Coloring relatives of intervals on the plane, I: chromatic number versus girth

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Abstract

For the intersection graphs of intervals, rays and strings on the plane, we estimate maximum chromatic number in terms of girth.

1 Introduction

Asplund and Grünbaum [1] and Gyárfás and Lehel [2] started studying many interesting problems on the chromatic number of intersection graphs of intervals and their relatives on the plane. A number of these problem can be obtained in the following framework. For a class $\mathcal{G}$ of intersection graphs and for positive integer k, $k \geq 2$ find or bound

(i) $f(\mathcal{G}, k)$ - the maximum chromatic number of a graph in $\mathcal{G}$ with the clique number at most k;

(ii) $g(\mathcal{G}, k)$ - the maximum chromatic number of a graph in $\mathcal{G}$ with the girth at least k (here we assume $k \geq 4$).

Note that $f(\mathcal{G}, 2) = g(\mathcal{G}, 4)$. In [3] we studied $f(\mathcal{G}, k)$ for several families $\mathcal{G}$, and the presented paper concerns $g(\mathcal{G}, k)$. Our objects are the families

$\mathcal{I}$ — intersection graphs of intervals on the plane;

$\mathcal{R}$ — intersection graphs of rays on the plane;

$\mathcal{S}$ — intersection graphs of such families of strings on the plane that the intersection of any two strings is a connected subset of the plane.

Obviously, $\mathcal{R} \subseteq \mathcal{I} \subseteq \mathcal{S}$ and thus $f(\mathcal{R}, k) \leq f(\mathcal{I}, k) \leq f(\mathcal{S}, k)$ and $g(\mathcal{R}, k) \leq g(\mathcal{I}, k) \leq g(\mathcal{S}, k)$ for every k. Note also that $f(\mathcal{G}, 2) = g(\mathcal{G}, 4)$ for every family $\mathcal{G}$. The results of S. McGuiness [5] on coloring intersection graphs of arcwise connected sets imply that $f(\mathcal{R}, k) < \infty$ for each integer k. No such facts are known for $f(\mathcal{I}, k)$ and $f(\mathcal{S}, k)$ and in fact the following problems motivated our research:

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Problem 1 (Erdös, Gyárfás [2]): Is $g(\mathcal{I}, 4) < \infty$?
Problem 2 (Kratochvil, Nešetřil; se e.g. [4]): Is $g(\mathcal{S}, 4) < \infty$?

These Problems are presently open.

Perhaps somewhat surprisingly we show here that already for $g(\mathcal{S}, 5)$ the situation is more optimistic:

Theorem 1

$$g(\mathcal{S}, k) \leq \begin{cases} 6, & k \geq 5; \\ 4, & k \geq 6; \\ 3, & k \geq 8. \end{cases} \tag{1}$$

Obviously the bound for girth ≥ 8 is the best possible but for remaining two bounds this is not clear. We state this as

Problem 3 Is $g(\mathcal{S}, 5) > 3$?

Explicitely: Does there exist a graph of girth ≥ 5 (i.e. triangle- and rectangle-free) which is an intersection graph of strings (or intervals) and which satisfies $\chi(G) > 3$?

For $\mathcal{R}$, we improve the bounds of Theorem 1 as follows.

Theorem 2 *For any integer $k \geq 6$,*

$$g(\mathcal{R}, k) = 3;$$

and $g(\mathcal{R}, 5) \leq 4$.

The following subfamilies of $\mathcal{I}$ and $\mathcal{R}$ will be also considered:

$\mathcal{I}_m$ — intersection graphs of intervals on the plane parallel to some m lines;

$\mathcal{R}_m$ — intersection graphs of rays on the plane parallel to some m lines.

In particular, we are able to prove stronger bounds of $g(\mathcal{I}_2, 5)$ and $g(\mathcal{R}_2, 5)$ than that of $g(\mathcal{I}, 5)$ and $g(\mathcal{R}, 5)$, respectively.

Theorem 3 $g(\mathcal{I}_2, 5) \leq 5$.

Theorem 4 $g(\mathcal{R}_2, 5) = 3$.

It is a bit surprising that long cycles do not belong to $\mathcal{R}_m$.

