Density via duality

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Abstract

We present an unexpected correspondence between homomorphism duality theorems and gaps in the poset of graphs and their homomorphisms. This gives a new proof of the density theorem for undirected graphs and solves the density problem for directed graphs.

1 Introduction

Given two graphs G_1 and G_2, where $G_i = (V_i, E_i)$, a *homomorphism* of G_1 into G_2 is a mapping $f : V_1 \mapsto V_2$ which preserves all the edges: $[f(x), f(y)] \in E_2$ whenever $[x, y] \in E_1$. We write $G_1 \to G_2$ if there exists a homomorphism from G_1 to G_2, and $G_1 \not\to G_2$ otherwise. The class $\mathcal{G}$ of finite graphs endowed with the relation $\to$ is the "skeleton" of the category of finite graphs. This is essentially an ordered set, modulo the equivalence relation $\sim$, where $G_1 \sim G_2$ means that G_1 is *homomorphically equivalent* to G_2, that is, $G_1 \to G_2$ and $G_2 \to G_1$.

This ordering of graphs from the point of view of homomorphisms has a fascinating structure. For almost every fixed graph H, the question "does the graph G admits a homomorphism into H" is NP-complete (see [5]), thus the relation $\to$ has an intricate local structure. However, it turns out that $\to$ induces a distributive lattice order on the classes of homomorphically equivalent graphs, so that the global structure is reasonably well behaved. One of the motivations of this paper is the "density theorem" of Welzl [12] which gives a flavour of the relation between these two aspects of graph homomorphisms.

Theorem 1 (Welzl [12]) *Let G, H be two finite graphs such that H is not bipartite and there exists a homomorphism from G to H but none from H to G. Then there exists a graph K such that there exist homomorphisms from G to K and from K to H, but none from H to K or from K to G.*

In other words, the relation $\to$ induces a dense quasiorder on $\mathcal{G}$, with the unique exception occuring at the bottom, between the graphs with no edges and the bipartite graphs. This is therefore a statement on the global structure of the category of graphs. However, the construction of the interjacent graph K must take into account the specific instances of G and H, and respect the conditions $G \to K \to H$ and $H \not\to K \not\to G$. These constraints being NP-hard, it is perhaps not surprising that the first proof found by Welzl was a complicated ad hoc argument.

However, Theorem 1 admits a simple natural proof, as was later found out independently by M. Perles and J. Nešetřil (see [9, 11]). The pleasing argument intertwines classical results of graph theory with categorial aspects of graph homomorphisms. The present paper is a brief exploration of the developments made possible by this approach. We will show how the argument transposes to the category of directed graphs, connecting the problem of density with an apparently unrelated topic, that is, homomorphism duality.

2 The disjoint union and the product

The disjoint union is the coproduct in the category of graphs. This looks like a trivial construction. Any graph is the disjoint union of its connected components, hence the typical argument "without loss of generality, we can assume that G is connected ..." shows that in many situations, one can dispose of coproducts without even noticing it. We have $\chi(G \cup H) = \max\{\chi(G), \chi(H)\}$, $\omega(G \cup H) = \max\{\omega(G), \omega(H)\}$ and $\gamma_{\mathrm{odd}}(G \cup H) = \min\{\gamma_{\mathrm{odd}}(G), \gamma_{\mathrm{odd}}(H)\}$, where $\chi, \omega, \gamma_{\mathrm{odd}}$ denote respectively the chromatic number, the clique number and the odd girth. These identities are consistent with the categorial properties of the coproduct:

$$G \cup H \to K \text{ if and only if } G \to K \text{ and } H \to K. \tag{1}$$

Thus the coproduct is a supremum with respect to the relation $\to$. Since the chromatic number and the clique number are increasing and the odd girth is decreasing with respect to $\to$, the above identities seem natural, and turn out to be easy to prove.

The product $G \times H$ of G and H has $V(G) \times V(H)$ as its vertex set, and $[(u,v),(u',v')]$ is an edge of $G \times H$ if and only if $[u,u']$ is an edge of G and $[v,v']$ is an edge of H. This is the infimum of G and H with respect to the relation $\to$:

$$K \to G \times H \text{ if and only if } K \to G \text{ and } K \to H. \tag{2}$$

The identities $\omega(G \times H) = \min\{\omega(G), \omega(H)\}$ and $\gamma_{\text{odd}}(G \times H) = \max\{\gamma_{\text{odd}}(G), \gamma_{\text{odd}}(H)\}$ follow from this characterisation. However, the chromatic number of a product of graphs is an outstanding problem in graph theory.

