Geometric representations of graphs

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1 Some characterization theorems

1.1 Classes of interest

Definition 1 *Let $\mathcal{M}$ be a family of sets. We denote by IG($\mathcal{M}$) the class of all graphs G for which there exists a mapping*

$$f : V_G \longrightarrow \mathcal{M}$$

such that for every pair of distinct $u, v \in V_G$,

$$uv \in E_G \textit{ if and only if } f(u) \cap f(v) \neq \emptyset.$$

The graphs of IG($\mathcal{M}$) are referred to as the intersection graphs of $\mathcal{M}$, *but in many cases they have their own specific names.*

Observation 1 *If $\mathcal{M}$ contains at least one nonempty set (a very natural assumption), then IG($\mathcal{M}$) contains all complete graphs.*

For every $\mathcal{M}$, IG($\mathcal{M}$) is closed in the induced subgraph order (i.e., if $G \in$ IG($\mathcal{M}$) and H is an induced subgraph of G, then necessarily $H \in$ IG($\mathcal{M}$)). Therefore each IG($\mathcal{M}$) can be characterized by forbidden induced subgraphs (minimal graphs which are not in the class), but usually such characterization involves infinitely many forbidden graphs. Explicit description of the forbidden graphs is known in only very few cases.

Numerous classes have been considered in the literature. We will be interested in the following ones:

INT = IG(intervals on a line) called the *interval graphs*

CA = IG(intervals on a circle) called the *circular arc graphs*

CIR = IG(chords of a circle) called the *cricle graphs*

PC = IG(polygons inscribed to a circle) called the *polygon-circle graphs*

PER = IG(segments with endpoints on two parallel lines) which are exactly the *permutation graphs*

FUN = IG(curves with endpoints on two parallel lines lying between the lines) = IG(graphs of continuous functions defined on $< 0, 1 >$) called the *function graphs*

IFA = IG(interval filaments) = IG(graphs of continuous nonnegative functions defined on closed intervals with zero-values on their endpoints) called the *interval filament graphs*

STRING = IG(curves in the plane) = IG(arc-connected sets in the plane) called the *string graphs*

OSTRING = IG(curves lying in a disk, with one endpoint tied to the boundary) called the *outer-string graphs*

CONV = IG(convex sets in the plane)

SEG = IG(straight-line segments in the plane)

2BOX = IG(isothetic rectangles in the plane) are the graphs of *boxicity* ≤ 2

GI = IG(vertical and horizontal segments in the plane) called the *grid intersetion graphs*

Remarks The standard definition of permutation graphs is the following: A graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ is a permutation graph if there exists a permutation π of $\{1, 2, \ldots, n\}$ such that for every $i \neq j$, $v_i v_j \in E_G$ if and only if

$$(i - j)(\pi(i) - \pi(j)) < 0.$$

The equivalence with the intersection definition is readily seen.

In the case of grid intersection graphs it is assumed that no two horizontal (and no two vertical) segments overlap. Thus all grid intersection graphs are bipartite.

Boxicity of graphs has been defined by Roberts as follows:

Definition 2 *The boxicity of a graph G, denoted by* $\mathrm{box}(G)$, *is the smallest dimension d such that G is an intersection graph of d-dimensional intervals (boxes in the d-dimensional euclidean space).*

Observation 2 *The following inclusions follow more or less directly from the definitions:*

$INT \subseteq CA \subseteq PC \subseteq IFA$
$INT \subseteq FUN \subseteq IFA \subseteq OSTRING \subseteq STRING$
$PER \subseteq CIR \subseteq PC$

PER $\subseteq$ *FUN*
GI $\subseteq$ *SEG* $\subseteq$ *CONV* $\subseteq$ *STRING*
GI $\subseteq$ *2BOX* $\subseteq$ *CONV*.

We further introduce two classes of graphs, namely *chordal* graphs (denoted by CHOR) and *comparability* graphs (denoted by COMP), which are not defined as intersection graphs, but which play important roles in this theory.