Theorem 5 *Cycle C_n of length n belongs to $\mathcal{R}_m$ if and only if $3 \leq n \leq 6m$.*

This fact together with Theorems 4 and 2 and the fact that $f_1(\mathcal{R}_2, 2) = 4$, proved in [3], gives the exact values of $g(\mathcal{R}_2, k)$ and exact values for $g(\mathcal{R}_m, k), k \geq 6$:

Corollary

$$(1) \quad g(\mathcal{R}_2, k) = \begin{cases} 4, & k = 4; \\ 3, & 5 \leq k \leq 11; \\ 2, & k \geq 12, \end{cases}$$

$$(2) \quad g(\mathcal{R}_m, k) = \begin{cases} 3, & 6 \leq k \leq 6m - 1; \\ 2, & k \geq 6m. \end{cases}$$

2 Notations and preliminaries

Let P be a Cartesian plane. All our strings, intervals, rays and lines are supposed to be subsets of P. By a *string* we call a non-closed continuous curve on the plane without self-crossings, i.e. the image $s = s(f)$ of a one-to-one continuous mapping $f : [0, 1] \to P$. It will be convenient to consider the point $f(0)$ as the *origin* $o(s)$ of s, and the point $f(1)$ as its *terminal* $t(s)$. Intervals also are considered with origins and terminals.

Each ray r can be represented by the quadruple (x, y, u, w), where the point (x, y) is its *origin* $o(r)$ and (u, w) is its *vector* $v(r)$. In other words, $r = \{(x, y) + \lambda \cdot (u, w) \mid \lambda \geq 0\}$.

Say that a family F of intervals, rays or lines is an *m-direction family* if there are m (straight) lines $l_1, \dots, l_m$ such that any member of F is parallel to some l_i, $1 \leq i \leq m$.

For a family F of subsets of P, its *intersection graph* $G = G_F$ is the undirected graph with the vertex set F such that for $r, p \in V$,

$$(r, p) \in E(G) \longleftrightarrow r \cap p \neq \emptyset.$$

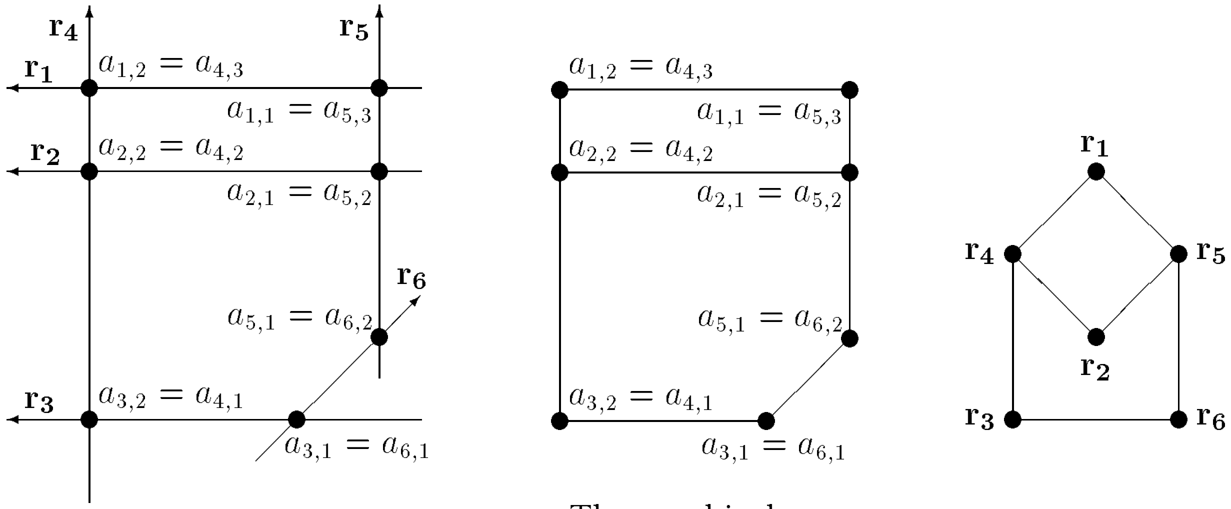
Certainly, for a graph G, there could be very different families F' and F'' such that $G = G_{F'} = G_{F''}$. Any such a family is called a *representation* of G.

We denote by $\mathcal{S}$ the collection of intersection graphs of string families on P such that the intersection of any two strings is a connected subset of P. Evidently, any intersection graph of intervals and rays belongs to $\mathcal{S}$. The intersection graphs of *L-shapes* described by Gyàrfàs and Lehel [2] also belong to $\mathcal{S}$.

