A Note on MaxFlow-MinCut and Homomorphic Equivalence in Matroids

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Abstract

In this note we point out that the validity of the max-flow-min-cut theorem in a matroid port M is equivalent to the homomorphic equivalence of the dual port M^* to a circuit in the category of matroid ports and strong port maps. As a consequence, restricting to the category of matroids with the strong integer MaxFlow-MinCut property, one has a linearly ordered set of equivalence classes and Menger's Theorem as the unique homomorphism duality.

1 Introduction

We assume familiarity with matroid theory, standard references are [5, 8]. All matroids will be finite and we will denote the ground set of a matroid M by $E(M)$ or sometimes just by E. We fix an element e of $E(M)$ throughout the paper. Then a matroid M such that $e \in E(M)$ is called a *matroid port* [4]. Let M and M' be two matroid ports on finite ground sets $E(M)$ resp. $E(M')$ and $o \notin E(M')$. A map $\sigma : E(M) \to E(M') \cup \{o\}$ is a called a *strong port map from M to M'* (or a homomorphism) if

SP1 $\sigma(e) = e$ and $\sigma^{-1}(\{e\}) = \{e\}$, (fixed ground point)

SP2 σ is a strong map from M to M', i.e. if $S' \subset E(M')$ is closed in M' then $\sigma^{-1}(S' \cup \{o\})$ is closed in M.

We denote the existence of a strong (port) map from M to M' by $M \to M'$ and by $M \not\to M'$ the non-existence of such a map.

2 From and to the Circuit

For any $k \in N \cup \{0\}$ let C_k denote the $(k+1)$-*circuit* that is the matroid port consisting of the circuit $\{e, a_1, a_2, \ldots, a_k\}$. Furthermore, we define C_∞ as the free matroid on $\{e, g\}$. The *girth of a matroid port M* is defined to be one less than the length of a shortest circuit containing e and is denoted by girth(M). If there is no such circuit we set the girth to infinity.

Theorem 1 *Let M be a matroid port and $k \in N$. Then*

$$C_k \to M \Leftrightarrow girth(M) \leq k.$$

Proof. If $C = \{e, b_1, \ldots, b_l\}, l \leq k$ is a circuit in M, then clearly the port map defined by $\sigma(e) = e$, $\sigma(a_i) = b_i$ for $i \leq l$ and $\sigma(a_i) = o$ for $l < i \leq k$ is strong. Assume on the other hand that $\sigma : C_k \to M$ is a strong port map and consider $S = \sigma(C_k) \setminus \{e\}$. Then $\sigma^{-1}(S)$ is not closed, thus, as σ is strong, e must be on a circuit in $S \cup \{e\}$. □

Note that $C_\infty \to M$ for any matroid port M. This gives rise to the following corollary:

Corollary 1 *Let $k, l \in N \cup \{\infty\}$. Then*

$$C_k \to C_l \Leftrightarrow k \geq l.$$

Since the restriction of a strong map is strong we also have:

Corollary 2 *If $M \to M'$ then girth$(M) \geq$ girth(M').*

Next we study the existence of strong port maps to the $(k+1)$-circuit. We will show that this is equivalent to the existence of k "disjoint" cocircuits containing $\{e\}$. One direction of this equivalence is:

Lemma 1 *Let M be a matroid port. Assume that M has k cocircuits $C_1^*, \ldots, C_k^*$ such that $\forall 1 \leq i < j \leq k : C_i^* \cap C_j^* = \{e\}$. Let $\sigma : M \to C_k$ denote the map defined by $\sigma(e) = e, \forall f \in C_i^* \setminus e : \sigma(f) = a_i$ and $\forall g \in E \setminus \bigcup_{i=1}^k C_i : \sigma(g) = o$. Then σ is a strong port map.*

