On Bounded Treewidth Duality of Graphs

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ABSTRACT

For a graph H, the H-coloring problem is to decide whether or not an instance graph G is homomorphic to H. The H-coloring problem is said to have bounded treewidth duality if there is an integer k such that for any graph G which is not homomorphic to H, there is a graph F of treewidth k which is homomorphic to G but not homomorphic to H. It is known that if the H-coloring problem has bounded treewidth duality then it is polynomial time decidable. We shall prove in this paper that for any integers m, k, there is an integer n_0 such that if G is a graph of girth $\geq n_0$ then any graph F of treewidth k homomorphic to G is also homomorphic to C_{2m+1}. It follows from this result that for non-bipartite graphs H, the H-coloring problems do not have bounded treewidth duality. We also present some classes of directed graphs H for which the H-coloring problems do not have bounded treewidth duality. In particular, there are oriented cycles H for which the H-coloring problems do not have bounded treewidth duality. This answers a question of Hell and Zhu [16].

Keywords: Graph homomorphisms, bounded treewidth duality, H-coloring, girth

1. INTRODUCTION

Any coloring (or scheduling) problem may be expressed as an appropriate graph-homomorphism problem. Recall that given two graphs $G = (V, E)$ and $G' = (V', E')$, a *homomorphism* $G \to G'$ is any map $f : V \to V'$

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satisfying

$$(x, y) \in E \Longrightarrow (f(x), f(y)) \in E'.$$

We say G is *homomorphic* to H if there exists a homomorphism from G to H. The question whether G is homomorphic to K_k (the complete graph with k vertices) is equivalent to the question whether $\chi(G) \leq k$. By fixing a target graph H (instead of the complete graph K_k) this leads to the following H-coloring problem:

Instance: Graph G

Question: Does there exist a homomorphism $G \to H$?

The complexity of the H-coloring problem was completely solved for undirected graphs by Hell and Nešetřil [10], while for directed graphs the problem seems to be presently intractable: only partial results are known, [2, 3, 4, 7, 8, 9, 11, 12, 13, 21, 24]. This inherent difficulty of the H-coloring problem for directed graphs was recently related to the constraint satisfaction problem and Datalog descriptions by Feder and Vardi [6].

The following notion has been suggested in [11] as a possible approach to the polynomial time decidability of H-coloring problems:

Definition. An H-coloring problem is said to have *bounded treewidth duality* if there exists a positive integer $k = k(H)$ such that the following holds:

G is homomorphic to H if and only if every graph F homomorphic to G with treewidth $\leq k$ is also homomorphic to H.

In other words: G fails to be H-colorable if and only if there exists a graph F with treewidth $\leq k$ such that $F \to G$ and F fails to be H-colorable.

Denoting the non-existing of homomorphisms by $\not\to$ and treewidth of G by $tw(G)$, we can schematically express the duality as

$$G \not\to H \quad \Longleftrightarrow \quad \exists F, F \to G, F \not\to H, \mathrm{tw}(F) \leq k.$$

The following has been proved independently in [6] and [11]:

Theorem 1. Every H-coloring problem with bounded treewidth duality is polynomial time decidable.

Presently Theorem 1 is the strongest tool for proving the polynomial time decidability of H-coloring problems. In fact, presently all known polynomial time decidable H-coloring problems are covered by Theorem 1 [6, 11, 15, 16]. On the other hand, if we assume $P \neq NP$, then all NP-complete H-coloring problems do not possess bounded treewidth duality. The purpose of this paper is to establish this directly (i.e., without assuming $P \neq NP$) for all instances of undirected graphs H and some classes of directed graphs. In particular, we prove the following result:

Theorem 2. For an undirected graph H, the H-coloring problem has no bounded treewidth duality if and only if H contains an odd cycle.

Homomorphism duality theorems may be viewed as an universal approach to min-max theorems and "good characterizations" (i.e., to the class $NP \cap coNP$, see [19, 20]). In [6, 11, 13, 15, 16, 19], several homomorphism duality theorems are listed. These examples are all examples of bounded treewidth dualities. However to prove that bounded treewidth duality does not hold for a particular H-coloring problem appears more difficult. A solution of this problem is the subject of this paper. The proof below may have some interesting combinatorial consequences. Particularly, we have the following corollary:

Corollary. For every k there exists a positive integer $g(k)$ such that if G is a graph of girth $\geq g(k)$ then any graph F of treewidth $\leq k$ which is homomorphic to G has chromatic number ≤ 3.

The paper is organized as follows: In Section 2 we deal with undirected graphs and prove Theorem 2. In Section 3 we modify our technique to include some classes of directed graphs. The central result here is Theorem 3, relating bounded treewidth duality for undirected and directed graphs. Section 4 contains some consequences and concluding remarks.