We are interested in triangle-free graphs. Clearly for any string-representation F of a triangle-free graph $G \in \mathcal{S}$, each point of P belongs to at most two strings. Moreover, if s_1 and s_2 are intersecting strings and $q = s_1 \cap s_2$ (q is a connected curve by the definition of S) then no point of q belongs to any other member of F. Thus, contracting q into a point we obtain a family with the same intersection graph. It follows that for each triangle-free graph $G \in \mathcal{S}$, there exists a string-representation $U(G)$ such that *any two strings have at most one point in common.* This refers also to rays-representations. We call such representations *U-representations* (by strings, intervals or rays).

Let $F = \{s_1, \dots, s_n\}$ be a U-representation of a triangle-free graph $G \in \mathcal{S}$. Let for $i = 1, \dots, n$, the points $a_{i,1}, \dots, a_{i,q_i}$ be the common points of s_i with other members of F numbered starting from its origin $o(s_i)$. Since G has no triangles, any $a_{i,j}$ lies on exactly two rays. Now we can define *the graphical graph* $H[F]$ as follows. The vertex set of H is the union $\bigcup_{i=1}^{n} \{a_{i,1}, \dots, a_{i,q_i}\}$ and the edge set of H is $\{(a_{i,j}, a_{i,j+1}) \mid 1 \leq i \leq n,\ 1 \leq j \leq q_i\}$. An example (for a ray-intersection graph) is given on Fig.1.

Note that $H(F)$ is considered as a *plane graph* i.e. a graph embedded into P and its edges are the parts of strings connecting $a_{i,j}$ with $a_{i,j+1}$. Let v, e and f denote



Family F | The graphical graph $H(F)$ | The intersection graph G_F

Figure 1: A ray family F and corresponding graphs

the number of vertices, edges and faces in $H(F)$, and m denote the number of edges in $G = G_F$. Each $a_{i,j}$ corresponds to an edge in G_F. Hence

$$v = m. \tag{2}$$

Each string s_i, $i = 1, \ldots, n$ produces $q_i - 1$ edges in $H(F)$, and so

$$e = \sum_{i=1}^{n} (q_i - 1) = 2v - n. \tag{3}$$

3 Coloring string graphs

If the class of string graphs $\mathcal{S}$ contains a k-chromatic graph of girth g then it contains also a k-vertex-critical graph G of girth g. Since its minimum degree is at least $k-1$, we have

$$m \geq n(k-1)/2. \tag{4}$$

Let $F = \{s_1, \ldots, s_n\}$ be a U-representation of G. Then each cycle in $H(F)$ corresponds to a closed walk in G such that any edge is traversed at most once. Hence the girth of $H(F)$ is at least g, and

$$2e \geq g \cdot f. \tag{5}$$

From Euler's formula, and (5) we obtain

$$v - e + 2e/g > 0.$$

Hence by (2) and (3),

$$m/n < \frac{g-2}{g-4},$$

and comparing this with (4), we have

$$k - 1 < \frac{2(g-2)}{g-4}. \tag{6}$$

For $g = 5$, (6) gives $k \leq 6$, for $g = 6$, it gives $k \leq 4$,and for $g = 6$ it gives $k \leq 3$. Theorem 1 is proved.

4 Coloring rays

Analogously to the previous section, let G be a k-vertex-critical ray intersection graph of girth g, and let $F = \{r_1, \ldots, r_n\}$ be an its U-representation by rays.

Lemma 1 *The size of the outer face of $H(F)$ is at least n.*

PROOF. We associate with each ray $r_i \in F$ an edge $d_i \in E(H(F))$ on the boundary of the outer face of $H(F)$ in the following way:
(i) If the degree of $a_{i,q(i)}$ in $H(F)$ is 2 then we put $d_i = (a_{i,q(i)-1}, a_{i,q(i)})$.
(ii) If the degree of $a_{i,q(i)}$ in $H(F)$ is 3 then for some $j \neq i$ and $2 \leq l \leq q(j) - 1$, $a_{i,q(i)} = a_{j,l}$. In this case, we put $d_i = (a_{j,l-1}, a_{j,l})$.

It is easy to see that all d_i-s are distinct. □

Now, instead of (5) by Lemma 1,

$$2e \geq g(f - 1) + n,$$

and hence

$$f \leq 1 + (4v - 3n)/g. \tag{7}$$

From Euler's formula and (3) we have

$$v - (2m - n) + f = 2,$$

which together with (7) gives $2 + (m - n) \leq 1 + (4m - 3n)/g$, and $g < (4m - 3n)/(m - n)$. In other words,

$$m/n < (g - 3)/(g - 4). \tag{8}$$

Comparing (8) with (4), we obtain

$$k - 1 < 2(g - 3)/(g - 4).$$

This gives $k \leq 4$ for $g = 5$ and $k \leq 3$ for $g \geq 6$. Thus, Theorem 2 is proved.