Definition 3 *A graph is* chordal *if it contains no induced cycles of length greater than three.*

Definition 4 Comparability *graphs are graphs expressing the comparability relations in partially ordered sets. I.e., G is a comparability graph if its vertex set can be partially ordered by a (transitive, antisymmetric, antireflexive) relation $<$ so that for any two vertices $u \neq v \in V_G$, $uv \in E_G$ iff $u < v$ or $v < u$.*

Observation 3 *A graph is a comparability graph if and only if its edges can be transitively oriented.*

1.2 Characterization of chordal graphs

Definition 5 *A vertex v of a graph G is called* simplicial *if its neighborhood $N_G(v)$ induces a complete subgraph. A* perfect elimination scheme *of G is an ordering of its vertices $v_1, v_2, \ldots, v_n$ such that for every $i = 1, 2, \ldots, n$, the vertex v_i is simplicial in the graph $G[v_1, \ldots, v_n]$.*

Lemma 4 *Every (inclusion-wise) minimal vertex cut in a chordal graph induces a complete subgraph.*

Lemma 5 *Every chordal graph which is not a complete graph contains at least two nonadjacent simplicial vertices.*

Definition 6 *A* clique tree *of a graph G is a tree T whose vertices are all (icnlusion-wise) maximal cliques of G, and which has the following property: For every vertex $u \in V_G$, the cliques containing it induce a connected subgraph in T.*

Not every graph allows a clique tree, and on the other hand, the clique tree is not necessarily unique (if it exists).

Theorem 6 (Intersection characterization of chordal graphs) *For every graph G, the following statements are equivalent:*

1. *G is a chordal graph;*
2. *G allows a perfect elimination scheme;*

3. G has a clique tree;

4. G is the intersection graph of a family of subtrees of a tree.

Proof Prove 1. $\to$ 2. $\to$ 3. $\to$ 4. $\to$ 1. qed

Observation 7 *Now we can see that CHOR $\subseteq$ PC.*

Remark It follows directly from the definition of interval graphs that they are exactly intersection graphs of subpaths in paths, and hence INT $\subseteq$ CHOR.

1.3 Characterization of interval graphs

A *clique path* is a clique tree which is a path.

If $\mathcal{A}$ is a class of graphs, we denote by CO-$\mathcal{A}$ the class containing the complements of graphs from $\mathcal{A}$, i.e., CO-$\mathcal{A} = \{\overline{G} | G \in \mathcal{A}\}$.

Theorem 8 (Characterization of interval graphs) *For every graph G, the following statements are equivalent:*

1. G is an interval graph;

2. G has a clique path;

3. $G \in$ CHOR $\cap$ CO-COMP.

Proof Prove 1. $\to$ 3. $\to$ 2. $\to$ 1. qed

1.4 Classes related to comparability graphs

Theorem 9 *Function graphs are exactly complements of comparability graphs.*

Proof FUN $\subseteq$ CO-COMP: Given a representation of a graph G by curves lying between two vertical parallel lines, orient every nonedge uv of the graph from u to v if the curve representing v lies above the curve representing u (this is well defined due to the Jordan curve theorem). This gives a transitive orientation of the complement of G.

CO-COMP $\subseteq$ FUN: Let P be a partial order on the vertices of G such that the comparability graph of P is the complement of G. Express P as the intersection of d linear orders (minimum such d is called the dimension of the partial order). Draw d vertical lines and place points corresponding to the vertices of G on each of the lines in one of the linear orders. For every vertex of G, connect the points corresponding to it by a piece-wise linear curve. These curves form a function representation of G. qed

Lemma 10 *Complements of permutation graphs are again permutation graphs, and hence these two classes coincide.*

Proof If π is he permutation realizing G, then π^{-1} realizes $\overline{G}$. Hence CO-PER $\subseteq$ PER and therefore PER = CO-CO-PER $\subseteq$ CO-PER. qed

Theorem 11 *A graph G is a permutation graph if and only if both G and its complement are comparability graphs.*

Proof PER $\subseteq$ CO-COMP follows from PER $\subseteq$ FUN = CO-COMP.