2. TREEWIDTH AND GIRTH FOR UNDIRECTED GRAPHS

A *tree decomposition* of a graph H is a pair (T, S) such that T is a tree, and $S = \{X_t \subseteq V(H) : t \in V(T)\}$ is a family of subsets of $V(H)$ indexed by the vertices of T satisfying

- for every edge (x, y) of H there is a $t \in V(T)$ such that $x, y \in X_t$, and
- for any vertex x of H, the set $\{t \in V(T) : x \in X_t\}$ induces a connected subgraph of T.

We call X_t the set (of vertices) associated to the vertex t of T. For a tree decomposition (T, S), the associated sets will be denoted by X_t; for a tree decomposition (T^{ij}, S^{ij}) the associated sets will be denoted by X_t^{ij}; etc.

The *width* of such a tree decomposition is $\max\{|X_t| - 1 : t \in V(T)\}$. A tree decomposition (T, S) of H of width $\leq k$ is also called a k-tree decomposition of H. The *treewidth* of a graph H is the minimum width of a tree decomposition of H. A graph of treewidth $\leq k$ is also called a *partial k-tree* [18].

Given a tree decomposition (T, S) of a graph H, if A is a subgraph of H, then the *restriction of* (T, S) *to* A is a pair (T', S'), where T' is the subgraph of T induced by vertices t with $X_t \cap V(A) \neq \emptyset$ and $S' = \{X_t \cap V(A) : t \in T'\}$. It is easy to verify that if A is connected, then T' is also connected (hence is a tree) and (T', S') is a tree decomposition of A. If A is not connected, T' may not be connected. However, by arbitrarily adding some edges to connect components of T', the resulting tree together with the associated sets will be a tree decomposition of A. Thus without loss of generality, we may assume that the restriction of a tree decomposition of a graph H to a subgraph A is a tree decomposition of A.

Theorem 1. Given two positive integers k and m. If G is a graph of girth $n > 2^{k+2}(4km)^{4km-1} + 2(k+1)$ then any partial k-tree homomorphic to G is also homomorphic to the odd cycle C_{2m+1}.

We now proceed to prove this theorem.

Let G be a graph of girth $n > 2^{k+2}(4km)^{4km-1} + 2(k+1)$, let H be a partial k-tree, and let $h : H \to G$ be a homomorphism. If $|V(H)| \leq n$, then $h(H)$ is a tree and hence is homomorphic to C_{2m+1}. By composition of homomorphisms we obtain a homomorphism of H to C_{2m+1}. We now assume that $|V(H)| > n$, and use induction on the number of vertices of H. We assume that any partial k-tree H' with fewer vertices than H which is homomorphic to G is also homomorphic to C_{2m+1}. Let (T, S) be a tree decomposition of H of width $\leq k$.

For a vertex t of T, we denote by $T_1(t), T_2(t), \cdots, T_{a(t)}(t)$ the connected components of $T - t$. Each $T_i(t)$ is called a *branch* of T at t. For $i = 1, 2, \cdots, a(t)$, we denote by $A_i(t)$ the set $\{x \in X_v : v \in T_i(t)\}$, which is a subset of $V(H)$. If $I \subseteq \{1, 2, \cdots, a(t)\}$ then we denote by $A_I(t)$ the union of $\{A_i(t) : i \in I\}$. It follows from the definition of tree decomposition that if $I, J \subset \{1, 2, \cdots, a(t)\}$ and $I \cap J = \emptyset$ then $A_I(t) \cap A_J(t) \subset X_t$. Moreover for $I = \{1, 2, \cdots, a(t)\}$ we have $V(H) - X_t \subset A_I(t)$.

Lemma 2. There is a vertex t of T, and a set $I \subset \{1, 2, \cdots, a(t)\}$ such that $(n/2) - (k+1) \leq |A_I(t)| \leq n - (k+1)$.

Proof. Assume that the lemma is false. Let t_0 be an arbitrary vertex of T. Then for any set $I \subset \{1, 2, \cdots, a(t_0)\}$, $|A_I(t_0)|$ is either less than $(n/2)-(k+1)$ or greater than $n-(k+1)$. In particular, for each $i \in \{1, 2, \cdots, a(t_0)\}$, $|A_i(t_0)|$ is either less than $(n/2) - (k+1)$ or greater than $n - (k+1)$. Assume first that $|A_i(t_0)| < (n/2) - (k+1)$ for all $i \in \{1, 2, \cdots, a(t_0)\}$. Let $I_j = \{1, 2, \cdots, j\}$. Then $|A_{I_1}(t_0)| = |A_1(t_0)| < (n/2) - (k+1)$, and $|A_{I_2}(t_0)| \leq |A_{I_1}(t_0)| + |A_2(t_0)| < n - (k+1)$. If $|A_{I_2}(t_0)| \geq (n/2) - (k+1)$ then the lemma would be true with $t = t_0$ and $I = \{1, 2\}$. Thus $|A_{I_2}(t_0)| < (n/2) - (k+1)$. Similarly we can show that $|A_{I_j}(t_0)| < (n/2) - (k+1)$ for $j = 3, 4, \cdots, a(t_0)$. This is contrary to the fact that $|A_{I_{a(t_0)}}(t_0)| =$

$|\cup_{i=1}^{a(t_0)} A_i(t_0)| \geq |V(H) - X_t| \geq n-(k+1)$. Therefore $|A_i(t_0)| > n-(k+1)$ for some i.

