sometimes called *hom-functor*.)

On each set $< A >$ define relations $R_1^A, \ldots, R_m^A$ as follows

$$(\varphi, \varphi') \in R_i \text{ iff } \varphi' = \varphi \circ f_i$$

(This is a generalization of the right translation from groups to categories.)
We shall prove that the above functor F is in fact an embedding $\mathcal{K} \longrightarrow REL(m)$.

For this it clearly suffices to prove:

1. For every $f_i : A \longrightarrow A'$ the mapping $F(f_i)$ is a homomorphism (in $REL(m)$)

$$(< A >, (R_i^A)) \longrightarrow (< A' >, (R_i^{A'}))$$

2. For every hommorphism

$$g : (< A >, (R_i^A)) \longrightarrow (< A' >, (R_i^{A'}))$$

there exists f_i such that $g = F(f_i)$.

However 1. is clear (as if $(\varphi, \varphi') \in R_j^A$ then $\varphi' = \varphi \circ f_j$ and $F(f_i)(\varphi) = f_i \circ \varphi$, $F(f_i)(\varphi') = f_i \circ \varphi' = f_i \circ \varphi \circ f_j$ and thus $(F(f_i)(\varphi), F(f_i)(\varphi') \in R_j^{A'})$.

For 2. we define f by $f = g(1_A) : A \longrightarrow A'$. It is a routine to check that $F(f) = g$. ♡

2.3.2 Arrow Construction (Amalgamation Technique)

The construction which we are going to introduce has many variants and many analogies in virtually any type of structures: algebraical, geometrical and combinatorial, see book [61] for many examples. But the nature of all these applications is similar, in many cases the same. So we can restrict ourselves to a simple illustrative example:

A graph I with two distinquished vertices a, b is called an *indicator*.

Given an oriented graph $G = (V, E)$ and an indicator (I, a, b) we define graph $G * (I, a, b) = (W, F)$ as follows:

$$W = (E \times V(I))/\sim$$

where the equivalence $\sim$ is generated by the following pairs:

$$((x, y), a) \sim (x, y'), a)$$
$$((x, y), b) \sim (x', y), b)$$
$$((x, y), b) \sim ((y, z), a)$$

Thus the vertices of G are equivalence classes of the equivalence $\sim$. For a pair $(e, x) \in E \times V(I)$ its equivalence class will be denoted by $[e, x]$.

We put $\{[e, x], [e', x']\} \in F \iff e = e'$ and $\{x, x'\} \in E(I)$.

This construction which is called an *arrow construction* is schematically indicated on Fig.5

From a homomorphism point of view the arrow construction has many convenient properties and in many instances one can guarantee that some properties of $G*(I, a, b)$

depend on the indicator (I, a, b) only. Particularly one can guarantee that for every oriented graph G in many cases holds that

$G \longrightarrow G'$ iff $G * (I, a, b) \longrightarrow G' * (I, a, b)$

Even more so: for every homomorphism $g : G * (I, a, b) \longrightarrow G' * (I, a, b)$

there is a homomorphism $f : G \longrightarrow G'$ such that

$$g([(u, v), x] = ([f(u), f(v)], x).$$

An indicator satisfying this property is called *rigid indicator.*

Examples of rigid indicators are easy to find. For example oriented graph I_1 on Fig.6

is rigid oriented indicator and also undirected graph I' on Fig.7 is an example of rigid indicator:

Let us prove this at least for the second graph $I = I'$ (see [22],[25]): The graph I has the following properties:

(i) $\chi(I) = 4$ and $\chi(I') < 4$ for every vertex deleted subgraph I' of I;

(ii) Every vertex x of I belonges to a triangle, moreover for every two vertices x and y or i there exists a path $x = x_0, x_1, x_2, \ldots, x_\epsilon = y$ such that $\{x_{i-1}, x_i, x_{i+1}\}$ forms a triangle in I.

(iii) Identity is the only automorphism of I.(such graphs are called *asymmetric*).

Combining (i) and (iii) we get that the only homomorphism $I \longrightarrow I$ is the identity. Such graph is called a *rigid* graph. Moreover, this together with (ii) gives that for every oriented graph G the graph $G * (I, a, b)$ has the following property:

For every homomorphisms $f : I \to G * (I, a, b)$ there exists an edge $e \in E(G)$ such that $f(x) = [e, x]$ for every $x \in V(I)$.

Now this clearly implies that I is a rigid indicator.

Combining the above definition with the rigidity of the (symmetric) indicator I we just constructed an embedding F of the category $REL = REL(1)$ of all finite relations into category GRA of all finite undirected graphs:

Given an oriented graph G we put $F(G) = G * (I, a, b)$ and for a homomorphism $f : G \to G'$ we put $F(f)([u, v], x) = (f(u), f(v)], x)$.

Observe further that many combinatorial properties of graphs $G * (I, a, b)$ are determined *for any* graph G:

(1) If I has a k-coloring so that the vertices a and b gets the same color, then $G * (I, a, b)$ is k-colorable too;

(2) If I has maximal clique size k and $a, b \notin E(I)$ then $G * (I, a, b)$ has maximal clique size is k

(3) If I has girth k and the distance of vertices a and b is $\geq 2k$ then $G * (I, a, b)$ has girth k.

Thus in fact we embedded the category REL into a category of 4-chromatic graphs with clique size 3. Or we could also say that we reduced relations and their homomorphisms to homomorphisms of undirected 4-chromatic graphs with clique size 3. This lead to study of rigid graphs and rigid indicators (more examples of rigid graphs are given e.g. in [8], [25], [53],[40]).

The situation is reminiscent to problems in Theoretical Computer Science where we often use such techniques to reduce one problem to another while preserving certain particular property.

Recall for example polynomial reductions which lead to NP-complete (and, say, isomorphism complete) problems. (In a sense the monograph [61] resembles [17] in that it provides a *catalogue* of structures and reductions between them.)

Let us return to our main theme: We do not have to use one indicator only. Suppose that $(I_1,a_1,b_1),(I_2,a_2,b_2),\ldots,(I_m,a_m,b_m)$ be indicators. Let $(X,(R_i;i=1,\ldots,m))$ be an m-relational system. Let $(X,(R_i;i=1,\ldots,r))*((I_i,a_i,b_i);i=1,\ldots,m)$ (or shortly $(X,(R_i))*(I_i,a_i,b_i))$ denotes the variant of arrow construction where we replace edge $e\in R_i$ by a copy of indicator (I_i,a_i,b_i). It is easy to modify the arrow construction (we use all the notation used above in the definition of arrow construction $G*(I,a,b)$):

$(X,(R_i))*(I_i,a_i,b_i)=(W,F)$ where $W=\bigcup_{i=1}^{r}(R_i\times V(I_i))/\sim$

where the equivalence $\sim$ is generated by the pairs
$((x,y),a_i)\sim((x,y'),a_j)$ where $(x,y)\in R_i,(x,y')\in R_j$
$((x,y),b_i)\sim(x',y),b_j)$ where $(x,y)\in R_i,(x',y)\in R_j$
$(x,y),b_i)=((y,z),a_j)$ where $(x,y)\in R_i,(y,z)\in R_j$.
(We could also briefly say that $((x,y),a_i)\sim((x,y'),a_j)$ whenever these expressions belong to W).

$$F=\{\{[e,x],[e',x']\};e=e'\in R_i,\{x,x'\}\in E(I_i)\}$$

We continue to proceede analogously to the above:

We say that the set of indicators $(I_1,a_1,b_1),\ldots,(I_m,a_m,b_m)$ is *rigid* if for every m-relational system $(X,(R_i))$ and for every i every homomorphism

$$f:I_i\longrightarrow(X,(R_i))*(I_i,a_i,b_i)$$

has form $f(x)=[e,x]$ for some pair $e\in R_i$.