PER $\subseteq$ COMP then follows from PER = CO-PER $\subseteq$ CO-FUN = COMP.

COMP $\cap$ CO-COMP $\subseteq$ PER: Let F_1 be a transitive orientation of the edges of G, and F_2 a transitive orientation of its nonedges. Then both $F_1 \cup F_2$ and $F_1^{-1} \cup F_2$ are linear orderings of the vertices of G. Place points corresponding to the vertices of G on two parallel lines in these orderings and connect pairs of points corresponding to the same vertex by straight segments. This is a permutation representation of G. qed

1.5 Interval filament graphs and mixed graphs

Definition 7 *Let $\mathcal{G}$ be a class of graphs. The class of $\mathcal{G}$-*mixed *graphs contains all graphs $G = (V, E)$ such that E can be partitioned into $E = E_1 \cup E_2$ and E_2 transitively oriented by a relation F in such a way that*

$(V, E_1) \in \mathcal{G}$, and

for all $x, y, z \in V$, $xy \in F$ and $yz \in E_1$ imply $xz \in E_1$.

Theorem 12 *Complements of interval filament graphs are exactly (co-interval)-mixed graphs.*

Proof If $G = (V, E)$ is an interval filament graph with filament representation $I_u, f_u, u \in V$, the nonedges of G are partitioned into

$$E_1 = \{uv | I_u \cap I_v = \emptyset\}$$

and

$$E_2 = \{uv | (I_u \subseteq I_v \text{ or } I_v \subseteq I_u) \text{ and } f_u \cap f_v = \emptyset\}.$$

Moreover, (V, E_1) is the complement of an interval graph (namely of the intersection graph of intervals $I_u, u \in V$) and the orientation F of E_2 given by

$$F = \{uv | I_u \subseteq I_v \text{ and } f_u \cap f_v = \emptyset\}$$

is transitive and satisfies the mixing property.

On the other hand, if G is a graph such that its complement $\overline{G}$ is (CO-INT)-mixed, we assume to have a partition of all pairs of vertices

$$\binom{V}{2} = E \cup E_1 \cup E_2$$

such that $(V, E \cup E_2)$ has an intersection representation by intervals $I_u, u \in V$, and such that E_2 has a transitive orientation F satisfying the mixing property with respect to E_1. We first show that the intervals may be chosen such that

$$uv \in F \text{ implies } I_u \subset I_v,$$

then in the first step define filaments f_u as halfcircles built above I_u, and then pull the filaments locally upwards to reach intersections $f_u \cap f_v \neq \emptyset$ when $I_u \subseteq I_v$ and $uv \notin E_2$. The mixing property guarantees that this is possible without creating undesired intersections. qed

The characterization theorems are summarized in the table below.

INT = CHOR $\cap$ CO-COMP
PER = CO $\cap$ CO-COMP
FUN = CO-COMP
CO-IFA = (CO-INT)-MIXED

1.6 Structural characterization of comparability graphs

Definition 8 *Let $G = (V, E)$ be an undirected graph. A binary relation $F \subseteq V \times V$ is called*

sensitive *if $xy \in F, yz \notin E$ and $xz \in E$ imply $xz \in F$ (and similarly $xy \in F, xz \notin E$ and $yz \in E$ imply $zy \in F$);*

nice *if $xy \in F$ implies $xy \in E$.*

Observation 13 *Every transitive orientation of the edges of a graph is sensitive and nice.*

Definition 9 *If $G = (V, E)$ is a graph and F a binary relation on $V \times V$, we denote by $< F >_S$ the smallest sensitive relation containing F.*

Theorem 14 *A graph $G = (V, E)$ is transitively orientable if and only if for every edge $xy \in E$, the relation $< xy >_S$ generated by the directed arc xy is a nice relation.*