Without loss of generality we assume that $|A_1(t_0)| > n-(k+1)$. Let t_1 be the neighbour of t_0 in $T_1(t_0)$. Let $T_i(t_1) \subset T_1(t_0)$ $(i = 1, 2, \cdots, \ell)$ be those branches of T at t_1 which does not contain t_0. It is obvious that $| \cup_{i=1}^{\ell} A_i(t_1)| \geq |A_1(t_0)| - (k+1) \geq n/2$. Similar to the argument in the previous paragraph, we can show that there is an $i \in \{1, 2, \cdots, \ell\}$ such that $|A_i(t_1)| > n-(k+1)$.

Repeating the argument, we shall find a sequence of distinct vertices $t_0, t_1, t_2, \cdots$, such that for each t_i, there is a branch, say $T_j(t_i)$, of T at t_i for which $|A_j(t_i)| > n-(k+1)$. This is contrary to the fact that T is a finite tree. Therefore the lemma is true. ∎

Fix a vertex t^* of T and a set $I \subset \{1, 2, \cdots, a(t^*)\}$ such that $(n/2) - (k+1) \leq A_I(t^*) \leq n-(k+1)$.

Let $A = A_I(t^*) \setminus X_{t^*}$, $B = A_I(t^*) \cup X_{t^*}$ and $C = V(H) \setminus B$. We shall also denote by A, B, C the subgraphs of H induced by A, B, C respectively.

It follows from the definition of a tree decomposition that there is no edge connecting a vertex of A to a vertex of C. Indeed, if $a \in A$, $c \in C$ and $(a, c) \in E(H)$, then there is a vertex t' of T such that $a, c \in X_{t'}$. Since $c \notin B$, we conclude that $t' \notin \cup_{i \in I} V(T_i(t^*))$. On the other hand, since $a \in A$, there is a vertex t'' in some branch $T_i(t^*)$ $(i \in I)$ such that $a \in X_{t''}$. Thus t' and t'' belong to two different branches at t^*. Therefore the path of T connecting t' and t'' contains the vertex t^*. This implies that the subgraph of T induced by the set $U = \{t \in V(T) : a \in X_t\}$ is not connected, because $t', t'' \in U$ and $t^* \notin U$. This contradicts the fact that (T, S) is a tree decomposition of H.

Let $R_1, R_2, \cdots, R_q$ be the connected components of $B - X_{t^*}$. For each R_i, let $Q(R_i)$ be the set of vertices of X_{t^*} which are attached to some vertices of R_i, i.e.,

$$Q(R_i) = \{x \in X_{t^*} : \exists y \in R_i, (x, y) \in E(B)\}.$$

We denote by R_i' the subgraph of H induced by $R_i \cup Q(R_i)$.

Lemma 3. For each $1 \leq i \leq q$, we have $|Q(R_i)| \leq k$.

Proof. Let R_i'' be the graph obtained from R_i' by adding edges to connect all pairs of vertices of $Q(R_i)$. Obviously R_i'' has treewidth $\leq k$. Indeed the k-tree decomposition of R_i' (inherited from H) is a k-tree decomposition of R_i''. Let $R_i^{\#}$ be the graph obtained from R_i'' by contracting all the edges of R_i. It is easy to see that the contraction of edges will not increase the treewidth of a graph. Therefore $tw(R_i^{\#}) \leq tw(R_i'') \leq k$. However $R_i^{\#}$ is a complete graph, with all the vertices of R_i contracted to a single vertex. Therefore $R_i^{\#}$ has cardinality $\leq k+1$, and hence $|Q(R_i)| \leq k$. ∎

For each of the graphs R'_i, consider the image $h(R'_i)$ of R'_i in G under the homomorphism h. As $|h(R'_i)| \leq |R'_i| \leq |R_i| + |Q(R_i)| \leq |A| + k < n$, we conclude that $h(R'_i)$ is a tree.

We shall (more or less) replace R'_i in H by the image $h(R'_i)$ to obtain a smaller graph H'. We want the new graph H' to have treewidth $\leq k$ and be homomorphic to G, so that we can use the induction hypothesis to conclude that H' is homomorphic to C_{2m+1}. Moreover, we want the new graph H' to be a homomorphic image of H. This would imply that H is homomorphic to C_{2m+1}.

Explicitly, for each R'_i we define the graph G_i on the vertex set $Q(R_i) \cup (h(R'_i) - h(Q(R_i))$ to consist of the following edges (u, v):

- $u, v \in Q(R_i)$ and $(u, v) \in E(R'_i)$;
- $u = h(x), v = h(y)$ and $(x, y) \in E(R'_i)$;
- $u \in Q(R_i), v = f(y)$ and $(u, y) \in E(R'_i)$.

