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ABSTRACT OF DOCTORAL THESIS

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d-Frames as algebraic duals of bitopological spaces

Department of Applied Mathematics (Charles University)

and

School of Computer Science (University of Birmingham)

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AUTOREFERÁT DISERTAČNÍ PRÁCE

Tomáš Jakl

d-Framy jako algebraické duály bitopologických prostorů

Katedra Aplikované Matematiky (Univerzita Karlova)

&

School of Computer Science (University of Birmingham)

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Contents

Abstract

Achim Jung and Drew Moshier developed a Stone-type duality theory for bitopological spaces, amongst others, as a practical tool for solving a particular problem in the theory of stably compact spaces. By doing so they discovered that the duality of bitopological spaces and their algebraic counterparts, called d-frames, covers several of the known dualities.

In this thesis we aim to take Jung's and Moshier's work as a starting point and fill in some of the missing aspects of the theory. In particular, we investigate basic categorical properties of d-frames, we give a Vietoris construction for d-frames which generalises the corresponding known Vietoris constructions for other categories, and we investigate the connection between bispaces and a paraconsistent logic and then develop a suitable (geometric) logic for d-frames.

1 Introduction

Point-free topology studies topological spaces in terms of an algebraic description of their lattice of open sets. The main objects of study are *frames*, that is, complete lattices $L = (L, \bigvee, \wedge, 0, 1)$ which satisfy

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\}$$

for all $A \subseteq L$ and $b \in L$. Any topological space (X, τ) gives rise to a frame $\Omega(X, \tau) = (\tau, \bigcup, \cap, \emptyset, X)$. Having such generalisation of the notion of space has turned out to be fruitful for a number of reasons. Apart from the fact that working in point-free setting often allows for cleaner and more descriptive proofs, also, many point-free proofs of are constructive (i.e. the Axiom of Choice or the Law of Excluded Middle are not required) as opposed to their point-set analogues.

An important feature of the point-free topology is the ability to relate frames to topological spaces and vice versa, that is, there is an adjunction between the category of topological spaces **Top** and the category of frames **Frm**, respectively:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathbf{Frm} \arrow[l, bend left, "\Sigma"] \end{tikzcd}$$

However, the main concern of the thesis is a study of bitopological spaces, also called *bispaces*, and their algebraic duals, called *d-frames*. Moving from spaces and frames to bispaces and d-frames is not just a mere generalisation. To see why, we first show that the latter appear naturally in a number of contexts and then we highlight some of the contributions of the thesis.

2 Motivations for studying bispaces and d-frames

2.1 Applications to program semantics

One motivation for studying bispaces comes from the fact that many known mathematical structures are naturally bitopological; although this often might not be mentioned explicitly. Basic examples include partially ordered spaces such as real line, unit interval or Priestley spaces. When working with those spaces, it is often practical to split the underlying topology into two simpler topologies: the topology of upper and lower opens. In general, we have:

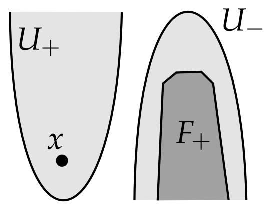
2.1 Definition. (X, τ_+, τ_-) is a *bitopological space* if (X, τ_+) and (X, τ_-) are topological spaces.

In the examples above we took the two topologies as two coarser topologies of an ambient topology. This, although very common, is not the only way bispaces arise. Another class of examples comes from the study of program semantics. Given a program fragment P, its denotational semantics $[\![P]\!]$ can be interpreted in a semantics space (X, τ), which is determined by the program's type. The semantics spaces one usually considers are *stably compact spaces*.

An important construction in the theory of stably compact spaces is taking the de Groot dual (X, τ^{d}) of a space, where τ^{d} is the dual topology of τ. We see that every stably compact space (X, τ) gives rise to a bitopological space $(X, \tau, \tau^{\mathrm{d}})$. Moreover, stably compact spaces can be identified with a class a bispaces which arise this way. Those bispaces can be topologically characterised precisely as those which are *d-compact*, *d-regular* (and T_0)[1].

2.2 Definition. A bispace (X, τ_+, τ_-) is *d-regular* if

1. Whenever $x \notin F_+$ for some τ_+-closed F_+, then there is a pair of disjoint open sets $U_+ \in \tau_+$ and $U_- \in \tau_+$ such that $x \in U_+$ and $F_+ \subseteq U_-$.
2. and symmetrically for $y \notin F_-$ where F_- is a τ_--closed set.