1.7 Grid intersection graphs and boxicity 2

Definition 10 *A* cross *in a 0-1 matrix M is a sixtuple of indices $i_1 < i_2 < i_3$ (for rows) and $j_1 < j_2 < j_3$ (for columns) such that $M_{i_1 j_2} = 1, M_{i_2 j_1} = 1, M_{i_2 j_2} = 0, M_{i_2 j_3} = 1, M_{i_3 j_2} = 1$.*

Theorem 15 *A connected bipartite graph $G = (A \cup B, E)$ is a grid intersection graph if and only if the rows and columns of its bipartite adjacency matrix can be permuted to destroy all crosses.*

Straightforwardly, every grid intersection graph has boxicity at most 2. Somewhat surprisingly, the converse is also true (for bipartite graphs).

Theorem 16 ([1]) *Every bipartite graph of boxicity 2 is a grid intersection graph.*

1.8 Outerstring graphs

Definition 11 *A* Constrained Outerstring *representation of a pair G, π, where π is an ordering of the vertices of G, is an outerstring representation of G in which the curves representing the vertices of G meet the boundary of the bounding disk in the order π.*

Theorem 17 *For $n \geq 4$, the complement of the cycle C_n with the natural cyclic ordering has no constrained outerstring representation.*

Proof by induction on n, using the fact that every outerstring graph has an outerstring representation with the same ordering of the endpoints meeting the bounding cycle, and with finite number of crossing points of the curves and no three curves passing through the same point. In the inductive step, if a representation of $\overline{C_n}$ exists, take one with minimum number of crossing points and shorten one curve to lose the last crossing point. That gives a representation of $\overline{C_k}$ for some $k < n$. qed

1.9 Exercises

1. Show that no odd cycle of length greater than three and no complement of a cycle of length greater than four is a comparability graph.

2. Show that all inclusions among INT, PER, CIR, CA, PC, CHOR, FUN and IFA mentioned in Observations 2 and 7 are strict, and that no other inclusions among these classes hold.

3. Prove that the boxicity (Definition 2) is well defined and that for every graph G, box(G) is the least number k for which there exists sets E_1, E_2, $\ldots$, $E_k \subseteq \binom{V_G}{2}$ such that

   - (V_G, E_i) is an interval graph for every $i = 1, 2, \ldots, k$, and

   - $E_G = \bigcap_{i=1}^{k} E_i$.

4. Show that a chordal graph on n vertices contains at most n maximal cliques.

5. A graph is a *proper interval graph* if it has an intersection representation by intervals such that no two of the intervals are in inclusion. A graph is a *unit interval graph* if it has an intersection representation by intervals of unit length. Show that these two classes of graphs coincide.

6. A set is called *topologically connected* if it cannot be partitioned into two subsets, both of them closed and open in the original set. Show that every graph is the intersection graph of a collection of topologically connected sets in the plane.

7. Construct a graph on 15 vertices which is not a string graph. (Hint: K_5 and $K_{3,3}$ are non-planar graphs.)

8. Show that every (finite) string graph has an intersection representation by simple curves such that every two of them have only finite number of crossing points.

9. Show that IG(curves in the plane) = IG(arc-connected sets in the plane).

10. Prove Theorem 14.

11. Prove the following "characterization" of co-chordal graphs: A graph G is chordal if and only if its complement $\overline{G}$ has a constrained outerstring representation for every ordering of its vertices.

2 Computational complexity of the recognition problems

Theorem 18 *The classes INT, CHOR, PER, FUN, CA and CIR can be recognized in polynomial time.*

Recognition algorithms for INT, CHOR, PER and FUN follow from characterization theorems proven in the previous section. Algorithms for CA and CIR are nontrivial and can be found in [14].