Note that although $R'_i - Q(R_i) = R_i$ is connected, the subgraph of G_i induced by $G_i - Q(R_i)$ may not be connected. Let R_{ij}, $j = 1, 2, \cdots, q_i$, be the connnected components of $G_i - Q(R_i)$. Then each vertex of $Q(R_i)$ is adjacent to at most one vertex in each of the components R_{ij}, for otherwise $h(R'_i)$ would contain a cycle. Let $Q(R_{ij})$ be the set of vertices of $Q(R_i)$ which are attached to some vertices of R_{ij}, and let G_{ij} be the subgraph of G_i induced by $Q(R_{ij}) \cup R_{ij}$.

We now glue all the graphs G_{ij} and the subgraph of H induced by $C \cup X_{t^*}$ together, by identifying different copies of vertices of X_{t^*}, i.e., identify each $Q(R_{ij})$ with the corresponding vertices in X_{t^*} (recall that $Q(R_{ij})$ is a subset of X_{t^*}). We denote the resulting graph by H'. To be precise the vertex set of H' is

$$C \cup X_{t^*} \cup_{i,j} (G_{ij} - Q(R_{ij})),$$

and with edge set consisting all the edges carried over from the different pieces.

Observe that by our construction there exists a homomorphism $h_0 : H \to H'$ and $H' \to G$. The proof would be complete if we can show that $H' \to C_{2m+1}$. For this purpose, We shall show that $tw(H') \leq k$. First we need a (surely folkloric) lemma:

Lemma 4. Let $T = (V, E)$ be a tree. Let X_0 be a set of end vertices of T and assume that $2 \leq |X_0| \leq k$. Then there exists a k-tree decomposition $(\bar{T}, \bar{S})$ of T such that there exists $t_0 \in V(\bar{T})$ with $\bar{X}_{t_0} = X_0$. Such a tree decomposition is called rooted, and t_0 is the root.

Proof. We use induction on the number of vertices of T. If $|V(T)| \leq k+1$ then the lemma is true, as we may take a single vertex tree as the tree decomposition of T. Assume now $|V(T)| \geq k+2$, and the lemma is true for all smaller trees. If $|X_0| = 1$, there is nothing to prove. Otherwise let $a, b \in X_0$ and let u be a vertex on the path of T connecting a, b. Let $T^1, T^2, \cdots, T^d$ be the connected components of $T - u$, and let $T'^i = T^i + u$. Then $|(X_0 \cup \{u\}) \cap T'^i| \leq k$. Using the induction hypothesis on T'_i with $X_0^i = (X_0 \cup \{u\}) \cap T'^i$, we obtain tree decompositions $\bar{T}_i$ of width $\leq k$ with root t_0^i associated to the set X_0^i. Take the disjoint union of all these trees, add a new vertex t_0, and edges $t_0 t_0^i$ $(i = 1, 2, \cdots, d)$, and associate to t_0 the set $X_0 \cup \{u\}$. It is routine to verify that the resulting system is the required tree decomposition of T. ∎

Note that if T' is obtained from T by adding edges that connect vertices of X_0, then the tree decompostion of T obtained in the above lemma is also a tree decomposition of T'. Also note that if we delete those edges of G_{ij} which connect vertices of $Q(R_{ij})$, we obtain a tree T in which all vertices in $Q(R_{ij})$ are end vertices. Therefore each of the graphs G_{ij} has a rooted k-tree decomposition (T^{ij}, S^{ij}) with root t_{ij} such that $X^{ij}_{t_{ij}} \supseteq Q(R_{ij})$.

Let $(T^{\#}, S^{\#})$ be the restriction of the tree decomposition (T, S) of H to the subgraph induced by $C \cup X_{t^*}$, which still contains the vertex t^* with associated set X_{t^*}. Then the tree decomposition (T', S') of H' will be built as the disjoint union of the tree decompositions $(T^{\#}, S^{\#})$ and (T^{ij}, S^{ij}), where the vertex t^* of $V(T^{\#})$ will be connected to each root t_{ij}.

We now show that the resulting pair is really a tree decomposition of H' of width $\leq k$. It follows from the definition that T' is a tree, and $|X'_t| \leq k+1$ for all $X'_t \in S'$, and every edge of H' is contained in some sets X'_t. It remains to check that for any $v \in V(H')$, the subgraph of T' induced by $W_v = \{t \in T' : v \in X'_t\}$ is connected. First we note that $W_v \cap V(T^{\#})$ induces a connected subgraph (possibly empty) of $T^{\#}$, and $W_v \cap V(T^{ij})$ induces a connected subgraph (again possibly empty) of T^{ij}, because $(T^{\#}, S^{\#})$ and (T^{ij}, S^{ij}) are all tree decompositions of some graphs. We need to show that the union of these subgraphs is also connected.

