# Density

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**Abstract**

We survey recent research of authors related to the study of homomorphisms between finite graphs from the point of view of the quasiorder defined by the existence (of homomorphism) between graphs. We demonstrate the surprising richness of this perhaps simplest case of graph comparision by proving the density theorem for both directed and undirected graphs in three different ways: Nešetřil-Perles proof using direct products (which works for undirected graphs) and two recently found proofs due to authors: a proof using arrow construction and a proof using a connection of density and homomorphism dualities. The later notion may be seen as a homomorphism approach to the class $NP \cap coNP$ and its connection to density problems may be seen as surprising. This analysis involves analysis of several other categorical notions most notably amalgamation and graph exponentiation.

## 1 Introduction.

We consider homomorphisms between both undirected and directed graphs: given two graphs $G = (V, E)$ and $G' = (V', E')$, a *homomorphism* $f$ of $G$ to

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$G'$ is any mapping $f : V \to V'$ which satisfies

$$(\star) \quad [f(x), f(y)] \in E' \text{ for all } [x, y] \in E.$$

If there exists a homomorphism of $G$ to $G'$ then we say $G$ is *homomorphic* to $G'$, and write $G \to G'$. Otherwise we write $G \not\to G'$.

If $G \to G'$ and $G' \to G$ then we say $G$ and $G'$ are *hom-equivalent*.

The graph homomorphisms were investigated in various context: graph coloring, [1, 2, 6, 7, 10, 11, 12, 16], graph products [8, 21], automata theory and formal languages [16, 27], various algebraical context (as a generalization of isomorphism; categories). Various of these questions can be conveniently expressed by considering the quasiorders (and partial orders) induced by the existence of homomorphism.

Here we consider the problem motivated by the existence of homomorphism between graphs:

One of the motivations of this paper is the following result due to Welzl [27]

**Theorem 1.1** *(density)*

*Given two finite graphs $G_1$ and $G_2$ satisfying $G_1 \to G_2 \not\to G_1$, with $G_2$ not bipartite, there exists a graph $G$ which satisfies $G_1 \to G \to G_2$ and $G_2 \not\to G \not\to G_1$.*

In the other words the quasiordering of the class of all (finite) graphs induced by the existence of a homomorphism is dense in the usual order-theoretic sense with the unique exceptional "gap-pair" $K_1 \to K_2 \not\to K_1$.

The original proof [27] is a complicated ad hoc argument. More recently M. Perles and J. Nešetřil independently found an alternative argument (see e.g. [19], [20] for the history of this) which simplified the proof and allowed to prove the density theorem for infinite graphs and some other classes of directed graphs [19]. We review this approach in Section 2. This proof is based on the direct product of graphs and it provided the motivation for other two proofs. So we decided to include it for a more clear and complete picture and we even give two variants of this proof. Let us also remark that recently there was a progress also in other directions and Tardif [26] solved a problem of Welzl (see [28]) related to density of vertex transitive graphs. However until recently for directed graphs the problem remained open. The strongest result in this direction is the following [25]

**Theorem 1.2** *The pairs* $(P_0, P_1), (P_1, P_2), (P_{k+1}, P_k), k = 3, 4, \ldots$ *are the only gaps for oriented paths (the graphs* $P_k$ *are depicted on Fig. 1).*

*Explicitely: Let* $P_1$ *and* $P_2$ *be oriented paths satisfying* $P_1 \to P_2 \not\to P_1$. *Let the pair* $(P_1, P_2)$ *be none of the above pairs. Then there exists a path* $P$ *which saisfies both* $P_1 \to P \to P_2$ *and* $P_2 \not\to P \not\to P_1$.

One can see easily the the above pairs form gaps also in the class of all oriented graphs. Thus for oriented graphs we have infinitely many gaps and their full description seemed to be not easily reachable even for the class of all oriented trees ([25]). The problem has been finally solved in [23] by relating it to a seemingly unrelated result of [13] about homomorphism dualities [17],[21]. This together with all the necessary notions will be introduced in Section 4. Before this we introduce in Section 3 one more proof of the density of both directed and undirected graphs. This proof is based on the iterated graph amalgamation known as *arrow construction.* This proof is perhaps the shortest and conceptually easiest proof of the density for both undirected and directed graphs. Moreover in the more general cases (when the homomorphism dualities are not characterized) this proof is an indispensable part of the proof.