Theorem 19 ([9]) *Recognition of SEG, CONV and STRING are NP-hard problems.*

Proof By a uniform reduction from Planar Satisfiability of formulas with at most 4 occurrences per variable, with 3-connected incidence graphs. The gadget for clauses is based on constrained outerstring non-representability of $\overline{C_6}$. qed

Theorem 20 ([12]) *Recognition of GI and 2BOX are NP-hard problems.*

Proof Similar reduction with different clause gadget. The constructed graph is bipartite and hence membership in GI and 2BOX coincides. qed

Theorem 21 *Recognition of CONV and SEG graphs are in PSPACE.*

Proof Using the fact that deciding if a system of polynomial inequalities is solvable in real numbers is a problem solvable in polynomial space. For CONV we introduce two variables

$$x_{uv}, y_{uv}$$

for every edge $uv \in E_G$, and three variables

$$a_{uv}, b_{uv}, c_{uv}$$

for every nonedge uv of G. If every vertex $u \in V_G$ is represented by the convex hull of $\{[x_{uv}, y_{uv}] | v \in V_G, uv \in E_G\}$, then all edges of G are represented by nonempty intersections of the corresponding sets. Then G is the intersection graph of these sets if and only if for every nonedge uv, the following inequalities hold:

$$a_{uv}x_{uw} + b_{uv}y_{uw} + c_{uv} > 0 \text{ for all } w \in V_G, uw \in E_G,$$

$$a_{uv}x_{vw} + b_{uv}y_{vw} + c_{uv} < 0 \text{ for all } w \in V_G, vw \in E_G$$

($a_{uv} + b_{uv} + c_{uv} = 0$ is the equation of a line separating the sets representing u and v).

Existence of a real solution to this system of inequalities can be tested in PSPACE (folklore). qed

The following difficult result is rather surprising, namely in comparison to Thereom 29.

Theorem 22 ([16]) *Recognition of STRING is in NP.*

2.1 Exercises

1. Show that chordal graphs can be recognized in polynomial time.
2. Derive a polynomial time algorithm for recognizing comparability graphs (you may use Theorem 14).
3. Design a direct polynomial time algorithm for recognizing interval graphs.
4. Prove that recognition of PC graphs is in NP.
5. Prove that recognition of IFA graphs is in NP.
6. Show that recognition of SEG graphs is in PSPACE.

2.2 Open problems

1. Is recognition of CONV (SEG) in the class NP?
2. How difficult is recognition of PC graphs?
3. How difficult is recognition of IFA graphs?

3 Complexity of some optimization problems restricted to classes of intersection graphs

3.1 Chromatic number

Chromatic number can be computed in polynomial time for chordal graphs and for function graphs, but it is NP-hard for circle and circular arc graphs. The former two classes of graphs are perfect and thus an algebraic polynomial time algorithm exists. However, simpler combinatorial algorithms exist as well.

Algorithm 1 (First Fit for chordal graphs)
Input: A chordal graph $G = (V, E)$ with a perfect elimination scheme $V = v_1, v_2, \ldots, v_n$.
Body
for i:=1 **to** n **do**
$\phi(v_i) := \min\{1, 2, \ldots, n\} \setminus \{\phi(v_j) | j < i \text{ and } v_i v_j \in E\}$.
Output coloring ϕ and chromatic number $\chi(G) = \max_{i=1}^{n} \phi(v_i)$.

Theorem 23 *Algorithm 1 finds the chromatic number of the input chordal graph in time linear in the number of edges. Moreover, it also identifies a clique of size $\chi(G)$.*

3.2 Maximum weight clique in mixed graphs

Maximum size of an independent set can be determined in polynomial time for all "small" classes - INT, FUN, PER, CA, CIR, CHOR, PC, IFA. Even the algorithm for IFA graphs can be generalized for mixed graphs of certain types.