2.3 Definition. A bispace (X, τ_+, τ_-) is *d-compact* if whenever

$$\bigcup_{i \in I} U_+^i \;\cup\; \bigcup_{j \in J} U_-^j \;=\; X,$$

for some $\{U_+^i\}_{i \in I} \subseteq \tau_+$ and $\{U_-^j\}_{j \in J} \subseteq \tau_-$, then there exist finite $F \subseteq_{\mathrm{fin}} I$ and $G \subseteq_{\mathrm{fin}} J$ such that $\bigcup_{i \in F} U_+^i \cup \bigcup_{j \in G} U_-^j = X$.

Identifying stably compact spaces with d-compact d-regular bispaces has technical advantages. The name "stably compact", although seemingly shorter, hides a much longer list of axioms when compared to only two bitopological ones. As a consequence, rewriting original results about stably compact spaces bitopologically leads to much shorter proofs (which is demonstrated in Chapter 5).

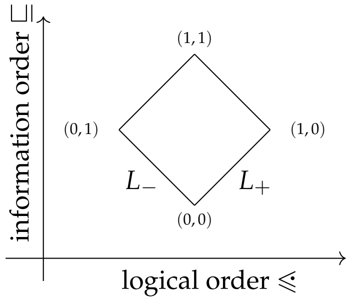
Furthermore, just like in the study of topological spaces one can consider many different topological notions or separation axioms apart from compactness and regularity. These are, however, not so important for the purpose of this short note.

2.2 Embedding of dualities

Just as frames are algebraic duals of topological spaces, bispaces also have their algebraic duals. It is no surprise that, because bispaces consist of two topologies, we will have two frames L_+ and L_- as the core of the structure of the algebraic duals of bispaces, called *d-frames* [JM06]. Before we give a full definition of d-frames, let us

[1] A bispace (X, τ_+, τ_-) is T_0 if, whenever $x \neq y$, then there is a $U \in \tau_+ \cup \tau_-$ such that $x \in U \not\ni y$ or $x \notin U \ni y$. We will often assume this axiom without mentioning.

take a look at some consequences of this. It is a general fact that the product of two lattices (or frames, in our case) $L_+ \times L_-$ introduces two orders which are somehow orthogonal to each other. Namely, for any $\alpha = (\alpha_+, \alpha_-), \beta = (\beta_+, \beta_-) \in L_+ \times L_-$ define



- *Information order:* $\alpha \sqsubseteq \beta$ if $\alpha_+ \leq \beta_+$ and $\alpha_- \leq \beta_-$, and
- *Logical order:* $\alpha \leqslant \beta$ if $\alpha_+ \leq \beta_+$ and $\alpha_- \geq \beta_-$.

In fact, those two orders introduce two bounded distributive lattices:

$$(L_+ \times L_-, \curlywedge, \curlyvee, tt, ff) \quad \text{and} \quad (L_+ \times L_-, \sqcap, \sqcup, \top, \bot).$$

To specify the interplay between L_+ and L_- we consider two relations on the product $L_+ \times L_-$. The *consistency relation* $\mathsf{con} \subseteq L_+ \times L_-$ expresses the fact that two open sets are *disjoint* and the *totality relation* $\mathsf{tot} \subseteq L_+ \times L_-$ expresses the fact that two open sets *cover the whole space*. Having this in mind we present the main definition:

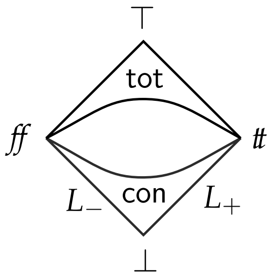
2.4 Definition. A *d-frame* is a quadruple $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ where L_+, L_- are frames, $\mathsf{con} \subseteq L_+ \times L_-$ and $\mathsf{tot} \subseteq L_+ \times L_-$ are such that

- (in the information order:)

$$\begin{aligned}
&(\mathsf{tot}\text{-}\uparrow) && \alpha \sqsubseteq \beta \text{ and } \alpha \in \mathsf{tot} \implies \beta \in \mathsf{tot},\\
&(\mathsf{con}\text{-}\downarrow) && \alpha \sqsubseteq \beta \text{ and } \beta \in \mathsf{con} \implies \alpha \in \mathsf{con},\\
&(\mathsf{con}\text{-}{\textstyle\bigsqcup}^{\uparrow}) && \sqsubseteq\text{-directed } A \subseteq^{\uparrow} \mathsf{con} \implies {\textstyle\bigsqcup}^{\uparrow} A \in \mathsf{con}
\end{aligned}$$

