Coloring and homomorphisms of degenerate and bounded degree graphs

A. Kostochka* J. Nešetřil P. Smolíková†

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Abstract

We are interested here in homomorphisms of undirected graphs and relate them to graph degeneracy and bounded degree property. Our main result reads that both in the abundance of counterexamples and from the complexity point of view the Brooks theorem (and first fit algorithm) are the "only" easy cases.

1 Introduction

Given two (undirected simple) graphs $G_i = (V_i, E_i)$, $i = 1, 2$, a homomorphism of G_1 into G_2 is any mapping $f : V_1 \to V_2$ which satisfies $\{x, y\} \in E_1 \Rightarrow \{f(x), f(y)\} \in E_2$. This is denoted by $f : G_1 \to G_2$ and the existence (the non–existence resp.) of a homomorphism is denoted by $G_1 \to G_2$ ($G_1 \not\to G_2$ resp.). Clearly the existence of a homomorphism $G \to K_k$ coincides with (regular) k–coloring and thus homomorphisms $G \to H$ are sometimes called H–colorings.

The existence of H–colorings was thoroughly discussed in several papers (see [GW], [HN]) and there seems to be a big gap between oriented and unoriented graphs here. However in this paper we are dealing with undirected graphs only. In this context let us quote the following two results of [HN] and [NZ].

Theorem 1.1 *The H–coloring problem is polynomial when H is bipartite and NP–complete otherwise.*

This has been strengthened (see [NZ]) as follows.

Theorem 1.2 *H–coloring problem has bounded treewidth duality if and only if H is bipartite.*

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(Here H–coloring problem has bounded treewidth duality if there is an integer k such that the following property holds for all graphs $G : G \not\to H$ if and only if there is a partial k–tree T such that $T \to G$ and $T \not\to H$.)

We are motivated here by these results and by classical Brook's theorem (see any book of graph theory) which claims that K_3–coloring problem is easy for cubic graphs. Moreover this problem is easy for 2–degenerate graphs. One of the main results of this paper says that this is the only exception:

Theorem 1.3 *Let H be a non–bipartite triangle free graph. Then the H–coloring problem is NP–complete for the class of all 2–degenerate graphs.*

We provide some further evidence that coloring 2–degenerate (and k–degenerate) graphs is a difficult problem.

For cubic graphs the situation is different. This question was investigated in [GHN], where the following has been proved :

Theorem 1.4 *The C_5–coloring problem is NP–complete for the class of all cubic graphs.*

Theorem 1.5 *For any integer b, there exists a triangle free graph U_b such that any triangle free graph G with degree at most b is homomorphic to U_b.*

Bellow we strengthen these result in various directions and under restrictions like girth and special H. However the following problem seems to be still open.

Problem 1.6 *Is it true that any cubic graph G with sufficiently large girth g is homomorphic to C_5?*

We show that necessarily $g > 7$. We complement this by showing that the answer is negative for C_k, $k \geq 11$.

The paper is organized as follows. In Section 2 we introduce all notions and state basic results. In Section 3 we prove the complexity results for graphs with bounded odd–girth, while in Section 4 we deal with cubic and in Section 5 with degenerate graphs. In Section 6 we list some open problems and comments.

2 Basic results

Definition 2.1 *Let H be a graph.*
(i) Let l be an integer. The (H, l)–coloring problem is the following decision problem:
Instance: A graph G, $girth(G) \geq l$.
Question: Is there any homomorphism from G to H?

(ii) Let l be an odd integer. The $(H, odd - l)$–coloring problem is the following decision problem:
Instance: A graph G, $C_{l-2} \not\to G$ (i.e. $odd - girth(G) \geq l$).
Question: Is there any homomorphism from G to H?

Lemma 2.2 ([NR]) *Let $\epsilon > 0$ be a real number, $l > 3$ be an integer and I, H be graphs such that $I \to H$. Then there exists a graph F with the following properties:*

(i) $girth(F) \geq l$

(ii) $V(F) = \bigcup_{i \in V(I)} F_i$ and there is a homomorphism $f : F \to I$ such that for every $i \in V(I)$, f maps F_i to i

(iii) there is α such that for all $i \in V(I)$, $|F_i| = \alpha$

(iv) for every homomorphism $h : F \to H$, for every edge $\{i, j\} \in E(I)$, and every sets A, B such that $A \subseteq F_i$, $B \subseteq F_j$, $|A| \geq \epsilon \cdot \alpha$, $|B| \geq \epsilon \cdot \alpha$, there is an edge in H between h/A and h/B.

We will call F the (l, ϵ, I)–majority graph for H.

In the following we need to review the concept of a core: a graph H is called a core if it does not admit a homomorphism to its proper subgraph. The previous lemma (in its probabilistic setting) may be strenghten to the following ("folkloristic") lemma (see e.g. [Z]).

Lemma 2.3 (On uniquely H–colorable graphs) *Let H be a core and l be an integer. Then there exists a graph $\hat{H}$ of girth at least l such that for any two homomorphisms f_1, $f_2 : \hat{H} \to H$, exists an automorphism h of H with $f_1 = h \circ f_2$.*

We shall need yet another technical refinement of this result:

Lemma 2.4 *For any two integers l, k and any core H with two specified vertices h_1, h_2, there exists a graph $B = B(H, h_1, h_2, l, k)$ of girth at least l with two vertices b_1, b_2 such that*

(i) $d_B(b_1, b_2) \geq k$

(ii) there exists a homomorphism f from B to H with $f(b_1) = h_1$, $f(b_2) = h_2$

(iii) for any two homomorphisms f_1, $f_2 : B \to H$ there exists an automorphism h of H with $f_1 = h \circ f_2$.

Proof. Let $\hat{H}$ be the graph satisfying Lemma 2.3 for H and l. Let $f : \hat{H} \to H$ be a homomorphism. It follows from the properties of the graph $\hat{H}$ that $|f^{-1}(h)| > 2$, for any $h \in V(H)$. Let x be a vertex from $f^{-1}(h_1)$ and let y, z be two vertices from $f^{-1}(h_2)$. Consider m disjoint copies of $\hat{H}$, where m is suficiently large ($m > \frac{k}{2} + 1$). For $j = 1, 2 \ldots m - 1$, identify the vertex z from the j–copy of $\hat{H}$ with the vertex y of the $j + 1$–copy of $\hat{H}$. Denote the arising graph B. Denote by b_1 the vertex x from the first copy and by b_2 the vertex z from the m–th copy of $\hat{H}$. $\square$

The first result is a technical statement which was isolated first in [NR] and which proved to be usefull in various contexts (see e.g. [Z]).

Further, we shall use three type of reductions. The first two reductions are similar to those defined in [HN] (which are reviewed and modified here for the sake of completeness). The third one is a new modification.