Theorem 24 ([6]) *Let $\mathcal{G}$ be a class of graphs for which the maximum weight clique problem can be solved in time $F(n, m)$. Then this problem can be solved in time $O(nF(n, m) + n + m)$ on $\mathcal{G}$-mixed graphs, provided a mixed partition is given as part of the input.*

Algorithm 2 (Max Weight Clique for $\mathcal{G}$-mixed graphs)
Input: A graph $G = (V, E_1 \cup E_2)$ and an orientation F of E_2 such that $(V, E_1) \in \mathcal{G}$ and F, E_1 satisfy the mixing property, and a weight function $w : E_1 \cup E_2 \longrightarrow Z^+$ on the edges of G.
Body
1. Order the vertices $V = v_1, v_2, \ldots, v_n$ in a linear extension of F.
2. WLOG assume that $v_i v_n \in F$ for every $i = 1, 2, \ldots, n-1$ (add a dummy vertex of weight 0 if necessary).
3. For every $i = 1, 2, \ldots, n$, define

$$W_i^* = \{v_j | j < i \text{ and } v_j v_i \in F\}, \text{ and}$$

$$W_i = W_i^* \cup \{v_i\}.$$

4. **for** $i := 1$ **to** n **do**
5. Let Q_j be a maximum weight clique in $G[W_j]$, and $c_j = w(Q_j)$, for every $j \in W_i^*$. (Recursive computation.)
6. Let Q be a maximum weight clique in $\widetilde{G}_i = (W_i^*, E_1 \cap \binom{W_i^*}{2})$ with respect to weights c_j per vertex $v_j \in W_i^*$.
7. Set $Q_i = \bigcup_{v_j \in W_i^*} Q_j \cup \{v_i\}, \quad c_i = \sum_{v_j \in W_i^*} c_j + w(v_i)$.
enddo
Output Q_n, c_n.

Observation 25 *Maximum weight clique in complements of interval (and more generally of function graphs) can be computed in polynomial time.*

Corollary 26 *Maximum weight independent set in IFA graphs can be computed in polynomial time.*

3.3 Exercises

1. Design a linear time algorithm for determining the maximum size of an independent set in a chordal graph.
2. Prove Observation 25.
3. Show that maximum weight clique in a comparability graph can be found in polynomial time.
4. * Show that the maximum size of an independent set in a comparability graph can be computed in polynomial time.
5. Given a finite family of intervals such that every two of them have a nonempty intersection, show that all intervals from this family share a common point (the Helly property).
6. Using the previous two exercises, show that the maximum size of a clique in an IFA graph can be found in polynomial time.
7. Show that CLIQUE is NP-complete for CONV graphs (use the fact that INDEPENDENT SET is NP-complete for planar graphs, but wait with the proof till the last section.)

3.4 Open problem

1. What is the complexity of CLIQUE for SEG graphs?

4 Sizes of representations

Natural way of showing that a problem is in the class NP is to provide a certificate for the YES answer that can be verified in polynomial time. Somewhat strangely some intersection classes do not allow straightforward certificates of this type. This is due to the fact that there are graphs that *require* representations of exponential size.

4.1 SEG graphs

Theorem 27 ([11]) *For every n, there exists a graph $G_n \in SEG$ with $O(n^2)$ vertices such that every intersection representation of G by segments with integer coordinates, has a coordinate $\geq 2^{2^n}$. (Hence already writing down this coordinate requires exponentially many bits with respect to the size of G_n.)*

4.2 STRING graphs

The reasoning of the proofs is easier for the so called *Abstract Topological Graphs.*

Definition 12 *An* abstract topological graph *is a pair $G = (V, E), R \subseteq \binom{E}{2}$. The at-graph G, R is called* realizable *if G can be drawn in the plane so that the drawings of two edges cross only if these two edges form a pair from R. If G, R is realizable, we denote by $cr(G, R)$ the minimum number of crossing points in a legal drawing of G.*

Remark Thus the pairs from R are pairs of edges that are allowed to intersect in a legal drawing. If $R = \emptyset$ then G must be a planar graph. If $R = \binom{V}{2}$, then $cr(G, R) = cr(G)$ the ordinary crossing number of G.

To define the complexity of representations and the maximum necessary complexity of representations of graphs with a given number of vertices, we introduce two functions:

$$f_{at}(n) = \max_{G,R:|E_G|=n} cr(G, R)$$

$$f_{st}(n) = \max_{G:|V_G|=n} \min_{\text{STRING representation S}} \#\text{ of crossings in } \mathcal{S}.$$

These two functions are polynomially equivalent, but both are at least exponential.