Conjecture 1 ([4]) *For any graphs G and H,*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

Very little progress has been made on this question since it has been formulated by Hedetniemi in 1966. This contrasts with the situation of the disjoint union, where the corresponding identity is trivial.

The density problem may be solved using products and coproducts. Suppose that we are given two graphs G and H such that H is not bipartite, $G \to H$ and $H \not\to G$. We want to find a graph K such that $G \to K \to H$ and $H \not\to K \not\to G$. Nešetřil and Perles proposed a solution of the form

$$K = G \cup (X \times H)\,.$$

For any choice of X, we then have $G \to K \to H$. The remaining conditions depend on the parameter X.

First of all, we note that H necessarily contains a connected (non bipartite) component H' such that $H' \not\to G$; otherwise we would have $H \to G$. It is possible to select X such that $H' \not\to X$, for instance by specifying that the odd girth of X should be larger than that of H'. We then have $H' \not\to (X \times H)$ whence $H \not\to G \cup (X \times H) = K$.

The remaining condition, $K \not\to G$, is equivalent to $X \times H \not\to G$. To make this condition more tractable, it is desirable to isolate X by dividing both sides by H:

$$X \times H \not\to G \text{ if and only if } X \not\to G \div H. \tag{3}$$

At least, this step would seem natural to an immature mathematical mind, but a mathematician would object that the division of graphs is not defined. However, there does indeed exist a graph $G \div H$ with the property described in (3). It has been used by Lovász [8] in his work on the cancellation law for relational structures, and it is also a fundamental object in the study of Hedetniemi's conjecture [1, 2]. The vertex set of $G \div H$ is the set of all functions from $V(H)$ to $V(G)$, and two functions f and g are joined by an edge if $[f(u), g(v)] \in E(G)$ for all $[u, v] \in E(H)$. This definition allows a natural correspondence between the homomorphisms from $X \times H$ to G and the homomorphisms from X to $G \div H$. It is customary to use an exponential notation and denote this graph G^H rather $G \div H$, because of the structure of its vertex set. Hence condition (3) is usually written as follows:

$H \times X \not\to G$ if and only if $X \not\to G^H$.

This meets the approval of mathematicians, but immature mathematical minds dissaprove. For our part, we will be satisfied with the functional characterisation of equation (3) and adopt the notation $G \div H$. The arithmetic is then consistent.

To sum up, the right choice of X must be "small enough" so that $H' \not\to X$ and "large enough" so that $X \not\to G \div H$. The classical result of Erdős garantees the existence of such a graph:

Theorem 2 ([3]) *There exist graphs with girth and chromatic number as large as we please.*

Selecting X such that $\gamma(X) > \gamma_{\text{odd}}(H')$ and $\chi(X) > \chi(G \div H)$ we then have $H' \not\to X \not\to G \div H$. Therefore

$$G \to G \cup (X \times H) \to H$$

and

$$H \not\to G \cup (X \times H) \not\to G.$$

This proves Welzl's density theorem.

3 Directed graphs

Homomorphisms of directed graphs preserve orientation as well as adjacency. In other aspects, the categorial setting remains essentially the same. The categorial constructions satisfying equations (1), (2) and (3) are readily defined in this context. These are the basic ingredients of the argument of the previous section. Hence it is worthwile to check whether this argument also proves a density theorem for directed graphs.

Suppose that we are given directed graphs $\vec{G}$ and $\vec{H}$ such that $\vec{G} \to \vec{H}$ and $\vec{H} \not\to \vec{G}$. For simplicity, we will also assume that $\vec{H}$ is connected. If there exists a directed graph $\vec{X}$ such that $\vec{H} \not\to \vec{X} \not\to \vec{G} \div \vec{H}$, then $\vec{G} \cup (\vec{X} \times \vec{H})$ is homomorphically interjacent to $\vec{G}$ and $\vec{H}$, just as in the undirected case. The only problem is that in the directed case, Erdős' Theorem 2 does not guarantee the existence of $\vec{X}$. Indeed, it can happen that $\vec{H}$ contains no odd cycles at all. The underlying base graph of $\vec{H}$ is then bipartite, and this is the exception in Welzl's theorem. Every undirected bipartite graph is homomorphically equivalent to K_2 or K_1, but the directed graphs with a bipartite underlying base graph are a nontrivial part of the category of directed graphs.