- (in the logical order:)

$$\begin{aligned}
&(\mathsf{tot}\text{-}\curlyvee, \curlywedge) && \alpha, \beta \in \mathsf{tot} \implies \alpha \curlyvee \beta, \alpha \curlywedge \beta \in \mathsf{tot},\\
& && tt, ff \in \mathsf{tot},\\
&(\mathsf{con}\text{-}\curlyvee, \curlywedge) && \alpha, \beta \in \mathsf{con} \implies \alpha \curlyvee \beta, \alpha \curlywedge \beta \in \mathsf{con},\\
& && tt, ff \in \mathsf{con},
\end{aligned}$$



- (interplay between con and tot:)

$$\begin{aligned}
&(\mathsf{con}\text{-}\mathsf{tot}) && \alpha \in \mathsf{con} \text{ and } \beta \in \mathsf{tot} \text{ such that}\\
& && (\alpha_+ = \beta_+ \text{ or } \alpha_- = \beta_-) \implies \alpha \sqsubseteq \beta
\end{aligned}$$

Every bitopological space (X, τ_+, τ_-) gives rise to a d-frame. Indeed, define $\Omega_d(X)$ to be the d-frame $(\tau_+, \tau_-, \mathsf{con}_X, \mathsf{tot}_X)$ where

$$\begin{aligned}
(U_+, U_-) \in \mathsf{con}_X &\quad \text{if and only if} \quad U_+ \cap U_- = \emptyset, \quad \text{and}\\
(U_+, U_-) \in \mathsf{tot}_X &\quad \text{if and only if} \quad U_+ \cup U_- = X.
\end{aligned}$$

As was the case for spaces and frames, also bispaces and d-frames are interlinked by a dual adjunction. This justifies our intuition that d-frames are, in fact, the *algebraic duals* of bispaces[2].

$$\begin{tikzcd} \mathbf{biTop} \arrow[r, bend left, "\Omega_d"] \arrow[r, phantom, "\perp"] & \mathbf{d\text{-}Frm} \arrow[l, bend left, "\Sigma_d"] \end{tikzcd}$$

Just like in frames, the dual adjunction between bispaces and d-frames restricts to the *dual equivalence* between the categories of d-compact d-regular bispaces and d-compact d-regular d-frames. Moreover, many of the previously known dualities embed into this duality. The following commutative diagram of categories expresses the fact that Stone duality, Priestley duality and the duality of compact regular spaces and frames, embed into the duality of d-compact d-regular bispaces and d-frames.

$$\begin{tikzcd}
 & \mathbf{Stone} \arrow[rrr, leftrightarrow, "\cong"] \arrow[ddl, hook] \arrow[dr, hook] & & & \mathbf{Bool} \arrow[ddl, hook] \arrow[dr, hook] & \\
 & & \mathbf{Pries} \arrow[rrr, leftrightarrow, "\cong"] \arrow[ddl, hook] & & & \mathbf{DLat} \arrow[ddl, hook] \\
\mathbf{KRegSp} \arrow[rrr, leftrightarrow, "\cong"] \arrow[dr, hook] & & & \mathbf{KRegFrm} \arrow[dr, hook] & & \\
 & \mathbf{biKReg} \arrow[rrr, leftrightarrow, "\cong"] & & & \mathbf{d\text{-}KReg} &
\end{tikzcd}$$

2.3 Logic of bispaces

The duality between Stone spaces and Boolean algebras has a logical reading. Since propositional logic is sound and complete with respect to Boolean algebras, a consequence of the duality is that Stone spaces are also adequate models of propositional logic. In other words, Stone duality provides a bridge between propositional logic and its topological semantics:

$$\text{Stone spaces} \quad \longleftrightarrow \quad \text{Boolean algebras} \quad \longleftrightarrow \quad \text{propositional logic}$$

A similar story can be retold also for the other two older dualities that appeared in the cube above. For example, Priestley duality provides a bridge for positive propositional logic and Priestley spaces.

[2] Also Banaschewski came up with structures, which he called *biframes*, to play the role of algebraic duals of bispaces [BBH83]. An advantage of d-frames over biframes is that one does not have to construct an ambient frame L_0 which contains both L_+ and L_- and is generated by them. Also, d-frames allow for an interesting logical reading, as we will see in the next section.

With this in mind, one can ask whether there is also a suitable logic for bispaces and d-frames. In particular, if there is a logic for which bispaces provide an adequate topological semantics via the dual adjunction between bispaces and d-frames.

To start with, we recall the work of Abramsky [Abr87], Scott [Sco70; Sco76] and their followers [Vic89; Smy83; Smy92; Esc04]. As was the case in the cases above, a *property* or *predicate* is interpreted as the set of models or states which satisfy it. However, in theoretical computer science we are rather interested in *observable properties*, which are those properties for which we can determine their validity in a state by inspecting only a finite amount of information about the state. In the terminology from computability theory, observable properties are exactly the semidecidable or recursively enumerable sets. Moreover, observable properties are closed under unions and finite intersections. In other words, the set of states (or models) equipped with the set of all observable properties forms a topological space.