5 Two directions families of girth five

Let a k-vertex-critical graph $G \in \mathcal{I}_2$ have girth five and F be an its U-representation by horizontal and vertical intervals. Since geometrically any face in $H(F)$ is a polygon with an even number of corners, each 5-face in $H(F)$ is a 4-gon which one side contains the common point of some two vertical or two horizontal members of F (See Fig.2a)).

Moreover, a common point of two vertical or two horizontal members of F can not belong to two 5-faces like on Fig. 2b); for otherwise the intervals $\mathbf{i_1}$, $\mathbf{i_2}$, $\mathbf{i_6}$ and $\mathbf{i_5}$ form a 4-cycle in G. Thus the number of 5-faces in $H(F)$ does not exceed the number p of common points of intervals in the same directions. Evidently, $p < n$. Consequently, instead of (5) we have

$$2e \geq 6f - p > 6f - n. \tag{9}$$

Now, proceeding along the lines of the proof of Theorem 1, we obtain $m/n < 5/2$ and then $k - 1 < 5$. This proves Theorem 3.

Let F be a U-representation of a 4-vertex-critical graph $G \in \mathcal{R}_2$ of girth at least five by horizontal and vertical rays. Any component of the subgraph of G induced by horizontal (resp., vertical) rays has either one or two vertices. In the latter case, the union of the corresponding rays must be a line, and the intersection must be a point. We will call such a line l an F-*line* with the origin $o(l)$ and two rays l^+ and l^-. The signs $+$ and $-$ are taken in accordance with orientations of axes of P.

By Lemma 1, in our case, inequality (9) can be refined as follows:

$$2e \geq 6(f-1) - p + n,$$

and hence

$$3n + p - 6 \geq 2m.$$

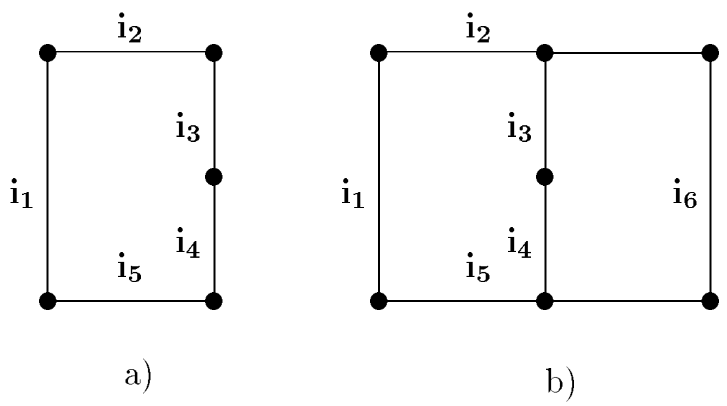


Figure 2:

The last inequality is equivalent to the following:

$$\sum_{x\in V(G)} (\deg_G x - 3) \le p - 6. \tag{10}$$

Recall that each summand $\deg_G x - 3$ is non-negative, since G is 4-critical.

Lemma 2 *If F forms at most two horizontal F-lines or at most two vertical F-lines then the chromatic number of G_F is at most three.*

PROOF. Consider the case when there are exactly two horizontal F-lines l_1 and l_2. The case when there is at most one horizontal F-line is analogous and even easier. Let $o(l_i) = (a_i, b_i)$, $l = 1, 2$. We may assume that $a_1 \le a_2$. For each vertical F-line l, one can choose a ray $r(l) \in \{l^+, l^-\}$ disjoint from both l_2^+ and l_1^-. We can color by 1 the set $\{l_2^+, l_1^-\} \cup \{r(l) \mid l \text{ is a vertical } F\text{-line}\}$. The remaining horizontal rays we color by 2, and vertical rays by 3. □

Lemma 3 *Let F have p_1 horizontal F-lines and p_2 vertical F-lines. Then* $\max\{p_1, p_2\} \le 4$.

PROOF. Assume that $p_1 \le p_2$, $p_2 \ge 5$. By Lemma 2, $p_1 \ge 3$.