Seeking analogues of Theorem 2 for directed graphs presents an interesting challenge. The girth is just the cardinality of a shortest cycle. but when an orientation is taken into consideration, a cycle is not defined just by its cardinality anymore. For instance the difference between the number of "forward edges" and the number of "backward edges" is an interesting feature. Other similar parameters can be defined. Generalizations of Theorem 2 may then restrict several of these parameters at once in directed graphs with high chromatic numbers.

It is possible, and perhaps natural, to try to complete the proof of a density theorem for directed graphs along these lines. Such an approach is bound to belittle the role of the particular instances of $\vec{H}$ and $\vec{G} \div \vec{H}$, reducing them to features such as their chromatic number, the structure of cycles, and so on. In contrast, our approach will put a great emphasis on the relationship between $\vec{H}$ and $\vec{G} \div \vec{H}$.

Suppose that there exists no directed graph $\vec{X}$ such that $\vec{H} \not\to \vec{X} \not\to \vec{G} \div \vec{H}$. This is an intriguing situation, as it implies a complementarity between homomorphisms *into* $\vec{G} \div \vec{H}$ and *from* $\vec{H}$: For any directed graph $\vec{X}$, $\vec{X} \not\to \vec{G} \div \vec{H}$ implies $\vec{H} \to \vec{X}$; and $\vec{H} \not\to \vec{X}$ implies $\vec{X} \to \vec{G} \div \vec{H}$. The option $\vec{H} \to \vec{X} \to \vec{G} \div \vec{H}$ can be ruled out as it implies $\vec{H} \to \vec{G} \div \vec{H}$ whence $\vec{H} \times \vec{H} \to \vec{G}$. This is contrary to our hypothesis since $\vec{H} \times \vec{H} \sim \vec{H}$. Hence, for any directed graph $\vec{X}$, we then have

$$\vec{X} \not\to \vec{G} \div \vec{H} \text{ if and only if } \vec{H} \to \vec{X}. \tag{4}$$

This suggests the following notion:

Definition 3 A couple $(\vec{A}, \vec{B})$ of directed graphs is called a *duality* if for every directed graph $\vec{X}$, we have

$$\vec{X} \not\to \vec{B} \text{ if and only if } \vec{A} \to \vec{X}.$$

Such dualities are the antithesis of our approach to density. If $(\vec{A}, \vec{B})$ is a duality, then $\vec{A} \to \vec{A}$ implies $\vec{A} \not\to \vec{B}$ whence $\vec{A} \not\to \vec{B} \times \vec{A}$. For any $\vec{K}$ such that $\vec{K} \to \vec{A}$ and $\vec{A} \not\to \vec{K}$, we then have $\vec{K} \to \vec{B}$ thus $\vec{K} \to \vec{B} \times \vec{A}$. In other words, $\vec{A}$ covers $\vec{B} \times \vec{A}$ in the sense that no directed graph lie strictly between $\vec{B} \times \vec{A}$ and $\vec{B}$. These covers will be called "gaps".

Definition 4 A couple $(\vec{G}, \vec{H})$ of directed graphs is called a *gap* if $\vec{G} \to \vec{H}$ and $\vec{H} \not\to \vec{G}$.