When we interpret the structure of a bitopological space (X, τ_+, τ_-) in these terms, we obtain that each of the topologies corresponds to a logical theory of observable properties. As suggested by the notation, τ_+ represents the frame of all *positive observations* and τ_- all *negative observations*. Then, performing an observation φ results in a pair of open sets $[\![\varphi]\!] = (U_+, U_-) \in \tau_+ \times \tau_-$ where U_+ determines the states where the examined property *observably holds* and U_- determines the states where the predicate *observably fails*.

For a state or model $x \in X$ and an observation $[\![\varphi]\!] = (U_+, U_-) \in \tau_+ \times \tau_-$, we distinguish four different options:

1. $x \in U_+ \setminus U_- \implies \varphi$ observably holds and does not fail in x, i.e. is *true*
2. $x \in U_- \setminus U_+ \implies \varphi$ observably fails and does not hold in x, i.e. is *false*
3. $x \in U_+ \cap U_- \implies \varphi$ is observably true and false in x, i.e. is *inconsistent*
4. $x \notin U_+ \cup U_- \implies \varphi$ is observably neither true nor false in x, i.e. *no-information*

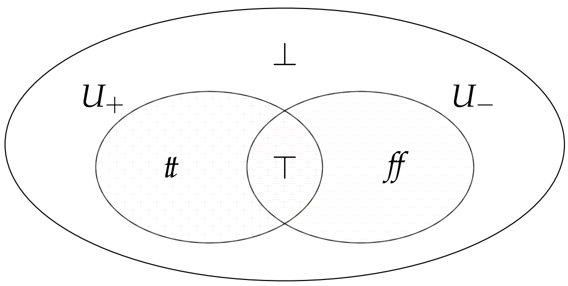


Figure 1: Four possible interpretation of the predicate (U_+, U_-)

This interpretation leads us to consider Belnap's paraconsistent logic for computer reasoning [Bel76; Bel77]. Belnap argued that it is in the very nature of computers to make decisions even in the presence of contradictions and, for that reason, a classical two-valued logic does not suffice.

3 Main contributions

In the previous section we have seen that the theory of bispaces and d-frames borders a wide range of other disciplines, ranging from program semantics and logic to various duality theories. Consequently, developing new tools and techniques for d-frames gives us tools for all those different fields at the same time.

In this section we introduce some of the theoretical results developed in the thesis and later we take a look at some applications to the disciplines mentioned in the previous section.

3.1 Categorical and algebraic constructions

Free construction is one of the key constructions in universal algebra. In general, it works as follows. Given a set of generators G and equations E, we (freely) construct an object in the category $\mathbf{A}\langle G \mid E\rangle$ such that the embedding $G \hookrightarrow \mathbf{A}\langle G \mid E\rangle$ preserves all equations in E. Moreover, we require $\mathbf{A}\langle G \mid E\rangle$ to be universal such, i.e. whenever a map $f\colon G \to A$ into another object in the category preserves all the equations in E, then there is a unique morphism $\widetilde{f}\colon \mathbf{A}\langle G \mid E\rangle \to A$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, "f"'] & \mathbf{A}\langle G \mid E\rangle \arrow[d, "\widetilde{f}"] \\ & A \end{tikzcd}$$

Even though frames are infinitary structures, free constructions of frames from the set of generators and equations is possible (written as $\mathbf{Fr}\langle G \mid E\rangle$). The challenge of defining a free construction for d-frames comes from the fact that d-frames are of a mixed nature. Namely, they consist of two algebraic components (i.e. the two frames) and two relational components (i.e. the two relations).

A d-frame presentation $E = (E_+, E_-, E_{\mathsf{con}}, E_{\mathsf{tot}})$ over two sets of generators G_+ and G_- consists of equations for

1. the positive frame component $E_+ \subseteq \mathbf{Fr}\langle G_+\rangle \times \mathbf{Fr}\langle G_+\rangle$,
2. the negative frame component $E_- \subseteq \mathbf{Fr}\langle G_-\rangle \times \mathbf{Fr}\langle G_-\rangle$,
3. the consistency relation $E_{\mathsf{con}} \subseteq \mathbf{Fr}\langle G_+\rangle \times \mathbf{Fr}\langle G_-\rangle$, and
4. the totality relation $E_{\mathsf{tot}} \subseteq \mathbf{Fr}\langle G_+\rangle \times \mathbf{Fr}\langle G_-\rangle$.