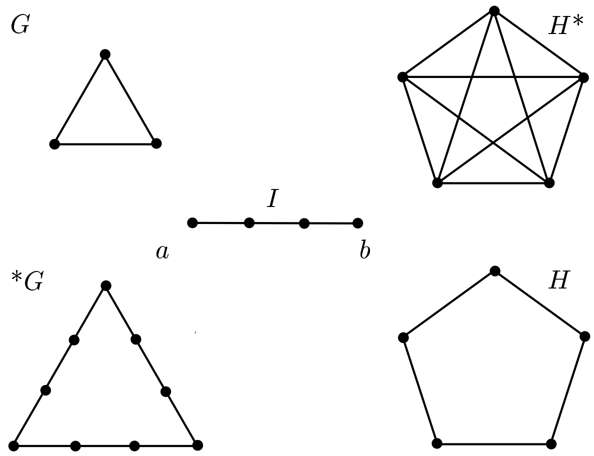

Figure 1: The indicator construction.

2.1 The indicator construction

Let I be a fixed graph, and let a and b be distinct vertices of I such that some automorphism of I interchanges vertices a and b. Transform a given graph H into the graph H^* defined to have the same vertex set as H and to have as the edge set all pairs $\{h, h'\}$ for which there is a homomorphism of I to H taking a to h and b to h'(see Fig. 1). Because of our assumption on I, the edges of H^* will be undirected.

Let $\mathcal{A}$ be a class of graphs. Given a graph $G \in \mathcal{A}$, let $^*G$ be the graph obtained from G by replacing each edge $\{g, g'\}$ by a disjoint copy of I, identifying a with g and b with g'. Define the class $\mathcal{A} * \mathcal{I}$ by $\mathcal{A} * \mathcal{I} = \{^*G; G \in \mathcal{A}\}$.

Lemma 2.5 *If the H^*–coloring problem is NP–complete for class $\mathcal{A}$, then so is the H–coloring problem for class $\mathcal{A} * \mathcal{I}$.*

(In applying Lemma 2.5 we need to be careful to ensure that H^* has no loops, i.e. that no homomorphism of I to H can map a and b to the same vertex, and that H^* fails to be a bipartite graph. Otherwise the H^*–coloring problem will not be NP–complete.)

Proof. It follows from the definitions that there is a homomorphism $^*G \to H$ if and only if there is a homomorphism $G \to H^*$. $\square$

2.2 The sub-indicator construction

Let J be a fixed graph with specified vertices a and $b_1, b_2 \dots b_k$. The sub-indicator construction (with respect to J, a and $b_1, b_2 \dots b_k$) transforms a given core H with specified vertices $h_1, h_2 \dots h_k$, to its subgraph $H^\sim$ induced by the vertex set

$$V^\sim = \{h';\ \text{there is } f : J \to H \text{ with } f(a) = h' \text{ and } f(b_1) = h_1, \dots f(b_k) = h_k\}.$$

Let $\hat{H}$ be the graph satisfying Lemma 2.3 for H and l. Let $\hat{h}_i$ for $i = 1, 2 \dots k$, be vertices of $\hat{H}$ such that there is a homomorphism $f : \hat{H} \to H$ with $f(\hat{h}_i) = h_i$.

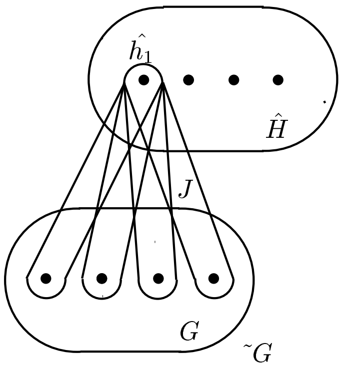

Figure 2: The sub-indicator construction.

Let $\mathcal{A}$ be a class of graphs. Given a graph $G \in \mathcal{A}$, define $\tilde{}G$ as the graph obtained from the disjoint union of G, $\hat{H}$ and $|V(G)|$ copies of J, by identifying the vertex b_i, for $i = 1, 2 \ldots k$, in each copy of J with the vertex $\hat{h}_i$ in $\hat{H}$, and by identifying each vertex g of G with the vertex a in the g–th copy of J (see Fig. 2). Let $\mathcal{A}\tilde{}I$ be the class of all graphs containing $\tilde{}G$.

Lemma 2.6 *If the $H\tilde{}$–coloring problem is NP–complete for class $\mathcal{A}$, then so is the H–coloring problem for class $\mathcal{A}\tilde{}I$.*

Proof. There is a homomorphism $G \to H\tilde{}$ if and only if there is a homomorphism $\tilde{}G \to H$. □

2.3 The majority-indicator construction

Let I be a fixed graph, and let a and b be non-adjacent distinct vertices of I which do not belong to any odd cycle of length less than $l - 2$ and such that some automorphism of I maps a to b and b to a. Let H^* be a graph defined as before by applying the indicator construction with the indicator I (and vertices a and b) to a given graph H.

Lemma 2.7 *Let l be an odd integer, $l > 3$. If the $(H^*, odd - (l-2))$–coloring problem is NP–complete, then the $(H, odd - l)$–coloring problem is also NP–complete.*

Proof. Let $\epsilon < 1/|V(H)|$. Let F be the (l, ϵ, I)–majority graph for H. Given a graph G of odd–girth at least $l - 2$, let $^\diamond G$ be the graph obtained from G by replacing each vertex g by a disjoint set N_g of α vertices and by replacing each edge $\{g, g'\}$ by a disjoint copy of F, identifying N_g with F_a and $N_{g'}$ with F_b.

Claim 1 *$odd - girth(^\diamond G) \geq l$.*

Proof. We know that $girth(F) \geq l$, $d_F(x, y) > 1$ for any two vertices $x \in F_a$, $y \in F_b$ (because a and b are non-adjacent in I), $odd - girth(G) \geq l - 2$ and $d_F(x, x') \geq 2$, $odd - d_F(x, x') \geq l - 2$ for any x, $x' \in F_a$ so as for any x, $x' \in F_b$ (because there is a homomorphism from F to I which maps both the vertices to the same vertex and the vertices a, b are not contained in I in any odd cycle of length less than $l-2$). Now it is easy to see that $odd - girth(^{\diamond}G) \geq l$. $\Box$

We shall make use of the following.

Definition 2.8 *Let h be a homomorphism from $^{\diamond}G$ to H. The majority set of a vertex $g \in V(G)$ with respect to h is the subset M_g of N_g with maximum cardinality such that $M_g \subseteq h^{-1}(v)$ for some vertex $v \in V(H)$.*

Notice that for any vertex $g \in V(G)$, $|M_g| \geq \frac{1}{|V(H)|} \cdot \alpha > \epsilon \cdot \alpha$.

Claim 2 *There is a homomorphism $^{\diamond}G \to H$ if and only if there is a homomorphism $G \to H^*$.*

Proof. Let $h :\ ^{\diamond}G \to H$ be a homomorphism. We will define a mapping $f : V(G) \to V(H^*)$ as follows. Let g be a vertex of G. Then put $f(g) = h/M_g$, where M_g is the majority set of g with respect to h. Then for any edge $\{g, g'\}$ in G, $\{f(g), f(g')\}$ is an edge in H^*.