Examples of such indicators are easy to obatin. For example if we use the above oriented indicator I we can get set I_i by enlarging the length of the cycle. See schematic Fig.8

It is a routine to prove that I_i have these properties. Note that an oriented rigid indicator (I, a, b) need not have a cycle and that the vertices a and b ma be on the same level. This implies that the arrow construction $G * (I, a, b)$ is a balanced graph for every graph G. An example of such a graph is given on the following Fig.9

Combining the above construction together with Theorem 12 we finally obtain Theorems 2.1, 2.2, 2.3. More concretely we also proved:

Theorem 2.9 *Every finite category $\mathcal{K}$ is representable by oriented balanced graphs of given girth.*

2.4 Poset of Homomorphisms - Independent Families

We can consider the finite graphs with the relation $\leq$ induced by the existence of a homomorphism:

(4) $$G \leq H \text{ if and only if } G \to H$$

The relation $\leq$ is a quasiorder on the set of all graphs. However we can think of this set as a partially ordered set if we restrict our attention to minimal graphs which are called cores: A graph G is called a *core* [23] if every homomorphism $f : G \to G$ is an authomorphism. One can prove easily that for every graph G there exists up to an isomorphism unique subgraph G' such that G' is a core and $G \to G'$. The graph G' is called the *core of* G. Not suprisingly most graphs are cores (and it is NP-complete to decide whether a given graphs is a core or not).

We denote this poset by $\mathcal{C}$. $\mathcal{C}$ is a countable poset which is very rich. In fact we have the following:

Theorem 2.10 *$\mathcal{C}$ is universal countable poset. Explicitely, every countable poset is isomorphic to a subposet of $\mathcal{C}$.*

This is non-trivial result due to Z.Hedrlín and L. Kučera, see [61]. No simple proof of this is presently known.

Note that for finite posets this is much easier result which follows from the previous section. For finite posets one can also prove stronger results for example the following result proved in [58]

Theorem 2.11 *Every finite poset may be represented by homomorphisms between finite oriented paths.*

An extension of this result to countable posets is presently unknown and as proved in [58] this is equivalent to the on-line representability of finite posets.

Some particular cases of universality property of $\mathcal{C}$ proved to be more usefull than other and they were studied intensively. Two of such examples - independency and density - are subject of this and of the following section.

We say that a set $G_1, G_2, \cdots, G_n, \cdots$ of graphs (finite or infinite) is *independent* if for no two graphs G_i, G_j, $i \neq j$ there is a homomorphism $G_i \to G_j$. Recall that a graph G is called *rigid* if the identity is the only homomorphism $G \to G$. We say that a set $G_1, G_2, \cdots, G_n, \cdots$ of graphs (finite or infinite) is *mutually rigid* if for no two graphs G_i, G_j there is a non-identical homomorphism $G_i \to G_j$.

We used the mutually rigid families in the above proof of Theorem 2.3 .

It is easy to construct an exponentially large set of mutually rigid graphs. This may be proved as follows:

Let $G = (V, E)$ be an undirected rigid graph with m edges. Let $G_1, \cdots, G_M$ be the all orientations of the graph G (i.e. $M = 2^m$). This set $G_1, \ldots, G_M$ is mutually rigid. This is easy to see as every homomorphism $f : G_i \to G_j$ is also a homomorphism $f : G \to G$ hence necessarily f is the identity and thus G_i is rigid and $i = j$.

Consequently we have exponentially many mutually rigid oriented graphs and if we want to have undirected graphs with the same property we can consider the set $G_1 * (I, a, b), \cdots, G_M * (I, a, b))$ for an undirected rigid indicator $(I, a, b,)$.

However this is not the end of the story and we can ask what is the maximal size of a set of mutually rigid graphs on a given set X (of vertices). This clearly depends on the size of the set X only and thus denote by $mr(k)$ the maximal size of the set of mutually rigid graphs on a set with k points. We have the following two basic results:

Theorem 2.12 (Mutually Rigid Families on Finite Sets)
$mr(k) = \binom{\binom{k}{2}}{\lfloor \binom{k}{2}/2 \rfloor}(1 + o(1))$

This is a paper on finite combinatorics. Let us make a exception at this moment and let us state an important infinite result related to our main theme.

Theorem 2.13 (Mutually Rigid Families on Infinite Sets)
$mr(k) = 2^k$ for every infinite k.

The first result is due to Koubek and Rödl [40] and uses probabilistic tools. The second result will follow easily again by *arrow construction calculus*, see also [52]. However we have to use the following [64]:

Theorem 2.14 *On every set there exists a rigid relation.*

Proof.[of Theorem 2.13] Let X be a set of cardinality k and let (X, R) be a rigid relation. Using an undirected rigid indicator we get a rigid undirected graph G again on the set X. Considering of all possible 2^k orientetions of G we obtain 2^k mutually rigid relations on the set X and if we want we can apply again (the same) undirected rigid indicator to obtain 2^k mutually rigid graphs on the set X. ♡

Particularly, there are continuum many countable graphs which are mutually rigid. This usefull fact is sometimes referred to as *Ulam's problem*, see [61].

Let us return to finite sets. We note that the above techniques have some further corollaries and for example one can costruct an infinite independent set of finite graphs G_i which have the following properties:

1. each of the graphs G_i is planar;

2. each of the graphs G_i has all vertices ≤ 3.

This should be compared with results mentioned in the section 2.3 (where we stated that neither planar, and more generally graphs with bounded genus, nor bounded degree graphs fail to represent all finite categories-even finite groups and finite monoids). See also the problems stated in the following section and in Section 3.5.4.

2.5 Density

Let us continue our study of the properties of the poset $\mathcal{C}$ induced by all finite graphs and the existence of homomorphisms between them. As we have seen this is a very rich poset and in a sense universal poset (compare Theorems 2.2 and 2.10).

Here we are going to proceede in yet another direction. As we are going to disscuss order-theoretic notions we shall denote graphs by capital letters $A, B, \ldots$.