Then, the freely generated d-frame $\mathbf{dFr}\langle G_+, G_- \mid E_+, E_-, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$ (or simply just $\mathbf{dFr}\langle G_\pm \mid E\rangle$) is constructed as follows. First, define the following operator on d-frame presentations:

$$\mathfrak{r}(E) \;=\; (E_+ \cup (E_{\mathsf{con}} \,;\, E_{\mathsf{tot}}^{-1}),\;\; E_- \cup (E_{\mathsf{con}}^{-1} \,;\, E_{\mathsf{tot}}),\;\; \mathbf{cls}_c(E_{\mathsf{con}}),\;\; \mathbf{cls}_t(E_{\mathsf{tot}}))$$

where $R\,;S$ denotes the relation composition $\{(x,z) \mid \exists y.\ xRySz\}$ and $\mathbf{cls}_c(R)$ and $\mathbf{cls}_t(R)$ denote the closure of the relation R under the logical and information axioms for the consistency and totality relation, respectively (see Definition 2.4).[3]

To obtain $\mathbf{dFr}\langle G_\pm \mid E\rangle$ starting from its presentation $E = (E_+, E_-, E_{\mathsf{con}}, E_{\mathsf{tot}})$, we iteratively (transfinitely many times) apply $\mathfrak{r}(-)$ until we reach a fixpoint $E^\infty = (E^\infty_+, E^\infty_-, E^\infty_{\mathsf{con}}, E^\infty_{\mathsf{tot}})$. Then, we assign

$$\mathbf{dFr}\langle G_\pm \mid E\rangle \quad = \quad (\mathbf{Fr}\langle G_+ \mid E^\infty_+\rangle,\ \ \mathbf{Fr}\langle G_- \mid E^\infty_-\rangle,\ \ q[E^\infty_{\mathsf{con}}],\ \ q[E^\infty_{\mathsf{tot}}])$$

where q is the pair of quotient maps $\mathbf{Fr}\langle G_+\rangle \times \mathbf{Fr}\langle G_-\rangle \to \mathbf{Fr}\langle G_+ \mid E^\infty_+\rangle \times \mathbf{Fr}\langle G_- \mid E^\infty_-\rangle$.

3.1 Theorem.
The procedure described above is a free construction, that is, the freely constructed object $\mathbf{dFr}\langle G_\pm \mid E\rangle$ *is a d-frame which has the required universal property.*

3.1.1 Applications

Properties of the category d-Frm. As is the case for universal algebra, with free constructions we obtain many other constructions for the category of d-frames for free. For example, a *quotient* of a d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ by a pair of relations $R_+ \subseteq L_+ \times L_+$ and $R_- \subseteq L_- \times L_-$ is obtained as

$$\mathbf{dFr}\langle L_+, L_- \mid E_+ \cup R_+,\ \ E_- \cup R_-,\ \ \mathsf{con},\ \ \mathsf{tot}\rangle$$

where E_+ and E_- are all equations that hold in the frame L_+ and L_-, respectively. [4]

Similarly, we can prove that the *coproducts* of d-frames exist and, moreover, we obtain:

3.2 Theorem.
The category of d-frames is complete, cocomplete, and admits a factorization system.

Specific free constructions. d-Frame presentations can be alternatively given *single-sorted*. This allows us to make use of the two orders: the information and logical order. For example, the embedding $\mathbf{DLat} \hookrightarrow \mathbf{d\text{-}Frm}$, which we mentioned on page 4, can be described as follows. We assign to a distributive lattice D the freely generated

[3] Note that, to simplify matters, we combined two definitions from the thesis into one; we treat *quotient structures* and *d-frame presentation* as the same notion here.

[4] In fact, in the thesis, we define free constructions in terms of quotients and not vice versa as we do here. Those two definitions are equivalent to each other.

d-frame specified as

$$\mathbf{dFr}\Big\langle \langle d\rangle : d \in D \;\Big|\; \langle d\rangle \mathbin{\vee\!\!\!\vee} \langle e\rangle = \langle d \vee e\rangle\,,\ \langle 0\rangle = \mathit{ff},$$
$$\langle d\rangle \mathbin{\wedge\!\!\!\wedge} \langle e\rangle = \langle d \wedge e\rangle\,,\ \langle 1\rangle = \mathit{tt},$$
$$(\forall d \in D)\ \langle d\rangle \in \mathsf{con}, \langle d\rangle \in \mathsf{tot}\Big\rangle.$$