Conversely let $f : G \to H^*$ be a homomorphism. Put $h(w) = f(g)$ for all $w \in N_g$. Then h can be extended to the remaining vertices of $^{\diamond}G$ naturaly. $\Box$

It follows from Claim 2 that if the H^*–coloring problem is NP–complete for the class of all graphs of odd–girth at least $l-2$, then the H–coloring problem is NP–complete for $^{\diamond}G$, i.e. for the class of all graphs of odd-girth at least l. This proves Lemma 2.7. $\Box$

(In applying Lemma 2.7 we need to check that H^* contains no loops and that H^* is non–bipartite, as in the former case of the indicator construction.)

3 The complexity results

3.1 $(H, odd - l)$–coloring problem

Theorem 3.1 *Let H be a non–bipartite graph. Then the $(H, odd - 7)$–coloring problem is NP–complete.*

Proof. Suppose to the contrary that l is the minimum odd integer such that there is a non–bipartite graph H for which the $(H, odd - l)$–coloring problem is not NP–complete. Let H be a graph with this property and such that the $(H', odd - l)$–coloring problem is NP-complete for any non–bipartite H' with

(i) fewer vertices than H, or

(ii) with the same number of vertices as H, but with more edges.

If the theorem does not hold then such an H must exist. In the spirit of [HN] we derive sufficiently many properties of this minimal counterexample. First, H has to be a core. Now, we will prove that $\omega(H) = 3$.

Claim 3 *H contains a triangle.*

Proof. Suppose to the contrary that the shortest odd cycle C of H has $k \geq 5$ vertices. Then the indicator construction with a path with 3 vertices and endpoints a and b (such a graph has unbounded girth) as the indicator will transform H into the graph H^* which is undirected, has no loops and contains more edges than H (consider some chord of C). Given a graph G of odd-girth at least l. The transformed graph $^*G$ has girth at least $3 \cdot l > l$. The $(H^*, odd - l)$–coloring problem is NP–complete according to the assumptions. Then by Lemma 2.5 the H–coloring problem is NP–complete on the set $\{^*G; G \text{ has odd girth at least } l\}$, i.e. $(H, odd{-}l)$–coloring problem is MP–complete; which is a contradiction. □

Claim 4 *H contains no K_4.*

Proof. Suppose to the contrary that H contains K_4. Let h be any vertex of H which belongs to K_4. Construct the sub-indicator J as follows:

Consider graph $B = B(H, h, h, l, \frac{l-3}{2})$ with the vertices b_1, b_2 satisfying properties declared in Lemma 2.4. Add a new vertex a and the edge $\{b_2, a\}$, and rename the vertex b_1 to b. The sub-indicator construction (with respect to J, a and b) transforms H to the graph $H\tilde{}$ which does not contain h, but does contain a triangle in its neibhborhood. Hence $H\tilde{}$ is non–bipartite graph with fewer vertices than H and according to the assumptions the $(H\tilde{}, odd - l)$–coloring problem is NP–complete.

Let G be a given graph of girth at least l. Then $\tilde{}G$ has girth at least l too (consider that J, $\hat{H}$, G have girth at least l and $d_J(a, b) \geq \frac{l-3}{2} + 1 = \frac{l-1}{2}$). Since by Lemma 2.6 the $(H, odd - l)$–coloring problem is NP–complete; this again contradicts our assumptions. □

Claim 5 *Each edge of H belongs to a triangle.*

Proof. We will show that any two vertices of H have a common neighbor. Let h be any vertex which belongs to a triangle. Consider the sub-indicator construction with the sub-indicator J defined as follows : Consider graph $B = B(H, h, h, l, \frac{l-5}{2})$ with the vertices b_1, b_2 satisfying Lemma 2.4. Add two vertices a, c and add edges $\{a, c\}$, $\{c, b_2\}$. Rename the vertex b_1 to b. Suppose to the contrary that there exists a vertex q in H which has no common neighbor with h. Then the transformed graph $H\tilde{}$ does not contain q, but does contain a triangle containing h. Hence $H\tilde{}$ is non–bipartite graph with fewer vertices than H and according to the assumptions the $(H\tilde{}, odd - l)$–coloring problem is NP–complete.

Let G be a given graph of girth at least l. Then $\tilde{}G$ has girth at least l too (consider that J, $\hat{H}$, G have girth at least l and $d_J(a, b) \geq \frac{l-5}{2} + 2 = \frac{l-1}{2}$). Then by Lemma 2.6 the $(H, odd - l)$–coloring problem is NP–complete; what is a contradiction.

Thus it suffices to prove that each vertex belongs to a triangle. Suppose to the contrary that a vertex p does not belong to any triangle and p is joined with some vertex s in triangle. Then p, s have a common neighbor t. Since p belongs to a triangle pst. □

Claim 6 *There is no homomorphism $S \to H$ (for the Erdös–Moser graph S given in Fig. 3).*

Proof. If such a homomorphism exists, let $u', v' \ldots$ be the images of the vertices $u, v \ldots$ of S in H. Let $B_1 = B(H, u', u', l, \frac{l-3}{2})$ with b_1^1, b_2^1 satisfy Lemma 2.4. Let $B_2 = B(H, v', v', l, \frac{l-3}{2})$ with b_1^2, b_2^2 satisfy Lemma 2.4. Rename b_1^1 to b_1 and b_2^2 to b_2. The

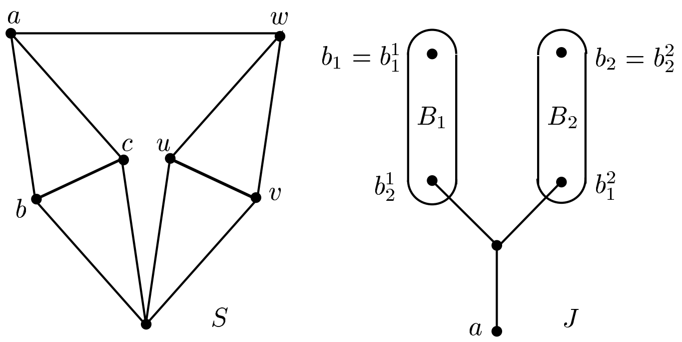

Figure 3: The graph S and the sub-indicator J.

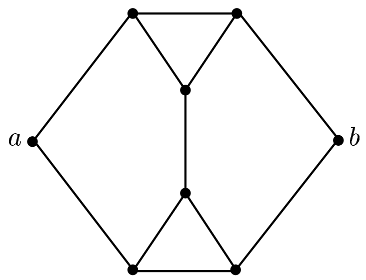

Figure 4: The graph U.

sub-indicator J of Fig. 3 yields a graph $\tilde{H}$ which contains the triangle $p'q'r'$ but does not contain the vertex w' (by Claim 4). Moreover for any given graph G of odd-girth at least l, the transformed graph $\tilde{G}$ has odd-girth at least l too. We obtain a contradiction as before. □

Suppose to the contrary that $l \leq 7$. Consider the graph U of Fig. 4. Notice that the vertices a, b do not belong to any triangle in U and that a, b can not be mapped to the same vertex in H (by Claim 6).