The key to this section is the following definition:

Definition 3 *A pair (A, B) of graphs is said to be a gap in $\mathcal{C}$ if $A < B$ and there is no graph C such that $A < C < B$.*

Similarly, for a subset $\mathcal{K}$ of $\mathcal{C}$, a pair (A, B) of graphs of $\mathcal{K}$ is said to be a gap in *K if $A < B$ and there is no graph $C \in \mathcal{K}$ such that $A < C < B$.*

The *Density Problem* for a class $\mathcal{K}$ is the problem of describing all gaps of the class $\mathcal{K}$. This is a challenging problem even in the simplest case of all undirected graphs. This question has been asked first by [47] in the context of the structure properties of classes of languages and grammar forms. The problem has been solved by E. Welzl [66]:

Theorem 2.15 (Density Theorem for Undirected Graphs)
The pairs (K_0, K_1) and (K_1, K_2) are the only gaps for the class of all undirected graphs. Explicitely, given undirected graphs G_1, G_2, $G_1 < G_2$, $G_1 \neq K_0$ and $G_1 \neq K_1$ there is a graph G satisfying $G_1 < G < G_2$.

In this surwey we give three proofs of the Density Theorem 2.15 which were recently found and which put this result in new context.

2.5.1 Probabilistic proof of undirected graph density

The proof is based on the following *Sparse Incomparability Lemma* first isolated in [53]:

Lemma 1
Let G, H be fixed graphs, H non-bipartite, ℓ a positive integer. Assume $G \to H$ and $H \not\to G$. Then there exists a graph G' with the following properties:
(i) $G' \to H$,
(ii) $G' \not\to G \not\to G'$, $H \not\to G'$,
(iii) G' has girth $> \ell$.

See Fig 10

This of course strengthens the classical Erdős result [11] on high chromatic graphs with given girth (say, for $H = K_k$ and $G = K_{k-1}$). Sparse Incomparability Lemma seems to be an useful tool and [53] originally applied this result to graphs without given symmetries and endomorphisms (the so called *rigid graphs*).

First, we show that Density of Undirected graphs follows easily from the Sparse Incomparability Lemma:

Proof.(First proof of 2.15)
Let $G_1 < G_2$ be given. Apply Sparse Incomparability Lemma for $\ell = |V(G_2)|$, $H = G_2$, $G = G_1$ to get a graph G' with stated properties. Put $G = G' \cup G_1$. Then G has all the desired properties: $G_1 \hom G$ obviously and $G \hom G_2$ by (i) of Sparse Incomparability Lemma. On the other hand $G_2 \not\to G$ by girth and $G \not\to G_1$ by (ii).
♡

An interested reader observed that we did not use the full strength of the condition (ii) of Sparse Incomparability Lemma. We used only the fact that H was non bipartite while G' contained no short *odd* cycles. This is much easier to guarantee and we shall return to this in the next section.

Proof. (of Sparse Incomparability Lemma)
Let graphs G, H be given, let ℓ be a given positive integer. Let t denotes the number of vertices of the graph G and, without loss of generality, let the set of vertices of H be $\{1, 2, ..., k\}$. For a (large) positive integer n consider pairwise disjoint sets $V_1, V_2, ..., V_k$, each of size n.

Let $\mathbf{G}$ be a random graph with vertex set $V = \cup_{i=1}^{k} A_i$ where the edges are chosen independently from the family $\{\{x, y\};\ x \in V_i,\ y \in V_j,\ \{i, j\} \in E(H)\}$, each with the probability $p = n^{\delta - 1}$, where $0 < \delta < 1/\ell$.

A set $A \subset V$ is said to be *large* if there are $i, j,\ 1 \le i < j \le k$, such that $|A \cap V_i| \ge n/t$ and also $|A \cap V_j| \ge n/t$. For evey large set A we consider all such pairs $\{i, j\}$ and we call them *good pairs* of A. For a large set A denote by $|\mathbf{G}/A|$ the minimum number of edges of $\mathbf{G}$ which lie in the set $\{\{x, y\};\ x \in V_i,\ y \in V_j\}$ for a good pair of A.

We first estimate probability

$$\alpha = Prob[A \text{ large implies } |\mathbf{G}/A| \ge n].$$

We have

$$1 - \alpha \le \sum_{A \text{ large}} Prob[|\mathbf{G}/A|] < n] \le 2^{kn} \cdot \binom{\binom{kn}{2}}{n} \cdot (1-p)^{\frac{n^2}{r^2}}.$$

Now bounding very roughly

$$\binom{\binom{kn}{2}}{n} \le \binom{k^2 n^2}{n} \le k^{2n} n^{2n} < e^{cn \log_2 n}$$

and

$$(1-p)^{\frac{n^2}{t^2}} \leq e^{-p\frac{n^2}{t^2}}$$

we obtain

$$1-\alpha < e^{cn\log_2 n - c'n^{1+\delta}}$$

for some positive constants c and c' which are independent on n.

Thus we get $Prob[A$ large implies $|\mathbf{G}/A| \geq n] = 1 - o(1)$.

On the other hand if we denote by $c(\mathbf{G})$ the number of edges contained in all cycles of length $3, 4, \ldots, \ell$ in $\mathbf{G}$ then by the linearity of expected value we have

$$E(c(\mathbf{G})) \leq 3!\frac{kn}{3}p^3 + 4!\frac{kn}{4}p^4 + \ldots + \ell!\frac{kn}{\ell}p^\ell = \ell \cdot \frac{k^l n^\ell n^{\delta\ell}}{n^\ell} < n^{\delta\ell} = o(n).$$

Thus there exists a graph G'' (an instance of the random graph $\mathbf{G}$) such that
(i) If i, j is a good pair of a large set A then G'' has at least n edges in the set $\{\{x, y\};\ x \in V_i,\ y \in V_j\}$,
(ii) There exists $n-1$ edges $e_1, e_2, \ldots, e_{n-1}$ such that the graph G' which we obtain from G'' by deleting edges $e_1, e_2, ..., e_{n-1}$ has girth $> \ell$.

We prove that the graph G' satisfies the above conditions of Sparse Incomparability Lemma. Properties (i) and (iii) are evident from the construction of G'. To prove (ii) let us suppose that f is a homomorphism $G' \operatorname{hom} G$. Define a mapping $g : V(H) \operatorname{hom} V(G)$ by $g(i) = y$ if $|f^{-1}(y) \cap V_i| \geq n/t$ (we could call g a *majority mapping*). Clearly for every i one can choose $g(i)$ (if there are more possibilities we choose one arbitrarily). It follows from the properties (i) and (ii) of graph G' that the majority mapping g is a homomorphism $H \operatorname{hom} G$ which is a final contradiction.

This is the only non-arrow which one has to prove for (ii) (the remaining non-arrows follow from the girth of graph G'). ♡

2.5.2 Constructive proof of undirected graph density via products

This proof is due to M. Perles and J. Nešetřil (see e.g. [49], the proof is implicite in [53]) and it is particularly simple. It uses product and the existence of high chromatic graphs without short odd cycles.