Proof. Let $A \subseteq C_k$ be a closed set. We have to verify that $\sigma^{-1}(A \cup \{o\})$ is closed. If $e \notin A$, then $\sigma^{-1}(A \cup \{o\}) = \bigcap_{a_i \notin A}(E \setminus C_i^*)$ is the intersection of closed sets and thus is closed. Thus, we may assume that $e \in A$ and, for a contradiction, that $\sigma^{-1}(A \cup \{o\})$ is not closed. Hence, there exists a circuit C in M such that $C \cap (E \setminus \sigma^{-1}(A \cup \{o\})) = \{g\}$. As $E \setminus \sigma^{-1}(A \cup \{o\}) = \bigcup_{a_i \notin A}(C_i^* \setminus e)$ there exists some i_0 such that $g \in C_{i_0}^*$. Hence $|C \cap C_{i_0}^*| \geq 2$ and thus $e \in C$. Therefore, C has to intersect each C_i^* where $a_i \notin A$ at least twice. We conclude that $\sigma^{-1}(A \cup \{o\}) = E \setminus C_{i_0}^*$ implying $A = C_k \setminus \{a_i\}$ contradicting the fact that A is closed. □

Theorem 2 *Let M be a matroid port and $k \in N \cup \{\infty\}$. Then $M \to C_k$ if and only if there exist k cocircuits $C_1^*, \ldots, C_k^*$ in M such that $C_i^* \cap C_j^* = \{e\}$ for $i \neq j$.*

Proof. Sufficiency has been proven in Lemma 1. Thus assume σ is a strong port map from $M \to C_k$. We set $\tilde{C}_i^* := \sigma^{-1}(\{e, a_i\})$. The claim follows if we can show that each $\tilde{C}_i^*$ contains a cocircuit C_i^* containing e. To see this note that $F := \sigma^{-1}(C_k \cup \{o\} \setminus \{e, a_i,\})$ is a closed set which is a proper subset of the groundset and does not contain e. Thus, there is a hyperplane H such that $F \subseteq H$ and $e \notin H$ and $C_i^* := E \setminus H$ is as required. □

Note that $M \to C_\infty$ if and only if e is a cocircuit in M.

3 MaxFlow-MinCut and Homomorphic Equivalence

Given a matroid port M, a *flow* is defined to be a set of circuits $C^1, \ldots, C^k$ such that $C^i \cap C^j = \{e\}$. (We do not consider capacities here, as they can be simulated by parallels, here.) The value of the flow is k and a MaxFlow is a flow of maximum value. By the results of the last section, the existence of a flow of value k in M is equivalent to $M^* \to C_k$ where M^* denotes the matroid dual of M. We have also shown that the existence of a cocircuit C^* containing e in M^* such that $|C * \setminus \{e\}| = l$ is equivalent to $C_l \to M^*$. As, obviously, there cannot be a flow of a value larger than such an l, we can formulate the well-known weak duality for matroid flows as

$$\max\{k \mid M^* \to C_k\} \leq \min\{l \mid C_l \to M^*\}.$$

If equality holds in the above inequality, we say that M has the *MaxFlow-MinCut property.* (Note, that this is weaker than Seymour's definition in [7]). Let us say that two matroid ports M, M' are *homomorphically equivalent* if $M \to M'$ and $M' \to M$. We will denote homomorphic equivalence by $M \leftrightarrow M'$. With this terminology we can summarize:

Theorem 3 *Let M be a matroid port. Then there exists a unique $k \in N \cup \{\infty\}$ such that $M^* \leftrightarrow C_k$ if and only if M has the MaxFlow-MinCut property.*

□

This also shows that the homomorphis equivalence is a well structured property.

4 Homomorphism Duality

In this section we study the category of matroid ports and strong maps from a homomorphism duality point of view as introduced [6], [1]. A homomorphism duality theorem for matroid ports is a class equation,

$$A \to = \not\to B,$$

with two ports A, B, meaning that that for any matroid port M either there is a strong port map from $A \to M$ or one from $M \to B$, but not both. We will show that in general there is no nontrivial such theorem, but restricting to the class of binary matroids without F_7-minor, where F_7 denotes the Fano-plane-matroid containing a special element marked e, the quasiordering defined by the existence of maps has an extremely simple structure. Furthermore, we can expose Menger's Theorem as the unique homomorphism duality in this category.

Theorem 4 *Assume that A, B are matroid ports such that $A \to = \not\to B$ is a homomorphism duality theorem. Then $A \leftrightarrow C_0$ and $B \leftrightarrow C_1$.*

We will derive Theorem 4 from the following lemma. For $n \geq k \in N$ let U_k^n denote the uniform matroid port of rank k with n elements.