Consider the majority-indicator construction with the indicator U. It transforms H into the graph H^* which is undirected (by the symmetry of I), has no loops (by claim 6) and is non–bipartite (each edge of H belongs to a triangle by Claim 5, and U admits a homomorphism onto a triangle which maps a and b to an edge; thus H^* contains all edges of H). Then $(H^*, odd - (l-2))$–coloring problem is NP–complete. According to Lemma 2.7 the $(H, odd - l)$–coloring problem is also NP–complete; this is a contradiction with the choice of H. Hence Theorem 3.1 is proved. □

3.2 Symmetric graphs

In this section we show that for many graphs H we can prove a result analogous to Theorem 3.1 in the full generality.

Definition 3.2 *We say that a graph H satisfies the symmetry condition if there is an odd $k > 1$ and the vertices $v_1, v_2 \ldots v_k$ of H with the following properties : for $i = 1, 2 \ldots k$,* *(1) there is an automorphism f_i of H with*

$$f_i(v_i) = v_{(i+1)}, \quad f_i(v_{(i+1)}) = v_{(i+2)};$$

(2) there is an automorphism g_i of H with

$$g_i(v_i) = v_{(i-1)}, \quad g_i(v_{(i-1)}) = v_i,$$

where the indices are counted modulo k.

Remark 3.3 *For any l and any non-bipartite graph H which satisfies the symmetry condition, the (H,l)–coloring problem is NP-complete.*

Proof. Let l be the minimum integer such that there is a non-bipartite graph H for which the (H,l)–coloring problem is not NP-complete. I.e. we assume that for any non–bipartite graph H' and any $l' < l$, the (H', l')–coloring problem is NP–complete. We know that $l > 3$ according to Theorem 1.1. Let $v_1, v_2 \dots v_k$ be the vertices of H which guarantee the symmetry condition for H. Consider the indicator construction with the indicator $B = B(H, v_1, v_2, l, 2)$ (the graph satisfying Lemma 2.4 for H), and with specified vertices $a = b_1$, $b = b_2$. Then the new graph H^* is undirected, has no loops and is non–bipartite (contains an odd cycle $v_1, v_2 \dots v_k$). It follows easily from the symmetry condition. Thus the $(H^*, l-1)$–coloring problem is NP–complete according to the choice of l. For any graph G of girth at least $l-1$, the transformed graph $^*G$ has girth at least l. Thus the (H,l)–coloring problem is NP–complete, which is a contradiction. □

However let us note that most graphs H are asymmetric.

4 Cubic graphs

4.1 (n, r)-configurations

For $r \geq 3$ and $n > r$, let $\mathbf{G}(n, r-reg)$ denote the set of all r-regular graphs with vertex set $V = \{1, 2, ..., n\}$. We always assume that $rn = 2k$ is an even number, and so k is the number of edges in a graph. We say that *almost all r-regular graphs have a certain property Q* if the portion of graphs in $\mathbf{G}(n, r-reg)$ not possessing Q is $o(|\mathbf{G}(n, r-reg)|)$. It is not too easy to calculate $|\mathbf{G}(n, r-reg)|$ (see e.g. [B]). In order to facilitate studying $\mathbf{G}(n, r-reg)$, Bollobás [BP] (for a more detailed description see [B]) introduced a very convenient model of (n, r)-configurations.
Let $W = \bigcup_{j=1}^{n} W_j$ be a fixed set of $2k = rn$ labeled vertices, where $|W_j| = r$ for each j. An (n, r)-*configuration* F is a partition of W into k pairs of vertices, called *edges* of F. Let Φ be the *set of* (n, r)-configurations. Clearly

$$|\Phi| = N(k) = (2k-1)!!.$$

(Recall that for any positive odd integer m, $m!! = m \cdot (m-2) \cdot \ldots \cdot 3 \cdot 1$.) For $F \in \Phi$, let $\phi(F)$ be the multigraph with vertex set $V = \{1, 2, ..., n\}$ in which each i and j are joined by the same number of edges as W_i and W_j are joined in F. In other words, $\phi(F)$ is obtained from F by merging each W_i into a vertex i. Clearly, $\phi(F)$ is an r-regular multigraph on

V (sometimes, with loops). Most important is the fact that the portion of $F \in \Phi$ such that $\phi(F)$ is a simple graph is at least c_r, where $c_r > 0$, for every sufficiently large n, and each simple graph on V corresponds to the same number of (n,r)-configurations (namely, to $(r!)^n$). Thus if we prove that almost all (n,r)-configurations have a certain property Q, then almost all $r-$regular graphs have Q, as well.

4.2 Short cycles

We shall start with a well known lemma.

Lemma 4.1 *The portion of $(n,3)$-configurations F such that $\phi(F)$ contains more than $10(3l)^l$ cycles of length l or less is at most 0.1.*

Proof.
The number of different possible cycles of length k on n vertices is $\binom{n}{k}(k-1)!/2$. For every such a cycle, 6^k different sets of k edges in an $(n,3)$-configuration can form this cycle. Then the number of $(n,3)$-configurations containing given set of k edges is $(3n-1-2k)!!$. It follows that the expected number of "short" cycles per an $(n,3)$-configuration is at most

$$\sum_{k=1}^{l}\binom{n}{k}\frac{(k-1)!}{2}\cdot 6^k\cdot\frac{(3n-1-2k)!!}{(3n-1)!!}\leq$$

$$\leq \sum_{k=1}^{l}\frac{(6n)^k}{2k}\cdot(3n-2k+1)^{-k}\leq\sum_{k=1}^{l}(3k)^k/2k<(3l)^l.$$

Hence at most ten percent of all $(n,3)$-configurations can have at least $10(3l)^l$ "short" cycles each. □

4.3 Almost bipartite subgraphs

Lemma 4.2 *For at most half of $(n,3)$-configurations F, there exists a bipartition (V_1,V_2) of V such that the total number of edges induced by $\phi(F)$ in both V_1 and V_2 is less than $0.091n$.*