Let l be a horizontal F-line. Then l intersects at least p_2 horizontal members of F, and the vertices l^+ and l^- in G are adjacent. Hence $\deg_G l^+ + \deg_G l^- \ge p_2 + 2$. Consequently, $\sum_{x\in V(G)}(\deg_G x - 3) \ge p_1(p_2 - 4)$, and by (10),

$$p_1(p_2 - 4) \le p_1 + p_2 - 6.$$

This is impossible for $p_1 \ge 3, p_2 \ge 5$. □

Denote by V_4 the set of vertices of G of degree at least 4. By Gallai's theorem on critical graphs, each block of $G - V_4$ is either a complete graph or an odd cycle. By Lemma 3 and (10), $|V_4| \le 2$. Thus, for any $x \in V(G) \setminus V_4$, $\deg_{G-V_4} x \ge 3 - 2 = 1$. Moreover, if there are two vertices x and y of degree 1 in $G - V_4$ then $|V_4| = 2$ and both x and y are adjacent to both elements of V_4. But this implies the existence of a 4-cycle in G, a contradiction.

Consequently, any pendant block in $G - V_4$ is an odd cycle. Let x, y and z be three consecutive vertices on such a cycle having degree two in $G - V_4$. Since $|V_4| \le 2$, there exists $u \in V_4$ adjacent to at least two of x, y and z. This again implies the existence of a 3-cycle or a 4-cycle in G, a contradiction. Theorem 4 is proved.

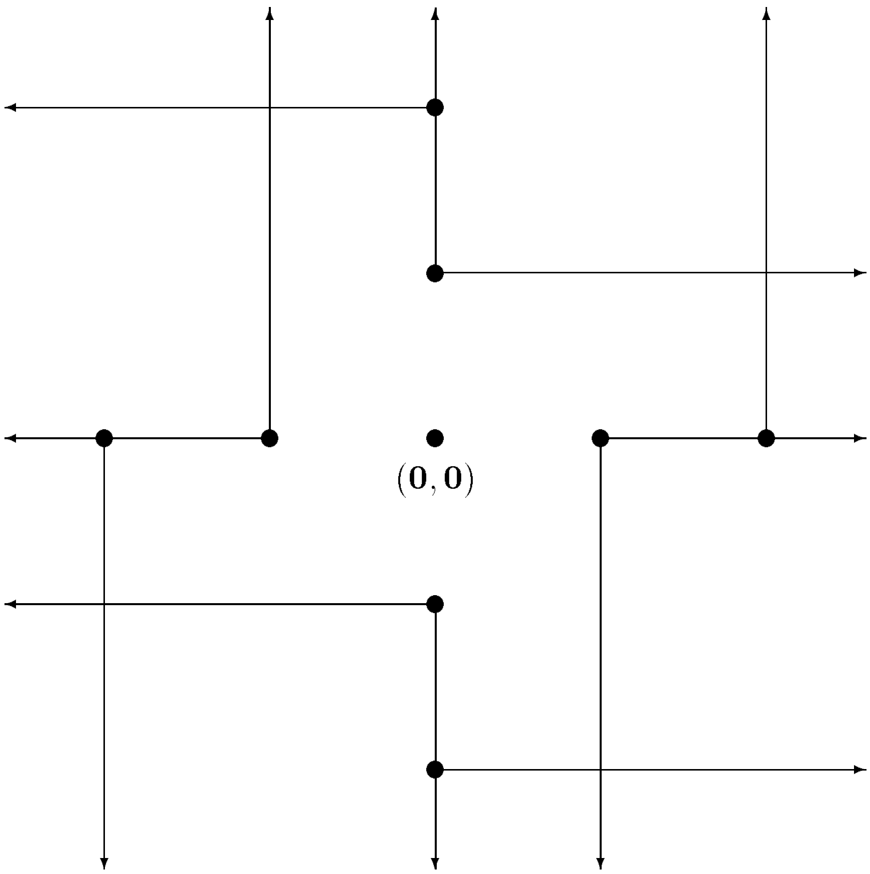


Figure 3:

6 Rays intersection families of large girth

As rays (unlike lines) are oriented configurations, we can consider *oriented directions*. Let $\mathcal{R}^m$ denote the set of intersection graphs of translates of some fixed m rays. Obviously, $\mathcal{R}_m \subset \mathcal{R}^{2m}$ for any m. Thus the 'only if' part of Theorem 5 is implied by the 'only if' part of the following fact.

Theorem 6 *Cycle C_n of length n belongs to $\mathcal{R}^m$, $m \geq 3$, if and only if $3 \leq n \leq 3m$.*

The 'if' parts of both theorems follow from the following construction.