"Density" is therefore the property of having no gaps. We have seen how the presence of gaps is linked to the presence of dualities. Algorithmically, the existence of dualities has great implications. For instance, if $(\vec{A}, \vec{B})$ is a duality, then the problem of deciding if a directed graph $\vec{X}$ admits a homomorphism into $\vec{B}$ is polynomial. We need only to check whether there exists a homomorphism from $\vec{A}$ to $\vec{X}$ and this can be done in polynomial time since $\vec{A}$ is fixed. The dualities in the category of directed graphs are characterised as follows:

Theorem 5 (Komárek [7]) *Given a directed graph $\vec{A}$, there exists a directed graph $\vec{B}_{\vec{A}}$ such that $(\vec{A}, \vec{B}_{\vec{A}})$ is a duality if and only if $\vec{A}$ is homomorphically equivalent to a directed tree.*

And with this result, our search finds an answer. The simple truth is that there is no density theorem for directed graphs. In the case of undirected graphs, an exception does occur at the bottom, between the graphs with no edges and the bipartite graphs. In the case of directed graphs, the exceptions are too numerous and too meaningful to be dismissed. The result of our investigations cannot be called a "density theorem" but rather a "correspondence theorem", outlining the correspondence between gaps and dualities.

Theorem 6 *The gaps in the category of directed graphs are the couples* $(\vec{G},\vec{H})$ *such that there exists a duality* $(\vec{A},\vec{B})$ *with*

$$\vec{B}\times\vec{A}\to\vec{G}\to\vec{B} \text{ and } \vec{H}\sim\vec{G}\cup\vec{A}.$$

Conversely, up to homomorphic equivalence, the dualities are the couples $(\vec{H},\vec{G}\div\vec{H})$ *such that* $\vec{H}$ *is connected and* $(\vec{G},\vec{H})$ *is a gap.*

Proof. The proof is essentially a summary of the arguments presented so far. We have seen that if $(\vec{G},\vec{H})$ is a gap and $\vec{H}$ is connected, then $\vec{H}$ and $\vec{G}\div\vec{H}$ must satisfy the condition (4) whence $(\vec{H},\vec{G}\div\vec{H})$ is a duality. Also, if $(\vec{A},\vec{B})$ is a duality, then $(\vec{B}\times\vec{A},\vec{A})$ is a gap. These are the main aspects of the correspondence between gaps and dualities. The only points that remain to be discussed are questions of unicity and connexity.

It is clear from the definition that up to homomorphic equivalence, one member of a duality $(\vec{A},\vec{B})$ uniquely determines the other member. Also, if $(\vec{A},\vec{B})$ is a duality, then we can assume that $\vec{A}$ is connected, for if $\vec{A}=\vec{A}_1\cup\vec{A}_2$ with $\vec{A}\not\to\vec{A}_1$ and $\vec{A}\not\to\vec{A}_2$, then $\vec{A}_1\to\vec{B}$ and $\vec{A}_2\to\vec{B}$ whence $\vec{A}\to\vec{B}$ which implies $\vec{A}\not\to\vec{A}$.

Therefore, we have a correspondence between all dualities and the gaps whose second member is connected: If $\vec{H}$ is connected and $(\vec{G},\vec{H})$ is a gap, then its corresponding duality is $(\vec{H},\vec{G}\div\vec{H})$, which corresponds to the gap $((\vec{G}\div\vec{H})\times\vec{H},\vec{H})$. However, $\vec{G}\to\vec{H}$ implies $(\vec{G}\div\vec{H})\times\vec{H}\sim\vec{G}$. This shows that the correspondence is one-to-one and onto.

The remaining gaps are those where the second member is disconnected. For any duality $(\vec{A},\vec{B})$ and any $\vec{G}$ such that $\vec{B}\times\vec{A}\to\vec{G}\to\vec{B}$, $(\vec{G},\vec{G}\cup\vec{A})$ is a gap. Conversely, for any gap $(\vec{G},\vec{H})$, there exists a connected component $\vec{A}$ of $\vec{H}$ such that $\vec{A}\not\to\vec{G}$. Since $(\vec{G},\vec{H})$ is a gap, there exists no directed graph $\vec{X}$ suvh that $\vec{G}\cup(\vec{X}\times\vec{A})$ is strictly between $\vec{G}$ and $\vec{H}$. From this follows that $\vec{H}\sim\vec{G}\cup\vec{A}$ and that $(\vec{A},\vec{G}\div\vec{A})$ is a duality, with $(\vec{G}\div\vec{A})\times\vec{G}\to\vec{G}\to\vec{G}\div\vec{A}$. ■

Nešetřil and Pultr [10] have shown that the only duality in the category of undirected graphs is (K_1,K_2). Note that modulo the algebraic machinery presented here, this already proves Welzl's density theorem.

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