# 2 Nešetřil - Perles Proof

**Proof.** Let $G_1$ and $G_2$ be given undirected graphs, let $f : G_1 \longrightarrow G_2$ be a homomorphism, and suppose there is no homomorphism $G_2 \longrightarrow G_1$. As this pair is not equivalent to the jump $(K_1, K_2)$, at least one component of the graph $G_2$ has chromatic number greater than 2. Also, at least one component of $G_2$ fails to be $G_1$-colorable, and this component may be assumed to be non-bipartite; let it contain an odd cycle of length $k$. Now choose a graph $H$ with the following properties: $H$ contains no odd cycle of length $k$ or less, and the chromatic number of $H$ is greater than $n^{n'}$, where n and $n'$ denote the number of vertices of the graphs $G_1$ and $G_2$ respectively. Such a graph exists by the celebrated theorem of Erdős [4]. Now let $G = G_1 \cup (H \times G_2)$. Here $\times$ denotes the direct product of two graphs and $\cup$ means the disjoint union. We shall prove that $G$ has the desired properties. Obviously $G_1 \longrightarrow G$ and $G \longrightarrow G_2$ follows as the second projection of $H \times G_2$ is a homomorphism into $G_2$. On the other hand there is no homomorphism from $G_2$ into $G$, as homomorphisms preserve odd cycles and they cannot increase the length of the shortest of them. Thus it suffices to prove that there is no homomorphism $G \longrightarrow G_1$. Let us suppose for the contradiction that there is a homomorphism $f : H \times G_2 \longrightarrow G_1$. Thus for any vertex $x$ of $H$ we have an induced mapping $f_x : V(G_2) \longrightarrow V(G_1)$ defined by $f_x(y) = f(x, y)$. (This mapping need not be a homomorphism.) As there are at most $n^{n'}$ such mappings there are vertices $x$ and $x'$ forming an edge of $H$ such that the mappings $f_x$ and $f'_x$ are identically equal, say to $g$. However in this case $g$ is a homomorphism of $G_2$ into $G_1$, contrary to our assumption. ∎

Given two graphs $G = (V, E)$ and $H = (V', E')$ one can define $G$ *power of* $H$ ,denoted by $H^G$, as the following graph $(W, F)$: $W = \{f; V \to V'\}$ and a pait $(f, g)$ forms an edge if $(f(x), g(x)) \in E'$ for every $x \in V$. (We define the $G$ power of $H$ by the same formula for both undirected or directed graphs.

This construction was isolated in the graph theoretic concept in [15], see also [3], [24]. The following is the crucial property which we use: For every graph $K$ holds $K \to H^G$ iff $K \times G \longrightarrow H$. (This is easy to see: Given $f : K \to H^G$ define $g : K \times G \longrightarrow H$ by $g(x, y) = f(x)(y)$. Conversely, given g we may define f by the same formula. One can easily check that $f$ is a homomorphism $K \to H^G$ iff $g$ is a homomorphism $K \times G \longrightarrow H$. Thus in the above proof we have $H \times G_2 \longrightarrow G$ iff $H \longrightarrow {G_1}^{G_2}$. Thus in the above proof we may assume that the chromatic number of $H$ is greater than the chromatic number of ${G_1}^{G_2}$. As $G_2 \not\to G_1$ there are no loops in ${G_1}^{G_2}$ and it is also clear that the chromatic number of ${G_1}^{G_2}$ is at most the number of vertices of ${G_1}^{G_2}$. (However we do not optimize at this point.)

# 3 Density via Amalgamation

Besides giving a new proof of undirected density theorem we prove the following result (which generalizes and solves a longstanding problem in this area). This proof appeared in [23].

**Theorem 3.1** *Let $G, H$ be directed graphs which are cores. Let $H$ be connected and assume that $H$ fails to an orientation of a tree. Further assume that $G \to H \not\to G$ holds. Then there exists a directed graph $H$ with $G \to K \to H$ and $H \not\to K \not\to G$.*

The proof is based on the following construction which goes back to [5] and [9], see also [18]:

Let $G = (V, E)$ be a directed graph, $I = (V', E')$ a graph with two distinguished vertices $a, b \in V'$. Denote by $G \star (I, a, b)$ the graph $(W, F)$ defined as follows:

$W = E \times V' \mid_{\sim}$

where the equivalence $\sim$ is generated by the following rules:

$((x, y), a) \sim ((x, y'), a)$, $((x, y), b) \sim ((x', y), b)$, $((x, y), b) \sim ((y, z), a)$.

The equivalence class containing $((x, y), z)$ will be denoted by $[((x, y), z)]$. We put

$F = \{([((x, y), z)], [((x, y), z')]; (x, y) \in E, (z, z') \in E'\}$.

Clearly $G{\star}(I, a, b)$ arises by replacing every arrow of $G$ by a copy of $I$.This construction is known as *arrow construction* (in some of the earlier papers also called šíp construction; šíp = arrow in czech). Note, that if $(I, a, b)$ is an undirected graph (considered as a symmetric relation), then $G \star (I, a, b)$ is also an undirected graph.