Further, we can also present the d-frame of reals $\mathcal{L}(\mathbb{R})$ as follows

$$\mathbf{dFr}\Big\langle \langle q\rangle : q \in \mathbb{Q} \;\Big|\; \langle q\rangle \mathbin{\vee\!\!\!\vee} \langle q'\rangle = \big\langle \max(q,q')\big\rangle\,,\ \langle q\rangle \mathbin{\wedge\!\!\!\wedge} \langle q'\rangle = \big\langle \min(q,q')\big\rangle\,,$$
$$\langle q\rangle = \bigsqcup_{q'<q} (\big\langle q'\big\rangle \sqcap \mathit{tt}) \sqcup \bigsqcup_{q<q''} (\big\langle q''\big\rangle \sqcap \mathit{ff}), \quad \top = \bigsqcup_{q} \langle q\rangle\,,$$
$$(\forall q, q' \in \mathbb{Q})\ \langle q\rangle \sqcap \big\langle q'\big\rangle \in \mathsf{con},\ \text{ if } q \neq q' :\ \langle q\rangle \sqcup \big\langle q'\big\rangle \in \mathsf{tot}\Big\rangle.$$

where a single generator $\langle q\rangle$ syntactically represents a pair of opens $((-\infty, q),\ (q, +\infty))$.

3.2 Vietoris constructions for bispaces and d-frames

A powerset-like construction for topological spaces was introduced by Leopold Vietoris in [Vie22] and its dual construction for the category of frames is due to Johnstone [Joh85; Joh82]. Over the years both spacial and frame versions of the Vietoris construction (and their variants) found many applications in logic, topology, the theory of coalgebras and program semantics.

In Chapter 4 of the thesis we present a Vietoris constructions for bispaces and d-frames and show that most of the basic properties of their monotopological variants can be recovered. First, define a Vietoris construction on d-frames as follows. Let $\mathcal{L}$ be a d-frame, then

$$\mathbb{W}_d(\mathcal{L}) \;\stackrel{\text{def}}{\equiv}\; \mathbf{dFr}\Big\langle \Box\alpha, \Diamond\alpha : \alpha \in \mathcal{L} \;\Big|\; (\Box \text{ distributes over } \mathbin{\wedge\!\!\!\wedge}, \mathit{tt}, {\textstyle\bigsqcup^{\uparrow}}),$$
$$(\Diamond \text{ distributes over } \mathbin{\vee\!\!\!\vee}, \mathit{ff}, {\textstyle\bigsqcup^{\uparrow}}),$$
$$\Box\alpha \mathbin{\wedge\!\!\!\wedge} \Diamond\beta \leqslant \Diamond(\alpha \mathbin{\wedge\!\!\!\wedge} \beta), \quad \Box(\alpha \mathbin{\vee\!\!\!\vee} \beta) \leqslant \Box\alpha \mathbin{\vee\!\!\!\vee} \Diamond\beta,$$
$$(\forall \alpha \in \mathsf{con}_{\mathcal{L}}/\mathsf{tot}_{\mathcal{L}}) \quad \Box\alpha, \Diamond\alpha \in \mathsf{con}/\mathsf{tot}\Big\rangle.$$

The assignment $\mathcal{L} \mapsto \mathbb{W}_d(\mathcal{L})$ is a functor **d-Frm** $\to$ **d-Frm**. Moreover, it is closed on the important subcategories of d-frames:

3.3 Theorem.
Let $\mathcal{L}$ be a d-frame. Then we have that,

1. *$\mathbb{W}_d(\mathcal{L})$ is d-regular iff $\mathcal{L}$ is,*
2. *$\mathbb{W}_d(\mathcal{L})$ is d-zero-dimensional iff $\mathcal{L}$ is; and*
3. *$\mathbb{W}_d(\mathcal{L})$ is d-compact if $\mathcal{L}$ is d-regular and d-compact.*

In the thesis we also define its topological dual $\mathbb{W}\colon \mathbf{biTop} \to \mathbf{biTop}$. Then, a lot of effort of Chapter 4 goes into showing that those two constructions are in fact dual to each other:

3.4 Theorem.
The functors $\mathbb{W} \circ \Sigma_d$ and $\Sigma_d \circ \mathbb{W}_d$ are naturally isomorphic, when restricted to the subcategories of d-compact d-regular bispaces and d-frames.

The upper and lower variants of the Vietoris constructions ($\mathbb{W}_\Diamond(\mathcal{L})$ and $\mathbb{W}_\Box(\mathcal{L})$, respectively) are also discussed as well as their relationship to $\mathbb{W}_d(\mathcal{L})$.

3.2.1 Applications

Notable applications to other disciplines, outside the theory of d-frames, are the following:

1. Because the category of stably compact frames and d-compact d-regular d-frames are equivalent, from our Vietoris construction for d-frames we obtain that the standard monotopological Vietoris endofunctor
$$\mathbb{V}_{\mathrm{Fr}}\colon \mathbf{Frm} \to \mathbf{Frm}$$
is closed on the category of stably compact frames. This is the first time a *choice-free* proof this fact has been presented.