If $v \in C$, then $W_v \cap V(T^{ij}) = \emptyset$ for all i, j. In other words, $W_v \subset T^{\#}$. Therefore W_v is connected. If $v \in V(G_{ij}) - Q(R_{ij})$, then $W_v \subset V(T^{ij})$, and hence W_v is connected. Otherwise $v \in X_{t^*}$. Then $t^* \in W_v \cap V(T^{\#})$. Furthermore for each T^{ij} either $v \in Q(R_{ij})$ and hence $W_v \cap V(T^{ij})$ contains the root t_{ij}, or $v \notin Q(R_{ij})$ and $W_v \cap V(T^{ij}) = \emptyset$. Since t^* is adjacent to all the roots t_{ij}, we see that W_v is connected. Thus $tw(H') \leq k$.

We now show that $H' \rightarrow C_{2m+1}$. Clearly we may assume without loss of generality that H' is a core (i.e., every homomorphism $H' \rightarrow H'$ is an automorphism). If $|V(H')| < |V(H)|$, then by the induction hypothesis, we have $H' \rightarrow C_{2m+1}$ (as $tw(H') \leq k$ and $H' \rightarrow G$). Thus we assume that $|V(H')| = |V(H)|$, and we consider two cases:

Case 1: Suppose one of the graphs G_{ij} contain a path, say P, from x to y of length $\geq 2m$ all of whose inner vertices (i.e., $\neq x, y$) belong to R_{ij} and have degree 2 in G_{ij}.

Let H'' be the graph obtained from H by deleting all the inner vertices of P. Then $|V(H'')| < |V(H)|$ and we may use the induction assumption to obtain a homomorphism $h'' : H'' \to C_{2m+1}$. However any such homomorphism can be extended to a homomorphism $h' : H' \to C_{2m+1}$.

Case 2: Suppose that no G_{ij} contains a path P as in Case 1.

By our above assumption all end-vertices of each tree G_{ij} belong to X_{t^*} (as otherwise H' is not a core). However, then we can think of each G_{ij} as being obtained from a tree, say T_{ij}, without degree 2 vertices by subdividing each of its edges by at most $2m-1$ points. If T_{ij} has k_{ij} end vertices (which form a subset of X_{t^*}), then it has at most $2(k_{ij}-1)$ edges. Therefore each G_{ij} has at most $2(k_{ij}-1)2m \leq 4km$ vertices.

Recall that $Q(R_{ij})$ is the set of end vertices of the tree G_{ij} which are contained in X_{t^*}. For each subset S of X_{t^*}, let $\mathcal{F}_S$ be the family of those G_{ij} for which $Q(R_{ij}) = S$. We claim that $|\mathcal{F}_S| \leq (4km)^{4km-2}$. Suppose to the contrary that $|\mathcal{F}_S| > (4km)^{4km-2}$. We label the vertices of all the trees with labels $\{1, 2, \cdots, 4km\}$ so that distinct vertices of a tree receive different labels, the vertices in S receive the same labels in each of the trees, and the other vertices of the trees are labeled arbitrarily. Since there are only $(4km)^{4km-2}$ labelled trees on $4km$ vertices, we conclude that there are two trees, say $F_1, F_2 \in \mathcal{F}_S$, which are isomorphic labeled trees. By identifying the corresponding vertices of F_1 and F_2, we obtain a homomorphism of H' to a proper subgraph of H', contrary to our assumption that H' is a core. Therefore $|\mathcal{F}_S| \leq (4km)^{4km-2}$ for each subset S of X_{t^*}. This then implies that there are at most $2^{k+1}(4km)^{4km-2}$ trees G_{ij} in total, and hence the number of vertices of all these trees is less than $2^{k+1}(4km)^{4km-1}$. Thus $|B| \leq 2^{k+1}(4km)^{4km-1} < n/2 - (k+1)$, contrary to our assumption. This completes the proof of Theorem 1.

Corollary. If a graph G has treewidth $\leq k$ and girth $> 2^{k+2}(4km)^{4km-1} + 2(k+1)$, then G is homomorphic to C_{2m+1}.

Corollary. An undirected graph H has bounded treewidth duality if and only if H is bipartite.

Proof. It is known that bipartite graphs have 2-treewidth duality. (Suppose H is a bipartite graph. If G is homomorphic to H then certainly every partial 2-tree homomorpic to G is also homomorphic to H. If G is not homomorphic to H, then there is a cycle which is homomorphic to G but not homomorphic to H, and cycles have treewidth 2.) Now we prove that any non-bipartite graph H does not have k-treewidth duality for any integer k. Suppose H contains an odd cycle C_{2m+1}. Let G be a graph with $\chi(G) > \chi(H)$ and girth at least $2^{k+2}(4km)^{4km-1} + 2(k+1)$. Then G is not homomorphic to H, because $\chi(G) > \chi(H)$. On the other hand any partial k-tree homomorphic to G is homomorphic to C_{2m+1} by Theorem 1, and hence is homomorphic to H. Therefore H does not have k-treewidth duality. ∎

3. THE METHOD OF INDICATORS FOR DIRECTED GRAPHS

Let (I, a, b) be a directed graph with two specified vertices a, b. We call (I, a, b) an *indicator*. The following indicator construction is a useful tool in various combinatorial (and algebraical) situations:

Given a directed graph $G = (V, E)$, we denote by $G * (I, a, b)$ (or $G * I$) the directed graph obtained from G by replacing each arc $(x, y) \in E$ by a copy of the indicator I in such a way that a is identified with x and b is identified with y. Explicitly: $V(G * I) = V \cup (V(I) - \{a, b\}) \times E$ and $E(G * I)$ consists of arcs of the form:

- $((x, e), (y, e)) : (x, y) \in E(I), e \in E$;
- $(x, (y, e)) : (a, y) \in E(I), x$ is the tail of e;
- $((x, e), y) : (x, b) \in E(I), y$ is the head of e.