Proof. (Second proof of Theorem 2.15)
Let G_1 and G_2 be given undirected graphs, let $f : G_1 \to G_2$ be a homomorphism, and suppose there is no homomorphism $G_2 \to G_1$. As this pair is not equivalent to the gap (K_1, K_2), at least one component of the graph G_2 has chromatic number greater than 2. Also, at least one component of G_2 fails to be homomorphic to G_1, and this component may be assumed to be non-bipartite; let it contain an odd cycle of length k. Now choose a graph H with the following properties: H contains no odd cycle of

length k or less, and the chromatic number of H is greater than $n_1^{n_2}$, where n_1 and n_2 denote the number of vertices of the graphs G_1 and G_2 respectively. Such a graph exists by the celebrated theorem of Erdős [11] but the existence follows much more easily and one can give also an easy construction of such graphs (shift graphs).

Now let $G = G_1 \cup (H \times G_2)$. Here $\times$ denotes the direct product of two graphs and $\cup$ means the disjoint union. We shall prove that G has the desired properties. Obviously $G_1 \to G$ and $G \to G_2$ follows as the second projection of $H \times G_2$ is a homomorphism into G_2. On the other hand there is no homomorphism from G_2 into G, as homomorphisms preserve odd cycles and they cannot increase the length of the shortest of them. Thus it suffices to prove that there is no homomorphism $G \to G_1$. Let us suppose for the contradiction that there is a homomorphism $f : H \times G_2 \to G_1$. Thus for any vertex x of H we have an induced mapping $f_x : V(G_2) \to V(G_1)$ defined by $f_x(y) = f(x, y)$. (This mapping need not be a homomorphism.) As there are at most $n_1^{n_2}$ such mappings there are vertices x and x' forming an edge of H such that the mappings f_x and f'_x are identically equal, say to g. However in this case g is a homomorphism of G_2 into G_1, contrary to our assumption. ♡

Note that this construction of graph G given in the proof can be used to prove that Sparse Incomparability Lemma holds for large *odd girth* (and in this setting this proof is implicite in [53]).

This is a good opportunity to review yet another construction:

Given two graphs G and H one can define G *power of* H, denoted by H^G, as the following graph: $V(H^G) = \{f : V(G) \to V(H)\}$ and a pair (f, g) forms an edge if $(f(x), g(y)) \in E(H)$ for every edge $(x, y) \in E(G)$. (We define the G power of H by the same formula for both undirected or directed graphs.)

This construction was isolated in the graph theoretic concept in [42], however this is also a standard category theory construction (where it is called "map object" or exponentiation, see e.g. [41]) and this led to the notion of *cartesian closed category.* The power graph construction plays an important role in the study of *Hedetniemi conjecture*, [68].

The following is the crucial property which we use (and which in fact defines the power construction):

$$\text{for every graph } K,\ K \to H^G \text{ holds if and only if } K \times G \to H.$$

This is easy to see: given $f : K \to H^G$, define $g : K \times G \to H$ by $g(x, y) = f(x)(y)$. Conversely, given g we may define f by the same formula. One can easily check that f is a homomorphism $K \to H^G$ if and only if g is a homomorphism $K \times G \to H$.

Thus in the above proof we have $H \times G_2 \to G$ if and only if $H \to {G_1}^{G_2}$. Thus in the above proof we may assume that the chromatic number of H is greater than the chromatic number of ${G_1}^{G_2}$. As $G_2 \not\to G_1$ there are no loops in ${G_1}^{G_2}$ and it is also clear that the chromatic number of ${G_1}^{G_2}$ is at most the number of vertices of ${G_1}^{G_2}$. (However we do not try to optimize at this point.)

5mm

Neither of these proofs solves the density problem for oriented graphs which remained the main open problem for several years. Until recently the best result in this direction was [58] where all the gaps were characterized for the class of all oriented paths. This result may look modest on the first glance but even the following problem is presently open:

Problem 2 (Tree Problem)
Describe all the gaps of finite oriented trees.

This problem is particularly interesting in view of the gap characterization which we state bellow as Theorem 3.12.

But even for classes of undirected graph the density presents challenging problems. On such problem (due to Welzl on the circulants, [66]) was recently solved by C. Tardif [62].

Let us list two more problems which probably call for a new method.

Problem 3
Describe all gaps for the class of all (undirected) planar graphs.

Problem 4 *Describe all gaps for the class of all undirected graphs with maximal degree bounded by a fixed number k (i.e. for k-bounded graphs).*

In both cases the only gap presently known is the trivial (K_1, K_2) gap.

3 Paradoxes of Complexity

We consider here the following decision problem called *H-coloring problem*:

Problem 5

Instance: A graph G
Question: Does there exists a homomophism $G \longrightarrow H$.

This problem covers many concrete problems which were and are studied :

(i) For K_k we get a k-coloring problem;

(ii) For graphs G_k^d we get *circular chromatic numbers*' see e.g. [69];

(iii) For Kneser graphs $K\left(\begin{array}{c} n \\ k \end{array}\right)$ we get to called *multicoloring*.

Further examples include so called T-colorings, see e.g.[69],[60],[7].

Equivalently, the H-coloring problem may be considered as a decision problem related to the following class of graphs:

$$\longrightarrow H = \{G; G \longrightarrow H\}$$

Such classes (sometimes denoted by $\mathcal{C}_H$) are called *color classes* and their structure is one of the leitmotifs of this paper. For example in the previous section we proved that with the unique exception the partial ordering defined by the inclusion of color classes is dense (for undirected graphs). We also know that we can restrict ourselves to those color classes $\longrightarrow H$ where H is a core.

3.1 Hard Cases

Here we deal with complexity issues. The situation is well understood for complete graphs: For any fixed $k \geq 3$ the K_k-coloring problem (which is equivalent to the deciding $\chi(G) \leq k$) is NP-complete. On the other hand K_1 - and K_2-coloring problems are easy. Thus, in the undirected case, we will always assume that the graph H is not bipartite.

Some other problems are easy to solve. For example, if $H = C_5$ then we can consider the arrow construction which we introduced in the previous section :

Let the indication (I, a, b) be path of length 3 with ith end vertices called a and b. It is then easy to prove that for any undirected graph G the following two statements are equivalent:

(i) $G \longrightarrow K_5$

(ii) $G * (I, a, b) \longrightarrow C_5$

(In fact in this case $G * (I, a, b)$ takes a very simple form : we subdivide every edge by two points.)

This example is not as isolated (the same trick may be used e.g. for any odd cycle). Using similar indicators (and subindicators, and edge - subindicators) the following has been proved by Hell and Nešetřil in 1987 [24]:

Theorem 3.1. *For a graph H the following two statements are equivalent :*

1) *H is non-bipartite ;*

2) *H-coloring problem is NP-complete.*

This theorem (and its proof) have some particular features which we are now going to explain:

(a) The result claimed by the theorem is expected. In fact the result has been conjectured in [47], and elsewhere, but it took nearly 10 years before the conjecture had been verified.