Proof.
Let (V_1,V_2) be a bipartition of V with $|V_1| = 0.5n-k$. If V_1 induces s edges of a cubic graphs G on V, then exactly $3(0.5n-k)-2s$ edges of G connect V_1 with V_2, and hence V_2 induces $3k+s$ edges. In particular, the total number of edges induced by both V_1 and V_2 is even if and only if k is even. Let $f(n,m,k)$ be the number of pairs $[F,(V_1,V_2)]$, such that F is an $(n,3)$-configuration and (V_1,V_2) is a partition of V with $|V_1| = 0.5n-k$ and the total number of edges induced by $\phi(F)$ in both V_1 and V_2 is equal to m. By the above, if $f(n,m,k)>0$, then $m+k$ is even. For such m and k we have

$$f(n,m,k)=\binom{n}{0.5n-k}\binom{1.5n-3k}{m-3k}(m-3k-1)!!\binom{1.5n+3k}{m+3k}(m+3k-1)!!(1.5n-m)!. \tag{1}$$

(Here $\binom{n}{0.5n-k}$ is the number of choices of V_1, $\binom{1.5n-3k}{m-3k}$ is the number of choices of $m-3k$ ends of $(m-3k)/2$ edges induced by V_1, $(m-3k-1)!!$ is the number of perfect matchings on these $m-3k$ ends, $\binom{1.5n+3k}{m+3k}$ and $(m+3k-1)!!$ are analogous numbers for V_2, and $(1.5n-m)!$ is the number of ways to put the remaining $1.5n-m$ edges so to connect V_1 with V_2.)

We will show that the function

$$f(n) = \frac{1}{(3n-1)!!} \sum_{m=0}^{\lfloor 0.091n \rfloor} \sum_{k=0} \lfloor m/3 \rfloor f(n,m,k) \tag{2}$$

tends to 0 when n tends to infinity. That would imply the validity of the lemma.

Claim 7 *For any $k < m/3 < n/30$, $f(n,m,k) \geq f(n,m,k+2)$.*

Proof. If $f(n,m,k+2) = 0$, we are done. Let $f(n,m,k+2) \neq 0$. Then

$$\frac{f(n,m,k)}{f(n,m,k+2)} =$$

$$= \frac{0.5n+k+1}{0.5n-k} \frac{(1.5n-3k)(1.5n-3k-1)\dots(1.5n-3k-5)}{(m-3k)(m-3k-1)\dots(m-3k-5)} \times$$

$$\times (m-3k-1)(m-3k-3)(m-3k-5) \frac{(m+3k+6)(m+3k+5)\dots(m+3k+1)}{(1.5n+3k+6)(1.5n+3k+5)\dots(1.5n+3k+1)} \times$$

$$\times \frac{1}{(m+3k+5)(m+3k+3)(m+3k+1)} =$$

$$= \frac{0.5n+k+1}{0.5n-k} \frac{(1.5n-3k)(1.5n-3k-1)\dots(1.5n-3k-5)}{(1.5n+3k+6)(1.5n+3k+5)\dots(1.5n+3k+1)} \times$$

$$\times \frac{(m+3k+6)(m+3k+4)(m+3k+2)}{(m-3k)(m-3k-2)(m-3k-4)} \geq$$

$$\geq \frac{0.5n+k+1}{0.5n-k} \left(1 - \frac{6k+6}{1.5n+3k+1}\right)^6 \left(1 + \frac{6k+6}{m-3k}\right)^3.$$

Under conditions of the claim, the last expression is greater than 1. This proves the claim. □

Claim 8 *For any odd $m < n/10$, $f(n,m,1) \leq f(n,m+1,0)$.*

Proof. Indeed,

$$\frac{f(n,m,1)}{f(n,m+1,0)} = \frac{\binom{n}{0.5n-1}\binom{1.5n-3}{m-3}(m-4)!!\binom{1.5n+3}{m+3}(m+2)!!(1.5n-m)!}{\binom{n}{0.5n}\binom{1.5n}{m+1}m!!\binom{1.5n}{m+1}m!!(1.5n-m-1)!} =$$

$$= \frac{0.5n \cdot (1.5n-3)!(m+1)!^2(1.5n-m-1)!^2(1.5n+3)!(m+2)(1.5n-m)!}{(0.5n+1)(m-3)!(1.5n-m)!^2(m+3)!m(m-2)\cdot(1.5n-m-1)!} =$$

$$= \frac{0.5n \cdot (m-2)(m-1)m(m+1)(m+2)(1.5n+3)(1.5n+2)(1.5n+1)}{(0.5n+1)(1.5n-m)1.5n(1.5n-1)(1.5n-2)m(m-2)(m+2)(m+3)} =$$

$$= \frac{(m-1)(m+1)(1.5n+2)(1.5n+1)}{(1.5n-m)(1.5n-1)(1.5n-2)(m+3)} < \frac{2(m-1)}{1.5n-m} < 1.$$

□

Claim 9 *For any even $m < n/10$, $f(n,m,0) \le 0.1 f(n,m+2,0)$.*

Proof. Indeed,

$$\frac{f(n,m,0)}{f(n,m+2,0)} = \frac{\binom{n}{0.5n}\binom{1.5n}{m}^2((m-1)!!)^2(1.5n-m)!}{\binom{n}{0.5n}\binom{1.5n}{m+2}^2((m+1)!!)^2(1.5n-m-2)!} =$$

$$= \frac{(m+2)^2(m+1)^2(1.5n-m)(1.5n-m-1)}{(1.5n-m)^2(1.5n-m-1)^2(m+1)^2} = \frac{(m+2)^2}{(1.5n-m)(1.5n-m-1)} < 0.1.$$

□

In view of the claims above, we derive from (2) that

$$f(n) \le \frac{1}{(3n-1)!!} \cdot n \cdot f(n, \lfloor 0.091n \rfloor, 0).$$

Denoting $\alpha = \lfloor 0.091n \rfloor / n$, we get

$$f(n) \le n \frac{\binom{n}{0.5n}\binom{1.5n}{\alpha n}^2((\alpha n-1)!!)^2(\frac{3n}{2}-\alpha n)!}{(3n-1)!!} < n \frac{2^n(1.5n)!^2}{(\frac{3n}{2}-\alpha n)!2^{\alpha n}(\frac{\alpha n}{2})!^2(3n-1)!!}.$$

Using Stirling formula we have for large n:

$$f(n) \le n \frac{2^{n-\alpha n}(\frac{3n}{2e})^{3n}}{(\frac{3n-2\alpha n}{2e})^{\frac{3n}{2}-\alpha n}(\frac{\alpha n}{2e})^{\alpha n}(\frac{3n}{e})^{1.5n}}.$$

Dividing both numerator and denominator by $(n/e)^{3n}$, we get

$$f(n) \le n \left(\frac{2^{1-\alpha}(3/2)^3}{(1.5-\alpha)^{1.5-\alpha}(\alpha/2)^\alpha 3^{1.5}} \right)^n = \left(\frac{3^{1.5}2^{-2}}{(1.5-\alpha)^{1.5-\alpha}(\alpha)^\alpha} \right)^n.$$

Observe that for $\alpha = 0.091$,

$$\phi(\alpha) = \frac{3^{1.5}2^{-2}}{(1.5-\alpha)^{1.5-\alpha}(\alpha)^\alpha} = \frac{0.25 \cdot 3^{1.5}}{1.409^{1.409} \cdot 0.091^{0.091}} \le 0.997,$$

and that the derivative

$$(\ln \phi(\alpha))' = (-(1.5-\alpha)\ln(1.5-\alpha) - \alpha \ln \alpha)' = \ln \frac{1.5-\alpha}{\alpha}$$

is positive for $0 < \alpha < 0.1$. This proves that $f(n) \to_{n\to\infty} 0$. Hence the lemma holds. □

4.4 C_{11} fails to be universal

Now we can prove the following fact.