We shall make use of the following obvious (but key) property of the arrow construction:

**Lemma 3.1** *Let $G$ and $H$ be directed graphs with $\chi(G) > |V(H)|$ and let every homomorphism $f : I \to H$ satisfies $f(a) \neq f(b)$. Then $G \star (I, a, b) \not\to H$.*

**Proof.** (Proof of Theorem 1.1) Let $G, H$ be directed graphs, $H$ non bipartite, with $G \to H \not\to G$. Clearly we may assume that $G$ and $H$ are cores. Let $e = (a, a') \in E(H)$ belongs to a circuit in $H$. Put $I = (H \setminus e) \cup \{a', b\}$ where $b \notin V(H)$. (Thus $I$ arises from $H$ by deleting the edge $e$, adding a new vertex $b \notin V(H)$ together with the edge $\{a', b\}$.)

It is clear that $I \to H$ (identifying vertices a and b) but any homomorphism $f : I \to G$ satisfies $f(a) \neq f(b)$ (for otherwise we get a contradiction with $H \not\to G$. Now let $F$ be any graph satisfying $\chi(F) > |V(G)|$ and let $F'$ be any orientation of $F$. Consider the arrow product $F' \star (I, a, b)$ and define the graph $K$ by $K = F' \star (I, a, b) \cup G$.

We prove that $K$ has properties claimed by Theorem 1.1: Clearly $G \to K$. We also have $K \to H$ as the mapping $f$ defined by $f([e, x]) = x$ for $x \in V(H)$ and $e \in E(F')$ and $f([e, b] = a$ is a homomorphism $K \to H$ (we preserve the above notation concerning the arrow construction $F' \star (I, a, b)$.) Further, by the above Lemma 3.1 $K \not\to G$ (as $\chi(F) > |V(G)|$). Thus it remains to be shown that $H \not\to K$. Suppose the contrary: Let $g : H \to F \star (I, a, b)$

be a homomorphism. Then $f \circ g : H \to H$ where $f$ is the above defined homomorphism $F \star (I, a, b) \to H$. As $H$ is a core $f \circ g$ is a homomorphism. Put $h = (f \circ g)^{-1}$. Then $f \circ g \circ h(x) = x$ for every $x \in V(H)$. Put $g \circ h(a) = [(e, a)]$ with $e = (u, v)$. Then the image $g \circ h(G)$ of $G$ is a connected subgraph of $F \star (I, a, b)$ which is (by the injectivity of the mapping $f \star g \star h$) contained in the set of all $[(e', x)]$ where $e'$ is incident with $u$ and $x \in V(I)$ (this set is the "star" induced by those edges of $F'$ which are incident with the vertex $u$). But then the edge $\{[g \circ h(a)], [g \circ h(a')]\}$ is a cut edge in the graph $g \circ h(G)$ which is the final contradiction as $a, a'$ was contained in a cycle of $H$. ∎

**Proof.**(of Theorem 3.1): Let $G, H$ satisfy the assumption of the theorem. let $H$ be a core and let $(a, a') \in E(H)$ belongs to a cycle in $H$. Put $I = H \setminus (a, a') + (b, a')$ where $b \notin V(H)$ (i.e. we first delete arc $(a, a')$ and the add a new vertex $b$ together with the arc $(b, a')$. Let $F$ be oriented graph with $\chi(F) > |V(G)|$ and consider the arrow construction $F \star (I, a, b)$. Put $K = G \cup (F \star (I, a, b))$. Then we have:

- $G \to K$ (by the inclusion map);
- $K \to H$ (by the same mapping as in the above proof);
- $K \not\to G$ (by the chromatic number assumption);
- $H \not\to K$ (as above in the above proof for undirected graph).

Thus the graph K has the desired properties. ∎

This proof can be generalize to finite models of general relational systems and thus it cunstitutes the essential part of [24]

# 4 Proof via homomorphism dualities

As we have seen Density Theorems 1.1, 3.1 do not hold without exceptional cases. To formalize this let us ay that a pair $(G, H)$ of graphs is a *gap* if $G \longrightarrow H$ and $H \not\longrightarrow G$ and every graph $K$ with $G \longrightarrow K \longrightarrow H$ is homomorphically equivalent to either $G$ or $H$. Thus any gap is a gap in the quasiorder induced by the existence of homomorphism (i.e. by $\longrightarrow$). In the above density theorems we want to characterize all gaps. It is easy to see that $(K_1, K_2)$ is a gap for undirected graphs and thus the Density Theorem 1.1 can be formulated as follows:

**Theorem 4.1** *$(K_1, K_2)$ and $K_0, K_1)$ are the only gaps for the class of indirected graphs.*

Also for directed graphs Theorem 3.1 covers all non-gaps pairs of oriented graphs. However this (perhaps a bit surprising) fact is harder to prove. We shall outline the proof, for full details see [23].