(b) How the statement is expected the proof is unexpected.

What one would expect in this situation?

Well we should prove first that C_{2k+1}-coloring is NP-complete (which is easy and in fact we sketched this above) and then we would "observe" that the problem is hereditary :
If H-coloring problem is NPC and $H' \supseteq H$ then also H'-coloring problem is NPC.

This statement may sound plausible but there is not known a direct proof of this statement. It is certainly a true statement (by virtue of Theorem 3.1) but the only known proof is again via proof of Theorem 3.1. In fact there may be here more than meets an eye : for oriented graphs the analogy of this statement does not hold (Gutjahr was first to give a counter example) .

(c) Having said that, we should point out that (as it stands) Theorem 3.1 fails to the true. We have to assume that all graphs are *undirected*. In this context this is not a technical assumption but rather deep and profound restriction:

One can construct easily an orientation $\vec{H}$ of bipartite graph H such that $\vec{H}$ -coloring problem is NP-complete. Even more so, one can construct a balanced oriented graph H for which H-coloring problem is NP-complete; an oriented graph is called balanced if every cycle has the same number of forwarding and backwarding arcs.

This can be done using again the indicator technique: Let (I, a, b) be the indicator which is a path oriented in such a way that

1) I has an automorphism which exchanges a and b;

2) Every homomorphism $G{*}(I,a,b) \longrightarrow H{*}(I,a,b)$ induces a homomorphism $G \longrightarrow H$ (G, H are undirected graphs).

Once these conditions are spelled out it is easy to satisfy them. In fact an example such an indicator is depicted on Fig.9 above.

But then as stated $G \longrightarrow H$ iff

$$G * (I,a,b) \longrightarrow H * (I,a,b)$$

and thus e.g. $K_3 * (I,a,b)$ is an NP-complete problem.

Now one can go further and in this way one can ommit in H all cycles (not a necessarily oriented) of short lengths. But it is perhaps bit surprising that one can ommit *all* cycles. Namely, one has the following proved in [18] and [28]:

Theorem 3.2. *There are oriented trees T (i.e. T is an orientation of an undirected tree) such that T-coloring problem is NP-complete.*

In [18] such a tree with 258 vertices has been constructed while in [28] a tree T_0 with 45 vertices with the same property has been found. This examples is schematically depicted on Fig.11

We see that T_0 has a very simple structure (and it is called *triad* in [28], [27]). We should remark that 45 is not such a large number here. T_0 has to be a core and this already implies that T_0 has at least 15 vertices and this is just a very first estimate.

(d) Thus for oriented graphs we face a much more complicated situation. Even for special classes, very special classes indeed. For example, the following are presently open problems :

Problem 6 *Characterise oriented trees T for which T-coloring problem is NP-complete.*

(This is open even for triads. Triads are in a way minimal examples as P-coloring problem is polynomial for every oriented path P, see [29] and the following section.)

It seems that the problem lies in "sparse" graphs. On the other side of the spectrum, it has been shown the following [3]

Theorem 3.3. *For a tournament T (i.e. T is an orientation of a complete graph) The following two statements are equivalent :*

(1) *T-coloring problem is NP-complete;*

(2) *T contains two directed cycles.*

(Bang- Jensen, Hell and MacGilliwray in fact prove the same result for "semi-complete graphs".)

(e) But in general the H-coloring problem seems to be a very hard problem. Presently there is no conjecture which should capture NP-completeness instances of H-coloring problem.

But may be there is even no such conjecture.

There is some evidence for this. For examples, as was shown in [15] the H-coloring problem for relational systems (i.e. we allow more relations on the same set) is reducible to the H-coloring problem for oriented graphs. (Motivation to [15] research commes from data base - theory.)

The following problem is posed [15]:

Problem 7 (Dichotomy) *Is it true that the H-coloring problem for any graph H is either polynomially solvable or NP-complete ?*

An important line of the research has been started with [10] where a characterization is given of those H-coloring problems for which the counting of the number $h(G,H)$ of homomorphisms $G \to H$ is a $\#P$- hard problem.

Let us finish with that and let us turn to polynomially solvable instances.

3.2 Polynomial Cases and Homomorphism Dualities

Coloring problems have to be solved, we usually do not walk away. But how to approach them?

A standard approach in a combinatrial setting is to look for *obstacles*, for configurations which are *obstructing* our goal, in our case the desired homomorphism $G \longrightarrow H$. These obstructions (i.e. "bad green dwarfs") can be special subgraphs as we have it in the bipartite case : $G \longrightarrow K_2$ iff G does not contain an odd cycle.

As we are interested in the existence of homomorphism $G \longrightarrow H$ these forbidden subgraphs (obstructions) are closed on homomorphism too:

$$\text{If } F \not\longrightarrow H \text{ and } F \longrightarrow F' \text{ then } F' \not\longrightarrow H.$$

Let us approach this more formally:

We introduced already the class $\longrightarrow H = \{G; G \longrightarrow H\}$. The complementary class $\{G; G \not\longrightarrow H\}$ will be denoted by $\not\longrightarrow H$. As we just observed $\not\longrightarrow H$ is closed on homomorphism:

$$F \in (\not\longrightarrow H), F \longrightarrow F' \text{ implies } F' \in (\not\longrightarrow H)$$

.

Thus there exists a set $\mathcal{F}$ of graphs such that $\not\longrightarrow H = \{G; F \longrightarrow G$ for some $F \in \mathcal{F}\}$. The latter class will be denoted by $\mathcal{F} \longrightarrow$. Explicitely, $\mathcal{F} \longrightarrow$ consists from all graph G for which there exists a graph $F \in \mathcal{F}$ which is homomorphic to G.

Similarily, we denote by $\mathcal{F} \not\longrightarrow$ the class of all graph G for which no member $F \in \mathcal{F}$ is homomorphic to G.

Thus we have equality

(5) $$\mathcal{F} \not\longrightarrow = \longrightarrow H.$$

We observe that for any H there exists a family $\mathcal{F}$ such that (5) holds. Simply take $\mathcal{F} =\not\longrightarrow H$. But our goal is more demanding : we would like to find a simple family $\mathcal{F}$, if possible such a family where the membership of the class $\mathcal{F} \longrightarrow$ would be easy to prove.

Theorems which have structure as in (5) are called *Homomorphism Duality Theorems*.

We shall give some examples to make the duality point of view more explicit:

A typical example is the case of oriented paths. According to [29], a digraph G is homomorphic to an oriented path P if and only if each oriented path P' homomorphic to G is also homomorphic to P. Thus in this case the obstructions are oriented paths P' homomorphic to G but not to P. To make this obstruction point of view more explict we restate the characterization (for the case when P is an oriented path) as follows: *A diagraph G is not homomorphic to P if and only if there exists an oriented path P' which is homomorphic to G but not to P.*

In the other words we have *Path Duality* (proved in [29])

Theorem 3.4. *Let P be an oriented path. Then*

$$\mathcal{P} \not\longrightarrow = \longrightarrow P$$

where $\mathcal{P}$ is the family of all paths P' which are not homomorphic to P.