Lemma 2 *Let B be a matroid port such that $k := girth(B) \geq 2$. Then there exists an n such that $U_k^n \not\to B$.*

Proof. Assume the lemma were false. Then, for n large enough and a strong map $\sigma : U_k^n \to B$, there is some element $f \in E(B)$ such that $|\sigma^{-1}(\{f, o\}) \geq k|$. Let F denote the closure of f in B. By assumption $e \notin F$, thus $\sigma^{-1}(F)$ is not closed, a contradiction. $\square$

Proof of Theorem 4: By Corollary 1 and Theorem 2 $C_0 \to M$ if and only if e is a loop in M and $M \to C_1$ if and only if e is on some cocircuit. Thus, $C_0 \to = \not\to C_1$ is a homomorphism duality theorem.

Now let $A \to = \not\to B$ be a homomorphism duality theorem. Consider the matroid A' which arises by extending A by an element $f \notin E(A)$ coparallel to e, i.e. we replace e in each circuit by $\{e, f\}$. Then $\text{girth}\,(A) < \text{girth}\,(A')$ and thus $A \not\to A'$ by Corollary 2. Again using Corollary 2 we conclude that $\text{girth}\,(B) \geq \text{girth}\,(A') > \text{girth}\,(A)$. By Lemma 2, if $k = \text{girth}\,(B) \geq 2$ there is some n such that $U_k^n \not\to B$. But as $\text{girth}\,(U_k^n) = k > \text{girth}\,(A)$ we cannot have $A \to U_k^n$ either. Thus $\text{girth}\,(A) = 0$ and $\text{girth}\,(B) = 1$ and the claim follows by Theorem 3. $\square$

It should be clear by now that, in the following, in order to derive some interesting homomorphism duality, we restrict our considerations to a class of matroids with a strong MaxFlow-MinCut duality. P. Seymour [7] has shown that a matroid port M and all its "replications" (replace each element by a specified number of parallels) have the MaxFlow-MinCut property if and only if e is neither on an F_7^* nor on a $U_{2,4}$-minor. Thus, we consider the*category* $\mathcal{M}$ of MaxFlow-MinCut ports, where the objects are the matroid ports such that e is neither on an F_7 nor on a $U_{2,4}$-minor and the maps are strong port maps.

In this category, the quasiorder defined by the existence of strong port maps on the classes of homomorphically equivalent ports has the simple structure of an infinite chain with minimum and maximum, since every port is equivalent to some C_k. We derive:

Theorem 5 *1. Let $k \in N$. Then $C_k \to = \not\to C_{k+1}$.*

2. Let A, B be two matroid ports and $A \to = \not\to B$ a homomorphism duality theorem in $\mathcal{M}$. Then there exists a unique k, such that $A \leftrightarrow C_k$ and $B \leftrightarrow C_{k+1}$.

Proof. Let $M \in \mathcal{M}$. Then e is neither on an F_7^*-minor nor on a $U_{2,4}$-minor in M^*. By Seymour's theorem ([7]) M^* has the MaxFlow-MinCut property. Hence, by Theorem 3 there is a unique l_0 such that $M \leftrightarrow C_{l_0}$. Corollaries 1 and 2 imply that $M \to C_k \Leftrightarrow l_0 \geq k$ and $C_k \to M \Leftrightarrow k \geq l_0$. For the second statement let $A \leftrightarrow C_{l_0}$. Then $A \not\to C_{l_0+1}$ and thus $C_{l_0+1} \to B$. By Theorem 3 there exists a unique $l \leq l_0 + 1$ such that $B \leftrightarrow C_l$. Since $A \to C_l$ for $l < l_0 + 1$, the claim follows. $\square$

We would like to point out that in graphic matroids the duality $C_k \to = \not\to C_{k+1}$ reflects the trivial fact that the length of a shortest st-path in a graph with unit weights limits the number of pairwise disjoint st-cuts. For cographic matroids we get the existence of either $k+1$ edge-disjoint st-paths or an st-cut of size k. Thus, the "$C_k \rightarrow = \not\rightarrow C_{k+1}$" is a formulation of Menger's Theorem as a theorem of the alternatives.

Remark:

This paper was motivated by the companion paper [2] where we considered dualities for strong maps of oriented matroids and showed that Farkas lemma is the only instance of such a duality. In the more restrictive context of matroid ports we obtained a richer spectrum of duality results.

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