Obviously for every arc $e = (x, y) \in E(G)$ the vertices $\{x, y\} \cup \{(z, e) : z \in V(I), z \neq a, b\}$ induce a subgraph of $G * I$ isomorphic to I. We denote this subgraph by $I_{x,y}$ (or $I_e, e = (x, y)$). If G is an undirected graph, then $G * I$ means $G' * I$ where G' is the symmetric orientation of the graph G (i.e., replace each edge of G by two opposite arcs). The treewidth of an oriented graph is defined to be the treewidth of its underlying graph.

We call an indicator I *good* if for every directed graph G the only homomorphisms $I \to G * I$ map I identically onto I_e for an arc $e \in E(G)$. We shall further assume the goodness of all indicators in the remaining part of this section without explicitly mentioning it. Of course in all concrete cases this goodness has to be proved.

The following result summarizes the usefulness of the indicator construction for our purposes:

Theorem 1. Let I be a good indicator. If the H-coloring problem is NP-complete then the $H * I$-coloring problem is NP-complete. Moreover the treewidth of $G * I$ is $\leq \max\{tw(G), tw(I + (a, b))\}$.

Proof. Since I is a good indicator, we have $G * I \to H * I$ if and only if $G \to H$. Therefore the $H * I$-coloring problem is NP-complete in case the H-coloring problem is NP-complete.

To see that the treewidth of $G*I$ is less than or equal to $\max\{tw(G), tw(I+(a, b)\}$, let a tree decomposition (T, S) of G of width $tw(G)$, and tree decompositions (T^e, S^e) of copies of $I_e + e$ of width $tw(I + (a, b))$ be given.

For each arc $e \in E(G)$, let $t(e) \in V(T)$ and $t^e(e) \in V(T^e)$ be such that $X_{t(e)} \supseteq \{e\}, X^e_{t^e(e)} \supseteq \{((a,e),(b,e))\}$. Form a new tree $\bar{T}$ by taking disjoint union of T and T^e, $e \in E(G)$, and adding edges $(t(e), t^e(e))$, $e \in E(G)$. Let $\bar{S}$ be the union of S and all S^e. One can check that $(\bar{T}, \bar{S})$ is a tree decomposition of $G * I$ of width $\max\{tw(G), tw(I + (a,b))\}$. ∎

We shall concentrate on good indicators which are paths as undirected graphs. We call them *indicator paths.* For example, let I be the path obtained from P_1 and P_2 in Fig. 1(a) by identifying their terminal vertices, i.e., the top vertices, and let a, b be their initial vertices, then it is straightforward to verify that (I, a, b) is a good indicator path.

In this section we prove the following

Theorem 2. Let (I, a, b) be a good indicator path. Let H be a non-bipartite undirected graph. Then the $H*I$-coloring problem has no bounded treewidth duality.

Further corollaries and particular cases are mentioned in the next section. Theorem 2 is a consequence of the following more technical statement which is patterned after Theorem 1 in Section 2.

Theorem 3. Let $k \geq 2, m \geq 3$ be fixed integers. Let (I, a, b) be a good indicator path. Then there exists $n = n(k, m, I)$ with the following property:

If G is a simple directed graph (i.e, containing no opposite arcs) of girth $\geq n$ then every directed graph with treewidth $\leq k$ which is homomorphic to $G * I$ is homomorphic to $C_{2m+1} * I$.

We prove this theorem (for directed graphs) along the same line as Theorem 1 in Section 2 (for undirected graphs). Thus we stress only the differences.

Proof. Let integers k, m and a good indicator path (I, a, b) be fixed. Put $n = n(k, m, I) = (2^{k+2}(4km)^{4km-1} + 2(k+1)) \cdot |I|$. Let G be a graph of girth n. Let H be any graph of treewidth $\leq k$ which is homomorphic to $G * I$. We prove by induction on $|V(H)|$ the following statement:

For any homomorphism $h : H \to G * I$ there exists a homomorphism $g : H \to C_{2m+1} * I$ such that for every $x \in V(H), h(x) \in V(G)$ if and only if $g(x) \in V(C_{2m+1})$. (i.e., calling the vertices of G in $G * I$ *old vertices*, g and h map the same vertices to old vertices both in $G * I$ and $C_{2m+1} * I$).
(*)

Given $h : H \to G * I$, denote by $h(H)_G$ the minimal subgraph G' of G such that the image $h(H)$ of H is a subgraph of $G' * I$. (Equivalently $h(H)_G$ consists of those edges $e \in E(G)$ for which $h(V(H)) \cap V(I_e) \neq \emptyset$. We could think of $h(H)_G$ as the G-shadow of $h(H)$.) If $|V(H)| < n$ then $h(H)_G$ does not contain a cycle and thus $h(H)_G$ maps to C_{2m+1} and hence H maps to $C_{2m+1} * I$.