Another class of digraphs with a similar characterization theorem is the class of unbalanced cycles. An unbalanced cycle is an oriented cycle in which the number of forward edges is different from the number of backward edges (with respect to some fixed traversal of the cycle). According to [30], *a digraph G is not homomorphic to an unbalanced cycle C if and only if there is an oriented cycle C' homomorphic to G but not homomorphic to C.*

In the other words we have *Cycle Duality* (proved in [30]):

Theorem 3.5. *Let C be an unbalanced cycle. Then*

$$\mathcal{C}\dagger \not\longrightarrow = \longrightarrow C$$

where $\mathcal{C}\dagger$ is the family of all cycles C' which are not homomorphic to C.

We know that H-coloring problems can be hard even when H is an oriented tree T. However, there are also many oriented trees T for which there is structure to the T-coloring problem, which can be exploited to find a polynomial algorithm. The class of digraphs homomorphic to such a "nice" oriented tree T can be often characterized by the absence of certain subtrees.

Specifically, we say that H-coloring problem has *tree duality* if the following property holds for all digraphs G: *A digraph G is not homomorphic to H if and only if there exists an oriented tree homomorphic to G but not to H.* Tree duality seems to be a surprisingly useful property. In particular, one can prove that if H has tree duality then the H-coloring problem is polynomial [27].

This fits to our scheme (4): H has a tree duality iff

$$\mathcal{T} \not\longrightarrow = \longrightarrow H$$

where $\mathcal{T}$ is the set of all trees F which are not homomorphic to H.

The class of digraphs H with polynomial H-coloring problems can be further enlarged by generalizing tree duality to Tree Width-k Duality and to Bounded Tree Width Duality.

First, let us give some definitions:

An undirected graph is a *k-tree* if its maximal clique is of size $k+1$ and it does not contain an induced cycle of length > 3. It follows from basic graph theory that k-trees have indeed a tree structure;a k-tree can be obtained from a $(k+1)$-clique. by repeatedly adding a vertex joined to existing vertices which form a k-clique. (Thus a tree is a 1-tree.) An undirected graph is said to *have treewidth k*, or to be a *partial k-tree*, if it is a subgraph of a k-tree. This is denoted by $tw(G) \leq k$. Partial k-trees have small separating sets and as a consequence they admit efficient algorithms for many hard computational problems see e.g. [36]. We say that an oriented graph has *treewidth k* (or is an oriented partial k-tree) if its underlying undirected graph has treewidth k.

Definition 4 *We say a digraph H has treewidth-k duality if the following property holds for all digraphs G: A digraph G is not homomorphic to H is and only if there exists an oriented partial k-tree homomorphic to G but not to H.*

An H-coloring problem is said to have bounded treewidth duality if there exists a positive integer $k = k(H)$ such that the following holds:

G is homomorphic to H if and only if every graph F homomorphic to G with tree width $\leq k$ is also homomorphic to H.

This fits to our scheme (4): H has k-treewidth duality iff

$$\mathcal{T}_k \not\longrightarrow = \longrightarrow H$$

where $\mathcal{T}_k$ is the set of all partial k-trees F which are not homomorphic to H. Explicitely: For every graph G the non-existence of a homomorphism $G \longrightarrow H$ is equivalent to the existence of an F, $tw(F) \leq k$ such that $F \to G$.

The following has been proved independently and in different context in [27] and [15]:

Theorem 3.6. *Every H-coloring problem with bounded treewidth duality is polynomial time decidable.*

Presently Theorem 3.6 is the strongest tool for proving the polynomial time decidability of H-coloring problems. In fact, presently all known polynomial time decidable H-coloring problems are covered by Theorem 3.6.

On the other hand, if we assume $P \neq NP$, then all NP-complete H-coloring problems do not possess bounded treewidth duality.

This line was pursered in [59] where the following has been allowed directly (i.e. without the assumption $P \neq NP$) :

Theorem 3.7. *For an undirected graph H, the H-coloring problem has no bounded treewidth duality if and only if H contains an odd cycle.*

Also for some directed graphs H one can obtain similar results. For example one can prove the following (see [59]) :

Theorem 3.8. *There exists an oriented cycle C such that C-coloring problem has no bounded tree width duality.*

However the following is still open:

Problem 8 *(Without $P \neq NP$) prove that there exists an oriented tree T such that T-coloring problem has no bounded tree width duality.*

Let us finish this part with another problem.

First let us recall another results proved in [59]:

Theorem 3.9. *Given two positive integers k and m. If G is a graph of girth $n > 2^{k+2}(4km)^{4km-1} + 2(k+1)$ then any partial k-tree homomorphic to G is also homomorphic to the odd cycle C_{2m+1}.*

In the language of circular chromatic number χ_c(or star chromatic number) this implies $\chi_c(G) \leq 2 + \varepsilon$ for any large girth graph with bounded tree width.

A related result in this direction is that any large girth planar graph is homomorphic to a given odd cycle. Quite surprisingly the similar results do not hold for bounded degree and even cubic graphs. We have the following (proved recently in [39]).

Theorem 3.10. *For any $g \geq 3$ and any $l \geq 10$ there exists a cubic graph G with the following properties:*

1) *G has girth $\geq g$*

2) *$G \not\longrightarrow C_{2e+1}$*

The non-constructive proof of [39] leaves the following open:

Problem 9 *Is it true that any large girth cubic graph G is homomorphic to C_5?*

(This problem was first discussed in [16].)

It is proved in [35] that the answer to this problem is negative for 4-regular graphs.

3.3 Gaps and Dualities

Let us return to our main theme:

All the above results fall into the framework of Homomorphism Dualities. The following is perhaps the simplest instance of such homomorphism duality:

A *Singleton Homomorphism Duality* is a pair of graphs (F, H) satisfying

$$F \not\longrightarrow = \longrightarrow H$$

.

Similarily, *Finitary Homomorphism Duality* is a pair $(\mathcal{F}, H)$, where $\mathcal{F}$ a finite set of graphs, satisfying

$$\mathcal{F} \not\longrightarrow = \longrightarrow H$$

.

In this case we also say that H has *Finitary Homomorphism Duality.*

At the first glance this scheme seems to be too restrictive and indeed it is, at least for undirected graphs. The following result was proved essentially in [51].

Theorem 3.11. *For an undirected graph H the following two statements are equivalent:*

1) *H has finitary homomorphism duality*

2) *Either $H = \phi$ or $H = K_1$.*

Thus for undirected graphs only trivial finitary homomorphism dualites exist:

$$K_1 \not\longrightarrow = \longrightarrow \phi \text{ and } K_2 \not\longrightarrow = \longrightarrow K_1$$

.