2. It has also turned out that our Vietoris constructions $\mathbb{W}$ and $\mathbb{W}_d$ are a common generalisation of the corresponding constructions for all the categories shown in the cube on page 4. For example, $\mathbb{W}_d$ generalises both $\mathbb{V}_{\mathrm{Fr}}$ and the well-known construction $\mathbb{M}\colon \mathbf{DLat} \to \mathbf{DLat}$ defined as
$$\begin{aligned} D \;\mapsto\; \mathbf{DL}\Big\langle \Box a, \Diamond a : a \in D \;\Big|\; & \Box(a \wedge b) = \Box a \wedge \Box b, \quad \Box 1 = 1, \\ & \Diamond(a \vee b) = \Diamond a \vee \Diamond b, \quad \Diamond 0 = 0, \\ & \Box a \wedge \Diamond b \leq \Diamond(a \wedge b), \quad \Box(a \vee b) \leq \Box a \vee \Diamond b \Big\rangle. \end{aligned}$$

3. Furthermore, it immediately follows that coalgebras $X \to \mathbb{W}(X)$ on the category of Priestley bispaces provide adequate models of positive modal logic. One can then rephrase this combinatorially and obtain a different description of the same structure:

3.5 Proposition. *Positive modal logic is sound and complete with respect to the triples $\langle X, R, \mathcal{A}_+ \rangle$, where $R \subseteq X \times X$ is a relation and $\mathcal{A}_+$ is a set of subsets of X, such that*

(JT-1) $\quad$ *$\mathcal{A}_+$ is closed under finite unions and intersections,*

(JT-2) $\mathcal{A}_+$ *is closed under* $\Box(-)$ *and* $\Diamond(-)$.

(JT-3) $x \neq y$ *in* X *iff* $x \in A \not\ni y$ *for some* $A \in \mathcal{A}_+ \cup \mathcal{A}_-$,

(JT-4) *if* $\forall A \in \mathcal{A}_+ \cup \mathcal{A}_-$, $y \in A$ *implies* $x \in \Diamond A$, *then* $(x,y) \in R$,

(JT-5) *for any* $\mathcal{M} \subseteq \mathcal{A}_+ \cup \mathcal{A}_-$ *with finite intersection property,* $\bigcap \mathcal{M} \neq \emptyset$,

where $\mathcal{A}_- = \{X \setminus A \mid A \in \mathcal{A}_+\}$ *and, for a subset* $M \subseteq X$,

$$\Box M = \{x \in X \mid \forall y.\ (x,y) \in R \text{ implies } y \in M\},$$
$$\Diamond M = \{x \in X \mid \exists y \text{ s.t. } (x,y) \in R \text{ and } y \in M\}.$$

3.3 Belnap-Dunn logic of bispaces

d-Frames are not the first type of structure that models Belnap's logic. In fact, algebraic structures called *bilattices* were introduced long before d-frames for this reason. In Chapter 6 we show that the category of bilattices embeds into the category of d-compact d-regular d-frames and, moreover, that most of axioms of bilattice logic are still valid even in this broader class:

3.6 Theorem.
The following axioms of four-valued logic are valid in any d-compact d-regular d-frame:

(Weak implication)

$(\supset 1)$ $\quad \varphi \supset (\psi \supset \varphi)$
$(\supset 2)$ $\quad (\varphi \supset (\psi \supset \gamma)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \gamma))$
$(\neg\neg R)$ $\quad \neg\neg\varphi \supset \varphi$

(Logical conjunction and disjunction)

$(\curlywedge \supset)$ $\quad (\varphi \curlywedge \psi) \supset \varphi$ *and* $(\varphi \curlywedge \psi) \supset \psi$
$(\supset \curlywedge)$ $\quad \varphi \supset (\psi \supset (\varphi \curlywedge \psi))$
$(\supset \mathtt{tt})$ $\quad \varphi \supset \mathtt{tt}$
$(\supset \curlyvee)$ $\quad \varphi \supset (\varphi \curlyvee \psi)$ *and* $\psi \supset (\varphi \curlyvee \psi)$
$(\curlyvee \supset)$ $\quad (\varphi \supset \gamma) \supset ((\psi \supset \gamma) \supset ((\varphi \curlyvee \psi) \supset \gamma))$
$(\supset \mathtt{ff})$ $\quad \mathtt{ff} \supset \varphi$

(Informational conjunction and disjunction)