Thus let $|V(H)| \geq n$, and let statement (*) be valid for all H' with $|V(H')| < |V(H)|$. Let (T, S) be a tree decomposition of H of width $\leq k$. Now in the same way as in Lemma 2 we find a vertex $t^* \in V(T)$ and set $I \subset \{1, 2, \cdots, a(t^*)\}$ such that $(n/2) - (k+1) \leq |A_I(t)| \leq n - (k+1)$. Then as in the proof of Theorem 1, put $A = A_I(t^*) \setminus X_{t^*}$, $B = A_I(t^*) \cup X_{t^*}$ and $C = V(H) \setminus B$, and let A, B, C denote the subgraphs of H induced by A, B, C respectively.

We proceed with the same construction as in the proof of Theorem 1.

Let $R_1, R_2, \cdots, R_q$ be the connected components of $B - X_{t^*}$, and for each R_i, let $Q(R_i)$ be the set of its attachment vertices in X_{t^*}. Put $R'_i = R_i \cup Q(R_i)$. We have again $|Q(R_i)| \leq k$. Thus $|V(R'_i)| \leq |V(R_i)| + k < n \leq \text{girth}(G * I)$. Therefore for each i, the image $h(R'_i)$ of R'_i in $G * I$ is a tree. Similar to the proof of Theorem 1, we obtain a new graph H' from H by replacing each R'_i with its image in $G * I$. One can also show that H' has treewidth $\leq k$. Moreover there is a homomorphism $h' : H' \to G * I$, and a homomorphism $h'' : H \to H'$ such that $h' \circ h'' = h$.

The proof will be complete if we can show that there is a homomorphism $h^* : H' \to C_{2m+1} * I$ such that for every $x \in V(H'), h'(x) \in V(G)$ if and only if $h^*(x) \in V(C_{2m+1})$. (Then $h^* \circ h''$ would be the required homomorphism of H to $C_{2m+1} * I$.) This can be done in the same way as in the proof of Theorem 1, by using induction. The only difference is that in Cases 1 and 2, instead of considering the graphs G_{ij}, we should consider the G-shadow of these graphs. ∎

Proof of Theorem 2: Let H be a non-bipartite undirected graph, and let $C_{2m+1} \subset H$. Fix k (treewidth). Find an undirected graph G such that $G \not\to H$ (e.g. $\chi(G) > \chi(H)$) and with girth $\geq n(k, m, I)$. Let $\vec{G}$ be a simple arbitrary orientation of G (i.e., assign a direction to each edge). Then $\vec{G} \not\to H'$ and $\vec{G} * I \not\to H' * I = H * I$. By Theorem 3, every graph $F \to \vec{G} * I$ with $tw(F) \leq k$ is homomorphic to $C_{2m+1} * I$ and hence homomorphic to $H * I$.

4. CONCLUDING REMARKS

Applying Theorem 2, we obtain many digraphs which do not have bounded treewidth duality. However all these digraphs are obtained from undirected graphs, and we are only using the properties of the undirected graphs. When the path indicator is chosen to be a single arc, Theorem 3 is essentially the same statement as Corollary 2.. However, the idea in the proof of Theorem 3 can be applied to obtain other digraphs without bounded treewidth duality. We illustrate this by a simple example.

Given an oriented path $P = [p_1, p_2, \cdots, p_n]$, the *length* $\ell(P)$ of a path P is the number of forward arcs (i.e., arcs of the form $p_i p_{i+1}$ of P minus the number of backward arcs (i.e., arcs of the form $p_{i+1}p_i$). (For the two paths P_1, P_2 as depicted in Fig. 1(a), $\ell(P_1) = \ell(P_2) = 4$. Here the bottom vertices are initial vertices and the top vertices are terminal vertices.) For the reverse $P^T = [p_n, p_{n-1}, \cdots, p_1]$ of P, we have $\ell(P^T) = -\ell(P)$. An oriented path is called *minimal* if it contains no proper subpath of the same length.

Let P_1, P_2 be minimal oriented paths of the same length such that

- $P_1 \not\to P_2$, $P_2 \not\to P_1$;
- There is a path P of the same length which is homomorphic to both P_1 and P_2.