Theorem 4.3 *For any $l \ge 3$, there exists a cubic graph of girth at least $l+1$ which has no homomorphism to C_{11}.*

Proof. By the above lemmas, for every sufficiently large n, there exists an $(n,3)$-configuration F such that:
(a) $\phi(F)$ contains at most $10(3l)^l$ cycles of length l or less;
(b) for every bipartition (V_1,V_2) of V, the total number of edges induced by $\phi(F)$ in both V_1 and V_2 is at least $0.091n$.

Let n be so large that $10(3l)^l < 10^{-5}n$. Let $H = \phi(F)$. Let M be a smallest set of vertices in H such that it has a vertex from every short cycle in H. By (a), $|M| < 10^{-5}n$. Let $H_1 = H - M$. By the construction, H_1 has no short cycles. Suppose that it has a homomorphism to C_{11}. Then $V(H_1)$ has a tripartition (W_1, W_2, W_3) such that every W_i is independent and $|W_3| \le |V(H_1)|/11$. Each vertex $w \in W_3$ is adjacent either to at most one vertex in W_1, or to at most one vertex in W_2. It follows that we can move each vertex from W_3 either to W_1 or to W_2 so that for the resulting bipartition (W_1', W_2') of $V(H_1)$, the total number of edges induced by both W_1' and W_2' is at most $|V(H_1)|/11 \le n/11$. Now, in the same manner, we can move each vertex from M to W_1' or W_2' so that on each step at most one edge is added to the subgraphs induced by W_1' and W_2'. For the resulting bipartition (V_1,V_2), the total number of edges induced by both V_1 and V_2 is at most

$$n/11 + |M| \le n(1/11 + 10^{-5}) < n(0.09091 + 0.00001) < 0.091n,$$

a contradiction to (b).

Thus, H_1 is a graph of girth at least $l+1$ with the maximum degree at most 3 which has no homomorphism to C_{11}. There are many ways to construct a cubic graph with the same girth containing H_1. One such a way is as follows. Let $k = 3|V(H_1)| - 2|E(H_1)|$. There exists a k-regular graph G_1 of girth $l+1$. Replacing each vertex in G_1 by a copy of H_1 and using for the edges of G_1 free valencies in the corresponding copies of H_1, we get the desired cubic graph G. □

Note that Theorem 4.3 strengthens a theorem of Bollobás (see [B]) who proved the result for C_{13}.

4.5 Homomorphisms to Petersen's graph and C_5

It has been conjectured [GHN] that any cubic graph of sufficiently large girth is homomorphic to C_5. In the previous sections we proved that a similar conjecture for C_{11} (instead of C_5) does not hold. Here we prove that there are girth 7 graphs which are not homomorphic to the Petersen's graph (and thus also to C_5; independently, M. Albertson and E. Moore recently constructed a cubic graph of girth 7 which is not homomorphic to C_5). We shall apply the following "no homomorphism lemma".

Definition 4.4 *The independence ratio of a graph G is*

$$i(G) = \frac{\alpha(G)}{|V(G)|}.$$

Lemma 4.5 ([AC]) *Let G, H be graphs such that H is vertex-transitive and $G \to H$. Then $i(G) \ge i(H)$.*

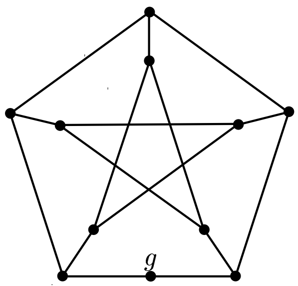

Figure 5: The graph G, $girth(G) = 5$, $G \not\to P_{10}$.

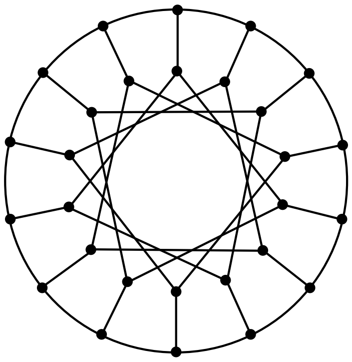

Figure 6: The graph G, $girth(G) = 7$, $G \not\to P_{10}$.

It is easy to construct a cubic graph of girth 5 which has no homomorphism to the Petersen's graph. We give once an example as a warm up. An example is depicted on Fig. 5. There is just one homomorphism from $G - \{g\}$ to the Petersen's graph up to the automorphism. This homomorphism maps the vertices a, b to the vertices which are joined with an edge. Thus there is no homomorphism from G to the Petersen's graph.

Proposition 4.6 *There exists a cubic graph of girth 7 which has no homomorphism to the Petersen's graph.*

Proof. The maximum number of independent vertices in the Petersen's graph is 4. Then $i(P_{10}) = \frac{4}{10}$. It suffices to show that there is a cubic graph G of girth 7 with $i(G) < \frac{4}{10}$. Hence $G \not\to H$ according to the Lemma 4.5. The desired graph G is depicted in Fig. 6.

It is easy to see that G is cubic and that G has girth 7. With respect to the fact that G has 28 vertices, it suffices to show that G does not have independent set with 12 vertices. The graph G is depicted with 14 vertices on the outside and 14 vertices on the inside. We will call the outside vertices *external* and the inside vertices *internal vertices* of G. The internal vertices constitute two cycles with 7 vertices. Then at most 6 of them are independent of each other, 3 in each of the cycles. Let S be an independent set with 7 external vertices. If it contains some internal vertices then all of them are from one 7-cycle. Thus it contains at most 3 internal vertices and at most 10 vertices together.