We include the proof as it uses one of the basic trics in this area:

Proof.

Clearly it suffices to prove that there are no other dualites. Assume the counary, so let $\mathcal{F} \not\longrightarrow = \longrightarrow H$ be a finitary duality for a non-bipartite H. Put $\mathcal{F} = \{F_1, \ldots, F_t\}$. Let l be larger than the shortest length of an odd cycle in any of graphs $F_1, \ldots, F_t$ (of course $F_i \not\longrightarrow H$). Now let $G = G_{k,l}$ be the Erdös graph with the following properties: $\chi(G) > k = |V(H)|$ and girth of $G > l$. Then both $G \not\longrightarrow H$ and $F_i \not\longrightarrow H$, which is a contradiction. ♡

However for other structures the singleton homomorphism dualities present more complex patterns and capture interesting theorems. For example for oriented matroids and (convenient version of) strong maps one can prove that singleton morphism duality give Farkas Lemma and one can even prove that no other such dualities (in this framework) exist, see [32].

Similarily, for ports with strong port mappings the only singleton duality is equivalent to Menger theorem, see [33].

Even for directed graph this situation is far more complicated and a full characterisation was achived (in a special case) by Komárek [38] and (in the full generality) by Nešetřil and Tardif [55, 57].

Theorem 3.12. (Characterisation of Singleton Dualities)

Up to a homomorphic equivalence the only singleton dualities for oriented graphs have the following form:

$$T \not\longrightarrow = \longrightarrow H_T$$

where is an oriented tree and the graph H_T is uniquelly determined by T.

Theorem 3.13. (Characterisation of Finitary Dualities)

Up to a homomorphic equivalence the only finitary hommomorphism dualites for oriented graphs have form

$$\mathcal{F} \not\longrightarrow = \longrightarrow H$$

where $\mathcal{F}$ is a finite set of trees $\{F_1, \ldots, F_t\}$ and $H = \prod_{i=1}^{t} H_{F_i}$ where $F_i \not\longrightarrow = \longrightarrow H_{F_i}$.

3.4 Duality and Density (and Gaps)

Nešetřil and Tardif stablished in [55] and [57] the following (perhaps surprising) connection of duality pairs. This provided the key to the characterization not only of the dualities but also to the characterization of gaps for classes of directed (and undirected) graphs.

We say that a gap (G, H) is *connected* if H is a connected graph. Observe that if (F, H) is a duality pair then F is necessarily connected (for if F is a core and F_1 and F_2 are distinct components of F then $F \not\to F_i$, and thus $F_i \to H$ for $i = 1, 2$. Thus $F \to H$, which a contradiction).

Theorem 3.14 *There is a one-to-one correspondence between duality pairs and gaps for the class of directed (and also undirected) graphs. Explicitely, given a duality pair (F, H) then $(F \times H, F)$ is a gap. Conversely, given a connected gap (G, H) then (H, G^H) is a duality pair.*

Proof. First, suppose that $G \to H \not\to G$ is a gap. We prove that for any graph K holds $H \not\to K$ iff $K \to G^H$. Thus let a graph K be such that $H \not\to K$ and suppose for the contrary that $K \not\to G^H$. Then obviously $G \to G \cup (G \times H) \to H$. If $H \to G \cup (K \times H)$ then (by the connectivity and the assumption $H \not\to G$) $H \to K \times H$ and thus $H \to K$,contrary to our assumpiton. If $G \cup (K \times H) \to G$ then $K \times H \to G$ and $K \to G^H$ again contrary to our assumption. Thus the class $H \not\to$ is a subclass of the class $\to G^H$. In order to prove the reverse inclusion let K be a graph satisfying $K \to G^H$ and $H \to K$. This implies $H \to H \times G^H \to G$ (as $H \times G^H \to G$ is equivalent to $G^H \to G^H$ and thus it always holds). Thus (H, H^G) is a duality pair.

Conversely, let (F, H) be a duality. We may clearly assume that F is a core (i.e. every homomorphism $F \to F$ is an automorphism) and further F is connected. Thus

$F \times H \to F$ and $F \not\to F \times H$ (as $F \to F \times H$ would imply $F \to H$). We claim that there is no graph K satisfying $F \times H \to K \to F$ and $F \not\to K \not\to F \times H$: If $K \to F \not\to F \times H$ then the duality implies $K \to H$ and thus $K \to F \times H$ which contradicts our assumptions. This completes the proof of theorem. ♡

This Theorem 3.14 leads to yet another proof of Density Theorem 2.15:
Proof. By 3.11 (K_2, K_1) and (K_1, K_0) are the only duality pairs. Thus $(K_2 \times K_1)$ (which is equivalent to (K_1, K_2)) and $(K_0 \times K_1, K_1)$ (which is equivalent to (K_0, K_1) are the only gaps ♡

Thus, modulo the above "arrow calculus" involved in the above proof of 3.14, the density theorem for undirected graphs has been known even before it has been formulated.

Finally, let us remark that the above also shows that Theorem 3.13 gives all non-gap pairs for directed graphs: Let (T, H_T) be all singleton duality pairs for oriented graphs (characterized by Theorem 3.12). Then $(T \times H_T, T)$ are exactly all gaps.

In neither direction is Theorem 3.13 an easy result:

The construction of the graph H_T is quite complicated and it has been approached from different sides in [38] and in [57].

Here we show the opposite direction (that there are no other gaps). This in fact solves the density problem for oriented graphs by a remarkably easy construction which we have already introduced in a different context - it is our indicator construction again.

Theorem 3.15
Let G, H be directed graphs which are cores. Let H be connected and assume that H fails to be an orientation of a tree. Further assume that $G \to H \not\to G$ holds. Then there exists a directed graph K with $G \to K \to H$ and $H \not\to K \not\to G$.

We shall make use of the following obvious (but key) property of the arrow construction:

Lemma 2
Let G and H be directed graphs with $\chi(G) > |V(H)|$ and let every homomorphism $f : I \to H$ satisfies $f(a) \neq f(b)$. Then $G \star (I, a, b) \not\to H$.

Proof. (Third proof of Theorem 2.15)
Let G, H be undirected graphs, H non bipartite, with $G \to H \not\to G$. Clearly we may assume that G and H are cores. Let $e = \{a, a'\} \in E(H)$ belong to a circuit in H.

Put $I = H - e + \{a', b\}$ where $b \notin V(H)$. (Thus I arises from H by deleting the edge e, adding a new vertex $b \notin V(H)$ together with the edge $\{a', b\}$.)

It is clear that $I \to H$ (identifying vertices a and b) but any homomorphism $f : I \to G$ satisfies $f(a) \neq f(b)$ (for otherwise we get a contradiction with $H \not\to G$). Now let F be any graph satisfying $\chi(F) > |V(G)|$ and let F' be any orientation of F. Consider the arrow construction $F' \star (I, a, b)$ and define the graph K by $K = (F' \star (I, a, b)) \cup G$.