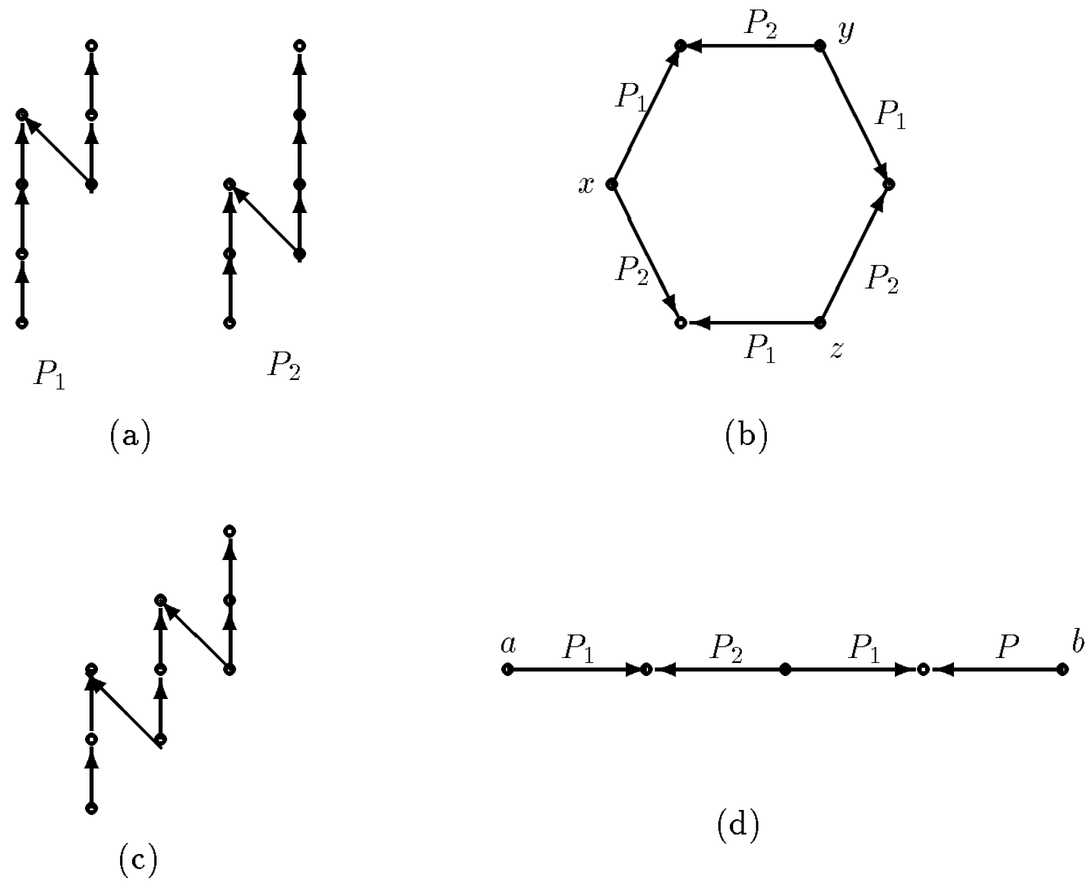

Figure 1

For example, the two paths P_1, P_2 depicted in Figure 1(a) satisfy this condition, with P being the path depicted in Fig. 1(c). Let C be the oriented cycle obtained from three copies of P_1 and three copies of P_2 as depicted in Fig. 1(b). Let (I, a, b) be the path indicator as depicted in Fig. 1(d). Then for any undirected graph G, and for any orientation $\vec{G}$ of G, we have $G \to K_3$ if and only if $\vec{G} * I \to C$. Indeed, if h is a homomorphism of G to K_3, then identifying the three vertices of K_3 with the three vertices x, y, z in C, we can view h as partial mapping of $\vec{G} * I$ to C. It is routine to verify that h can be extended to a homomorphism of $\vec{G} * I$ to C. On the other hand, if h' is homomorphism of $\vec{G} * I$ to C then the vertices of G must be mapped to the three vertices x, y, z, and if (u, v) is an edge of G (i.e., either (u, v) or (v, u) is an arc of $\vec{G}$), then $h'(u) \neq h'(v)$. Thus the restriction of h' to $V(G)$ can be viewed as a homomorphism of G to K_3. This, in particular, proves that the C-coloring problem is NP-complete.

Following the proof of Theorem 3, one can verify that if G is a graph of large girth (say, of girth $\geq 2^{k+2}300k^{12k-1} + 20(k+1)$) and $\chi(G) > 3$, then for any orientation $\vec{G}$ of G, the digraph $\vec{G} * I$ is a witness that C does not have treewidth-k duality. In particular C does not have cycle duality, i.e., there is a digraph D which is not homomorphic to C, and yet every oriented cycle homomorphic to D is homomorphic to C. This answers a question asked in [16]. (In [16], it was proved that every unbalanced cycle, i.e., cycles of non-zero length, has cycle duality).

This example can be modified to give other oriented cycles which do not have bounded treewidth duality. In [6], a complete classification of NP-complete and polynomial C-coloring problems is given, for oriented cycles C. All the polynomial cases have bounded treewidth duality. If $NP \neq P$, then all NP-complete cases should not have bounded treewidth duality. However a direct proof of this (i.e., without assuming $NP \neq P$) is not known.

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