Let S be an independent set which contains 6 internal vertices. Number the external vertices according to the figure. Then no three successive odd vertices and no three successive even vertices are in S all at once. It follows that at most 5 external vertices are in S. Hence S has at most 11 vertices. No other case is possible. $\square$

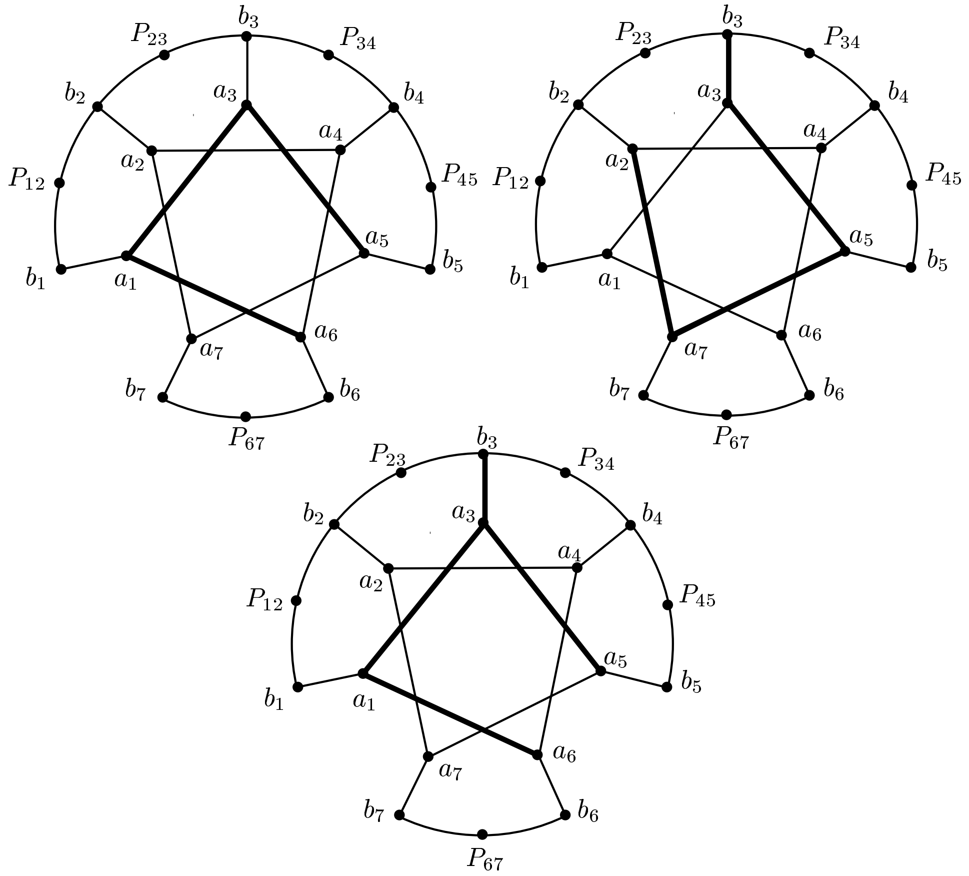

Figure 7: C_5–coloring of $I(5)$.

Lemma 4.7 ([GHN]) *Let C_k be an odd cycle with k vertices. The C_k–coloring problem is NP–complete for the class of all cubic graphs of girth k.*

We strengthen this result as follows.

Theorem 4.8 *Let $k \geq 5$ be an odd integer. Then the C_k–coloring problem is NP-complete for the class of all cubic graphs of girth $k+2$.*

Proof. Let G be a given graph of girth at least k. We will define a new graph $^*G$ with the following properties :

a) $\Delta(^*G) \leq 3$
b) $girth(^*G) = k+2$
c) $^*G \to C_k$ if and only if $G \to C_k$
d) $|V(^*G)| \leq c|V(G)|$, where c depends only on k and $|V(H)|$.

The C_k-coloring problem is NP-complete for the class of all cubic graphs of girth at least k according to Lemma 4.7. Then the C_k–coloring problem must be NP–complete for $^*G$ (i.e. for cubic graphs of girth at least $k+2$).

Construction of the graph $^*G$ (see Fig. 7 for illustration): First we shall construct a graph $I(k)$. Consider two disjoint sets of vertices $\{a_1, a_2 \ldots a_{k+2}\}$ and $\{b_1, b_2 \ldots b_{k+2}\}$. Add the edges $\{a_i, b_i\}$ and $\{a_i, a_{(i+2) \bmod (k+2)}\}$ for $i = 1, 2 \dots k+2$. For every even $i \in \{2, 4 \dots k+1\}$ join the vertices b_i and b_{i+1} by the path $P_{i,i+1}$ with $(k-1)/2$ edges (b_i is initial and b_{i+1} terminal vertex of $P_{i,i+1}$). Join the vertices b_1, b_2; b_3, b_4 respectively; by the path $P_{1,2}$; $P_{3,4}$ respectively; both with $(k-1)/2$ edges. Denote e the first edge of the path P_{12} and f the last edge of the path P_{45}. Denote by $I(k)$ the arising graph. Call touch vertex one of the vertices of $I(k)$ which is not incident with e, nor with f, and which has degree 2.

Let v be a vertex of G. Take $deg(v)$ copies of $I(k)$, for every odd $i \in \{1, 3 \dots deg(v)-1\}$ identify the edge f of i–th and $(i+1)$–th copy and for every even $i \in \{2, 4 \dots deg(v)-1\}$ identify the edge e of i–th and $(i+1)$–th copy. Call the arising graph $I(k, deg(v))$. Denote by $T(v)$ the set of all touch vertices in $I(k, deg(v))$. We call the vertices of degree 2 in $T(v)$ free touch vertices of v. Replace every vertex v of G by the graph $I(k, deg(v))$. If $\{u, v\}$ is an edge of G then conect one of the free touch vertices of v with one of the free touch vertices of u. (These used touch vertices become non-free.)

Now we will verify the three properties of $^*G$ announced above.

a) It follows clearly from the construction of $^*G$. Moreover $^*G$ can be similarly transformed to the cubic graph with the core $^*G$ (by hanging a copy of a cubic graph of degree at least $k+2$ which is homomorphic to C_k on the vertices of degree 2 in $^*G$ and deleting some edges).

b) The graph $I(k)$ is of girth $k+2$ and length of the shortest path between 2 of the touch vertices of any vertex v of G is 6. Thus $girth(^*G) = \min\{7 \cdot girth(G), k+2\} \geq \min\{7 \cdot k, k+2\} = k+2$.

c) We shall show that each homomorphism of $^*G$ into C_k maps all the touch vertices of some vertex $v \in V(G)$ to the same vertex. Number the vertices of C_k by $1, 2 \dots k$ in such a way that the successive vertices are joined with an edge. Let f be a homomorphism from $I(k)$ to C_k. We claim that in every cycle of lengh $k+2$ in $I(k)$ just 3 edges are mapped to the same edge and these 3 edges are successive. This is now the main rule. Images of the remaining edges differ from each other. Consider all the neighbours of the vertex a_3 in $I(k)$ (i.e. a_1, a_5 and b_3). At least two of them have the same image. We will consider the 3 cases. In Fig. 7 we mark with a bold line the edges $\{u, v\}$ in $I(k)$ which are mapped in f to the edge $\{f(a_3), f(a_5)\}$. We call the edges $\{a_i, a_{(i+1) \bmod (k+2)}\}$, $i = 1, 2 \dots k+2$, the inner edges of $I(k)$.