We prove that K has properties claimed by Undirected Graph Density: Clearly $G \to K$. We also have $K \to H$ as the mapping f defined by $f([e, x]) = x$ for $x \in V(H)$ and $e \in E(F')$ and $f([e, b]) = a$ is a homomorphism $K \to H$ (we preserve the above notation concerning the arrow construction $F' \star (I, a, b)$). Further, by the above Lemma 2, $K \not\to G$ (as $\chi(F) > |V(G)|$). Thus it remains to be shown that $H \not\to K$. Suppose the contrary and let $g : H \to F \star (I, a, b)$ be a homomorphism. Then $f \circ g : H \to H$ where f is the above defined homomorphism $F \star (I, a, b) \to H$. As H is a core $f \circ g$ is a homomorphism. Put $h = (f \circ g)^{-1}$. Then $f \circ g \circ h(x) = x$ for every $x \in V(H)$. Put $g \circ h(a) = [(e, a)]$ with $e = (u, v)$. Then the image $g \circ h(H)$ of H is a connected subgraph of $F \star (I, a, b)$ which is (by the injectivity of the mapping $f \circ g \circ h$) contained in the set of all $[(e', x)]$ where e' is incident with u and $x \in V(I)$ (this set is the "star" induced by those edges of F' which are incident with the vertex u). But then the edge $\{[g \circ h(a)], [g \circ h(a')]\}$ is a cut edge in the graph $g \circ h(G)$ which is the final contradiction as a, a' was contained in a cycle of H. $\heartsuit$

Proof. [of Theorem 3.15]
Let G, H satisfy the assumption of the theorem. Let H be a core and let $(a, a') \in E(H)$ belong to a cycle in H. Put $I = H - (a, a') + (b, a')$ where $b \notin V(H)$ (i.e. we first delete arc (a, a') and then add a new vertex b together with the arc (b, a')). Let F be an oriented graph with $\chi(F) > |V(G)|$ and consider the arrow construction $F \star (I, a, b)$. Put $K = G \cup (F \star (I, a, b))$. Then we have:

- $G \to K$ (by the inclusion map);
- $K \to H$ (by the same mapping as in the above proof);
- $K \not\to G$ (by the chromatic number assumption);
- $H \not\to K$ (as above in the proof for undirected graphs).

Thus the graph K has the desired properties. $\heartsuit$

3.5 Final View

Our approach to H-coloring problem may be put in various context. We list in these closing remarks three such approaches.

3.5.1 Good Characterizations

Finitary good characterizations are examples of *Good Characterizations* in the sense of Edmonds [12]: Given a finitary duality

$$\mathcal{F} \not\to = \to H \tag{6}$$

we can proof easily that a given graph G is not H-colorable. We simply check which graph F of the finite set F permits a homomorphism $F \to G$. This obviously takes polynomially many steps (and in fact one can do so in $O(n^{k\omega/3})$ steps where ω is the fast matrix multiplication constant and $k = max\{|V(F)|; F \in \mathcal{F}\}$, see [50]). On the other side the existence of an H-coloring is easy to verify.

In the previous sections (Theorem 3.13) we characterized all finitary dualities for coloring problems for graphs. Note that the main result may be extended to relational systems and even to the relational structures of a given type (i.e. to finite models of a given type).

Despite of this generality we see that Theorem 3.13 is very special (as one can "forbid" relational trees only). This is in a sharp contrast with the abundance of finitary dualities if other "morphisms" are allowed. For example (as follows from Robertson - Seymour - Thomas project) every minor closed property has "finitary duality" (with morphisms beeing minors). However note that most of these results are related to undirected graphs only. Other examples for matroids are given in [32], [33].

3.5.2 hom - Universal Graphs

One of the fundamental results of P. Erdös [11] can be formulated as follows:

Theorem 3.16
For a finite graph F the following two statements are equivalent:

1) *There exists $k = k(F)$ such that any graph G with $\chi(G) \geq k$ contains F as a subgraph.*

2) *F is a forest.*

This can be expresed also as a weaker form of homomorphism duality:
For a finite graph F the following two statements are equivalent:

1') There exists a graph H such that $\{G; F \not\subset G\} \subset \{G; G \to H\}$.

2') F is a forest.

If the condition $(1')$ is valid then we say that the graph H is *hom-universal* for the class of all F-free graphs.

In this setting Theorem 3.12 presents an extension of the Erdös result to forbidden homomorphism: In the case of a tree F the class $\{G; F \not\to G\}$ has not only an universal graph but this class can be *defined* by homomorphisms *into* a fixed graph. More precisely, there exists a graph H_F such that

$$F \not\to = \{G; F \not\to G\} = \{G; G \to H\} = \to H.$$

(Let us stress at this moment that this holds in the full generality for finite structures. It is perhaps a bit surprising that by forbidding homomorphisms, i.e. by forbidding a graph F together with all its homomorphic images we get so much more structure.)

3.5.3 Bounded Degree Graphs

Universal graphs obviously exist for bounded degree graphs: if $\Delta(G) \leq k$ then $G \to K_{k+1}$. It has been proved by R. Häggkvist and P.Hell in [19] that for *any* graph F there exists a graph $U_{F,k}$, $F \not\to U_{F,k}$ such that $G \to U_{F,k}$ for any graph G with $\Delta(G) \leq k$ and $F \not\to G$.

This has been extended recently [9] as follows:

Theorem 3.17 *For every choice of graphs F, H, $F \not\to H$ there exists a graph $U_{F,k,H}$ such that*

1) *$F \not\to U_{F,k,H}$ and $U_{F,k,H} \to H$.*

2) *If G is a graph with $\Delta(G) \leq k$, $F \not\to G$, and $G \to H$ then $G \to U_{F,k,H}$.*

Particularly, there exists a 3-chromatic triangle free graph U such that $G \to U$ for every traingle free, cubic, 3-chromatic graph G.

The universal graphs for cubic graphs are related to several interesting problems. The following one attracted recently some attention:

Problem 10 Pentagon Problem *Is it true that there exists a constant k such that for any cubic graph with girth $\geq k$ there is a homomorphism $G \to C_{2k+1}$?*

In this context one should note that this problem has a negative solution for homomorphisms into C_{11} (instead of C_5) as showed in [39] and also for graphs with maximal degree 4 as showed in [35].

The Pentagon Problem is motivated both by some complexity considerations [16] and by attempts to solve the Density problem for cubic graphs. Particularly, even the following seems to be presently open:

Given a cubic graph G, $C_5 < G$, prove that there exists a cubic graph H such that $C_5 < H < G$.

From the negative solution of the Pentagon Problem follows the existence of H easily (along the lines of the Second Proof of Density of Undirected Graphs).

Let us finish this survey by the following recent problem raised in [9]:

Problem 11 *Does there exists a triangle free graph U such that any triangle free planar graph G is homomorphic to U: $G \to U$.*

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