$(\sqcap \supset)$ $\quad (\varphi \sqcap \psi) \supset \varphi$ *and* $(\varphi \sqcap \psi) \supset \psi$
$(\supset \sqcap)$ $\quad \varphi \supset (\psi \supset (\varphi \sqcap \psi))$
$(\supset \top)$ $\quad \varphi \supset \top$
$(\supset \sqcup)$ $\quad \varphi \supset (\varphi \sqcup \psi)$ *and* $\psi \supset (\varphi \sqcup \psi)$

($\sqcup \supset$)	$(\varphi \supset \gamma) \supset ((\psi \supset \gamma) \supset ((\varphi \sqcup \psi) \supset \gamma))$
($\supset \bot$)	$\bot \supset \varphi$

(Negation)

($\neg \curlywedge$ L)	$\neg(\varphi \curlywedge \psi) \subset \neg\varphi \curlyvee \neg\psi$
($\neg \curlyvee$)	$\neg(\varphi \curlyvee \psi) \equiv \neg\varphi \curlywedge \neg\psi$
($\neg \sqcap$)	$\neg(\varphi \sqcap \psi) \equiv \neg\varphi \sqcap \neg\psi$
($\neg\sqcup$ L)	$\neg(\varphi \sqcup \psi) \subset \neg\varphi \sqcup \neg\psi$
($\neg\supset$ R)	$\neg(\varphi \supset \psi) \supset \varphi \curlywedge \neg\psi$

where $\varphi \equiv \psi$ is a shorthand for $(\varphi \supset \psi) \curlywedge (\psi \supset \varphi)$. Furthermore, the rule of Modus Ponens is sound:

(MP)	$\varphi, (\varphi \supset \psi) \vdash \psi$

Furthermore, it is also shown that a modal extension of bilattices can be also modelled in the category of d-compact d-regular d-frames as algebras of the following type:

$$\mathbb{W}_d(\mathcal{L}) \oplus \mathbb{W}_d(\mathcal{L}) \to \mathcal{L}$$

3.3.1 Belnap-Dunn geometric logic

Belnap, inspired by Scott's [Sco70], argued that (when paraphrased),

($\star$) *predicates ought to be constructed as directed joins of their (finite) approximations* (§81.1 and §81.3.2 in [Bel76]).

A directed join in the information order is understood as a computation which generates its output gradually, in a limiting process. Requiring ($\star$) simply means that all predicates must be somehow computable, even though some predicates may only be represented by an infinite computation which produces them.

This requirement, although justified philosophically, is not fulfilled by bilattices. On the other hand, d-compact d-regular d-frames are better suited to model Belnap's logic since they satisfy ($\star$) automatically. In the last section, we introduce a logic of d-frames and prove its soundness and completeness:

3.7 Theorem (Completeness)**.**
If a judgement φ in d-frame logic is true for all d-frames which satisfy Γ, i.e. $\Gamma \vDash \varphi$, then φ is provable from Γ, i.e. $\Gamma \vdash \varphi$.

Proof sketch. Because $\mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma\rangle \vDash \Gamma$, it is also the case that $\mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma\rangle \vDash \varphi$ (where $\sigma\Gamma$ is the closure of Γ under all substitutions). Then, we obtain the proof of $\Gamma \vdash \varphi$ by unfolding the iterative procedure which produces $\mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma \rangle$; every application of $\mathfrak{r}$ produces one derivation in the proof. $\square$

4 List of Publications

Journal papers

Richard N. Balla, Bernhard Banaschewski, Tomáš Jakl, Aleš Pultr, Joanne Walters-Waylande: *Tightness relative to some (co)reflections in topology*, Quaestiones Mathematicae, 2015.

Conference proceedings

Tomáš Jakl, Achim Jung, Aleš Pultr: *Bitopology and four-valued logic*, Proceedings of the 32nd Annual Conference on Mathematical Foundations of Programming Semantics (MFPS XXXII), Electronic Notes in Theoretical Computer Science, 2016.

Tomáš Jakl, Achim Jung: *Free constructions and coproducts of d-frames*, Proceedings of the 7th Conference on Algebra and Coalgebra in Computer Science (CALCO 2017), ed. Leibniz International Proceedings in Informatics, vol. 72, 2017, pages 14:1-14:15

In preparation/submitted

Tomáš Jakl, Achim Jung, Aleš Pultr: *Quotients of d-frames* (submitted).

Tomáš Jakl, Achim Jung: *A Vietoris construction for bispaces and d-frames*.

Future work

During the final stages of the thesis preparation the author opened a number of new topics which deserve their own dedicated publication. Namely, in the thesis we set out solid grounds for papers with working titles:

1. Point-Free Presentation of the Bitopological Real Numbers (Section 3.5.3)
2. New Models for Positive Modal Logic (Section 4.5.1)
3. Logic of d-Frames (Section 6.3)

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