Let $F, H$ be graphs. We say that $(F, H)$ is a duality pair if for every graph G the following statement holds:

$G \longrightarrow H$ iff $F \not\longrightarrow G$.

Obviously in this case $F \not\longrightarrow H$ and thus we can also define a duality pair as follows: For any graph $G$ we have either a homomorphism from $G$ to $H$ or a homomorphism from $F$ to $G$ while these two possibilities are mutually exclusive. This statement is also called *homomorphism duality theorem.*

Any duality pair is an example of a good characterization (in the sense of Edmonds) and in fact this was the original motivation of this notion [21], [17].

The duality pairs (and thus homomorphism duality theorems) were characterized in [21] for the class of undirected graphs and in [14] and [24] for the class of directed graphs:

**Theorem 4.2** *Up to a homomorphism equivalence $(K_2, K_1)$ and $(K_1, K_0)$ are the only duality pairs for the class of undirected graphs.*

**Theorem 4.3** *Up to a homomorphism equivalence the only homomorphism dualities for directed graphs are of the following form $(T, H_T)$ where $T$ is an oriented tree and for each tree $T$ the graph $H_T$ is uniquelly determined.*

Nešetřil and Tardif found in [23] and [24] the following (perhaps surprising) connection of duality pairs. This provided the key to the characterization of gaps for classes of directed (and undirected) graphs. We say that a gap$(G,H)$ is *connected* if $H$ is a connected graph. Observe that if $(F,H)$ is a duality pair then F is necessarily connected (for if $F$ is a core and $F_1$ and $F_2$ are distinct components of $F$ then $F \not\to F_i$ , and thus $F_i \to H$ for $i = 1,2$. Thus $F \to H$, which a contradiction).

**Theorem 4.4** *There is a one-to-one correspondence between duality pairs and gaps for the class of directed (and also undirected ) graphs. Explicitely, given a duality pair $(F,H)$ then $(F \times H, F)$ is a gap. Conversely, given a connected gap $(G,H)$ then $(H, G^H)$ is a duality pair.*

**Proof.** First, suppose that $G \to H \not\to G$ is a gap. We prove that for any graph $K$ holds $H \not\to K$ iff $K \to G^H$. Thus let a graph $K$ be such that $H \not\to K$ and suppose for the contrary that $K \not\to G^H$. Then obviously $G \to G \cup (G \times H) \to H$. If $H \to G \cup (K \times H)$ then (by the connectivity and the assumption $H \not\to G$) $H \to K \times H$ and thus $H \to K$ ,contrary to our assumpiton. If $G \cup (K \times H) \to G$ then $K \times H \to G$ and $K \to G^H$ again contrary to our assumption. Thus the class $H \not\to$ is a subclass of the class $\to G^H$. In order to prove the reverse inclusion let $K$ be a graph satisfying $K \to G^H$ and $H \to K$. This implies $H \to H \times G^H \to G$ (as $H \times G^H \to G$ is equivalent to $G^H \to G^H$ and thus it always holds). Thus $(H, H^G)$ is a duality pair.

Conversely, let $(F,H)$ be a duality. We may clearly assume that $F$ is a core (i.e. every homomorphism $F \to F$ is an automorphism) and further $F$ is connected. Thus $F \times H \to F$ and $F \not\to F \times H$ (as $F \to F \times H$ would imply $F \to H$). We claim that there is no graph $K$ satisfying $F \times H \to K \to F$ and $F \not\to K \not\to F \times H$: If $K \to F \not\to F \times H$ then the duality implies $K \to H$ and thus $K \to F \times H$ which contradicts our assumptions. This completes the proof of theorem. ∎

This Theorem 4.4 leads to yet another proof of Density Theorem 1.1:

**Proof.** By 4.2 $(K_2, K_1)$ and $(K_1, K_0)$ are the only duality pairs. Thus $(K_2 \times K_1)$ (which is equivalent to $(K_1, K_2)$) and $(K_0 \times K_1, K_1)$ (which is equivalent to $(K_0, K_1)$ are the only gaps ∎

Thus, modulo the above "arrow calculus" involved in the above proof of 4.4, the density theorem has been known even before it has been formulated.

Finally, let us remark that the above also shows that Theorem 3.1 gives all non-gap pairs for directed graphs: Let $(T, H^T)$ be all duality pairs for oriented graphs (characterized by Theorem 4.3). Then $(H^T, T)$ are exactly all gaps. This gives also yet another proof of Theorem 3.1.

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