Construction. Choose m oriented directions (vectors) $d_i = (\cos \frac{2\pi i}{m}, \sin \frac{2\pi i}{m})$, $i = 0, \ldots, m-1$. Note that for even m, the vector d_i is opposite to $d_{i+m/2}$, $i = 0, \ldots, m/2-1$. The origins of the three rays in the direction d_i, $i = 0, \ldots, m-1$ are $(\cos \frac{2\pi i}{m}, \sin \frac{2\pi i}{m})$, $(\cos \frac{2\pi(i+1)}{m}, \sin \frac{2\pi(i+1)}{m})$ and $(2\cos \frac{2\pi(i-1)}{m}, 2\sin \frac{2\pi(i-1)}{m})$. This gives a realization of C_{3m}. It is easy to change it slightly to obtain realizations of C_{3m-1} and C_{3m-2}.

An example (for $m = 4$) is given on Fig.3.

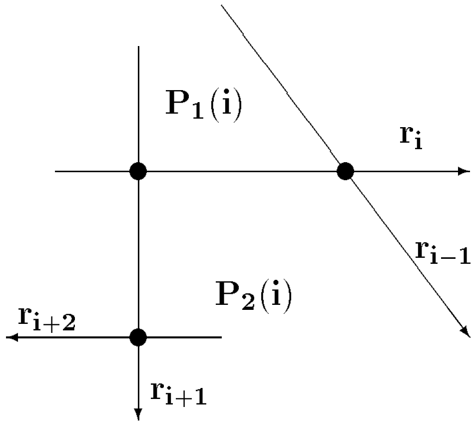


Figure 4:

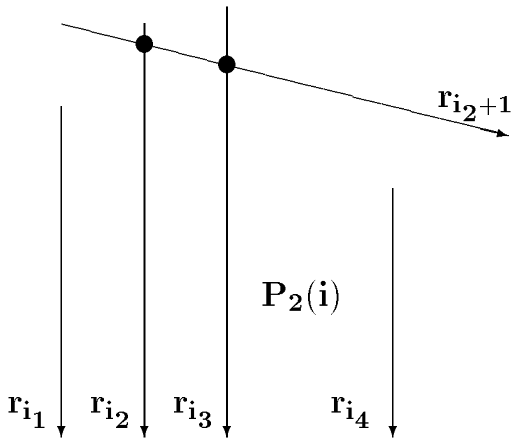


Figure 5:

To prove the 'only if' part of Theorem 6, assume that the rays $r_1, \ldots, r_n$ in m oriented directions form a representation of C_n, $n > 4$. For each i, $i = 1, \ldots, n$, the set $P \setminus (r_i \cup r_{i+1})$ has two connected parts: $P_1(i)$ (containing the origins of r_i and r_{i+1}) and $P_2(i)$. An example is given on Fig.4.

The following fact is obvious but helpful.

Observation. *Either all r_j, $j \in \{1, \ldots, n\} \setminus \{i-1, i, i+1, i+2\}$ are contained in $P_1(i)$ or all of them are contained in $P_2(i)$.*

Now we prove the final lemma which yields the 'only if' part of Theorem 6.

Lemma 4 *At most three rays go in each oriented direction.*

PROOF. Assume that there are four rays, $r_{i_1}, r_{i_2}, r_{i_3}$ and r_{i_4} (numbered from left to right), going downward.

CASE 1. The first coordinate of the vector of r_{i_2+1} is positive (i.e. it goes to the right hand of r_{i_2})(See Fig.5).

Since G is 2-regular, r_{i_2+1} does not meet either r_{i_3} or r_{i_4}. So one of them (say, r_{i_4}) is contained in $P_2(i_2)$. But then either r_{i_1} is contained in $P_1(i_2)$ or (if $i_1 = i_2 + 2$) r_{i_1+1} is contained in $P_1(i_2)$. This contradicts the Observation.

Because of the symmetry between $i_2 + 1$ and $i_2 - 1$, it is enough now to consider the following situation.

CASE 2. The first coordinates of vectors of both r_{i_2+1} and r_{i_2-1} are negative (i.e. the rays go to the left hand of r_{i_2}).

Since $n > 4$, at least one of r_{i_2+1} and r_{i_2-1} (say, r_{i_2+1}) misses r_{i_1}. It also must miss one of r_{i_3} and r_{i_4} (say, r_{i_4}). But then r_{i_1} is contained in $P_2(i_2)$ and r_{i_4} is contained in $P_1(i_2)$. This is the final contradiction.

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