1) $f(a_1) = f(a_5) \neq f(b_3)$

Then $\{a_1, a_3\}$, $\{a_3, a_5\}$ need to be bold and $\{a_3, b_3\}$ cannot be bold. By the main rule applied to the cycle $\{a_1, a_3 \dots a_{k+1}\}$ just one of the edges $\{a_1, a_{k+1}\}$ and $\{a_5, a_7\}$ is bold. Without loss of generality let it be $\{a_1, a_{k+1}\}$. None of the other inner edges is bold. Then there are only 2 successive bold edges in the cycle $\{b_3 \dots b_4, a_4, a_6 \dots a_1, a_3\}$. This contradicts the rule.

2) $f(a_1) \neq f(a_5) = f(b_3)$

Then $\{a_3, b_3\}$, $\{a_3, a_5\}$ are bold, $\{a_1, a_3\}$ is not bold. By the main rule applied to the cycle $\{b_2 \dots b_3, a_3, a_5 \dots a_{k+1}, a_2\}$ either $\{a_5, a_7\}$ or the last edge of $P_{2,3}$ is bold. But it could not be the last edge of $P_{2,3}$, because it would make a contradiction with the main rule applied to the cycle $\{b_1 \dots b_2 \dots b_3, a_3, a_1\}$. Thus $\{a_5, a_7\}$ is bold. Now we will apply the main rule to the cycle $\{a_1, a_3 \dots a_{k+1}\}$. We get that $\{a_7, a_{9 \bmod k}$ is bold. Then there are 4 succesive bold edges in the cycle $\{b_2 \dots b_3, a_3, a_5 \dots a_{k+2}, a_2\}$. This contradicts the main rule.

3) $f(a_1) = f(a_5) = f(b_3)$

Then $\{a_3, b_3\}$, $\{a_1, a_3\}$ and $\{a_3, a_5\}$ need to be bold. By the main rule applied to the cycle $\{a_1, a_3 \dots a_{k+1}\}$ just one of the edges $\{a_1, a_{k+1}\}$ and $\{a_5, a_7\}$ is bold. Without loss of generality let it be $\{a_1, a_{k+1}\}$. No one of the other inner edges is bold. According to the rule applied to the cycle $\{b_3 \dots b_4, a_4, a_6 \dots a_1, a_3\}$ no other edge of this cycle is bold and coloring of the rest is unique. From the cycles $\{b_2 \dots b_3 \dots b_4, a_4, a_2\}$, $\{b_2 \dots b_3 \dots b_4, a_4, a_2\}$ and $\{b_1 \dots b_2 \dots b_3, a_3, a_1\}$ we get the unique coloring (up to symmetry of $I(k)$) of the path $\{b_1 \dots b_2 \dots b_3 \dots b_4 \dots b_5\}$ which is increasing or decreasing. For us of most importance is the fact that the colors of the vertices of the edge e or f give unambiguously the color of the touch vertex (and vise versa).

Let g be a homomorphism from $^*G$ to C_k. We know that the touch vertices of any vertex v has the same color. For every edge $\{u, v\}$ of G there is an edge between a touch vertex of u and a touch vertex of v. Thus g gives a homomorphism from G to C.

Conversely let f be a homomorphism from G to C. For any $w \in V(^*G)$, put $g(w) = f(v)$, $w \in T(v)$. Then g can be similarly extended to the rest of vertices (these vertices are the in $I(k, deg(v)))$.

d) It follows easily from the construction of $^*G$. □

5 Degenerate graphs

If we replace the cubic graphs by k–degenerate and even 2–degenerate graphs then we have a nearly complete solution.

Theorem 5.1 *Let H be a non-bipartite graph. Then the H-coloring problem is NP–complete for the class of all $\omega(H)$–degenerate graphs.*

Proof.

Let H be a non-bipartite graph. We will use the indicator construction for $l = 3$. Let $k = \omega(H)$. Let I be the graph obtained from two copies of K_{k+1} in the following way: denote a, c two arbitrary vertices of one copy of K_{k+1} and b, d two arbitrary vertices of the other copy of K_{k+1}. Delete the edges $\{a, c\}$ and $\{b, d\}$ and add the edges $\{a, d\}$ and $\{c, b\}$. Then I is homomorphic to H because it is homomorphic to K_k. The indicator construction with I (and vertices a and b) transforms H into the graph H^* which is undirected by the symmetry of I. We will show that H^* has no loops.

Assume to the contrary that there is a homomorphism $f : I \to H$ which maps a and b to the same vertex v. The vertex a is mapped to a vertex of a k–clique in H, because f maps a clique of I to a clique of H with the same number of vertices. The vertex $f(c)$ must be joined with each vertex of this k–clique, except $f(a)$. But $f(a) = f(b)$ and $f(c)$ is joined with $f(b)$. Since $f(c)$ is joined with each vertex of a k–clique in H and it constitutes with this k–clique a $(k + 1)$–clique in H. This is a contradiction.

For proving that H^* is not bipartite we will consider the two following cases: If $k = 2$ then H^* contains all the edges of H. Hence it must be non–bipartite. If $k \geq 3$ then the indicator I is homomorphic to H in such a way that a, b are mapped to an arbitrary edge of a k–clique. Since H^* contains K_k. Thus H^* is non–bipartite.

Let G be an arbitrary graph. The H^*–coloring problem is NP-complete for G according to Theorem 1.1. Then the H-coloring problem is NP–complete for the transformed graph G^* according to Lemma 2.5. It remains to verify that the new graph $^*G$ is k–degenerate. We will define the numbering of vertices of $^*G$ in the following way : first we number the original vertices of G arbitrarily (this set of vertices is independent in $^*G$), afterwards we number the rest of vertices arbitrarily (these vertices are the in copies of I - they have degree just k). This numbering demonstrates that $^*G$ is k–degenerate, i.e. $\omega(H)$–degenerate. □

6 Concluding remarks

Related problem was considered in [NZ], where the following statement was proved: Given two positive integers k and m. If G is a graph of girth $g > 2^{k+2}(4km)^{4km-1} + 2(k+1)$ then any partial k–tree homomorphic to G is also homomorphic to the odd cycle C_{2m+1}.

Let us note that the cubic graphs which do not map to C_{11} have fractional chromatic number at least $\frac{13}{6} = 2 + \frac{1}{6}$. (It follows from a result of Bollobás: if the maximum independent set of a graph G is at most $\frac{6 \cdot |V(G)|}{13}$ then the fractional chromatic number of G is at least $\frac{13}{6}$.) Moreover we can prove that the star chromatic number of these graphs is at least $\frac{11}{5} = 2 + \frac{1}{5}$. One can check easily that the star chromatic number of a graph G is at most $2 + \frac{1}{m}$ if and only if there is a homomorphism $G \to C_{2m+1}$.

It is not clear how to prove that Petersen's graph–coloring problem is NP–hard for cubic graphs of girth 5 (a problem mentioned already in [GHN]).

On the positive side it follows from [C1],[C2] that the following problem is polynomial for any k and any graph H:

Instance: a partial k-tree G.

Question: does there exists a homomorphism $G \to H$?

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