Theorem 28 ([10])

$$f_{at}(n) \leq f_{st}(2n) - 2n,$$

$$f_{st}(n) \leq f_{at}(n^2 - 2n) + 2n^2.$$

Theorem 29 ([10])

$$f_{at}(n) \geq 2^{\frac{n-7}{4}}.$$

For a long time no recursive upper bound has been known, and neither a finite recognition algorithm. This has changed recently.

Theorem 30 ([15])

$$f_{at}(n) \leq \frac{n}{2}(2^{n-1}).$$

Proof is based on this following clever claim. If G, R is an abstract topological graph and D a legal drawing with minimum number of crossing points, then every edge crossed by k other edges carries at most $2^k - 1$ crossing points. qed

5 Chromatic number versus the clique number

5.1 Perfect graph classes

Definition 13 *A graph G is called* perfect *if $\chi(G') = \omega(G')$ holds true for every induced subgraph G' of G.*

Perfect graphs have been introduced by Berge, who conjectured that a graph is perfect if and only if it contains no odd cycles of length greater than three nor their complements as induced subgraphs. This so called Strong Perfect Graph Conjecture has been recently proved by Chudnovsky, Robertson, Seymour and Thomas [2]. It implies that a graph is perfect if and only if its complement is perfect, a result proven 30 years ago by Lovász (and before that referred to as the Weak Perfect Graph Conjecture of Berge). In the case of intersection graph classes we will be able to avoid this heavy machinery. Note that for a class of graphs $\mathcal{A}$ closed in the induced subgraph order (and such are all intersection classes), it suffices to check that $\chi(G) = \omega(G)$ holds for all graphs $G \in \mathcal{A}$ in order to show that all graphs from $\mathcal{A}$ are perfect (i.e., we do not need to show the equality for all induced subgraphs of the graphs of $\mathcal{A}$).

Theorem 31 *Chordal graphs are perfect.*

Proof The First Fit method applied in a Perfect Elimination Scheme ordering of the vertices produces a coloring using $\omega(G)$ colors. qed

Theorem 32 *Comparability graphs are perfect.*

Proof This statement is equivalent to the fact that for every partial order, the size of the longest chain equals the minimum number of antichains that cover all elements. qed

Theorem 33 *Complements of comparability graphs are perfect.*

Proof This is equivalent to the fact that in every partial order, the maximum size of an antichain equals the minimum number of chains that cover all elements. And this is the well known Dilworth theorem on posets. qed

5.2 Near-perfect graph classes

Informally we call a class $\mathcal{A}$ of graphs *near-perfect* if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{A}$. Problems of this type have been studied by Gyarfás and Erdős. The former proved that CIR graphs are near-perfect. We improve his bound a little bit and show that it holds for a more general class of graphs.

Theorem 34 ([8]) *For every $G \in PC$, $\chi(G) \leq 50 \cdot 2^{\omega(G)}$.*

Suppose we have an alternating representation of $G \in \mathrm{PC}$, i.e., for every vertex $u \in V_G$, we have a pair I_u, M_u such that I_u is an interval and M_u is a finite number of points in I_u which contains both endpoints of I_u. The representation is such that for every two vertices $u, v \in V_G$, $uv \in E_G$ iff some two points of M_u alternate with some two points of M_v.

Let x_0 be the vertex whose interval I_{x_0} has minimum left endpoint. We partition the vertex set of G into levels V_i by the distance from x_0.

Lemma 35 *If $\chi(G[V_i]) \leq k$ for all i, then $\chi(G) \leq 2k$.*

Lemma 36 *If $i > 0$ and $u \in V_i, U \subseteq V_i$ are such that*

$$I_u \subseteq \bigcap_{x \in U} I_x,$$

then V_{i-1} contains a vertex z such that

$$zx \in E_G \text{ for all } x \in U \cup \{u\}.$$

Lemma 37 *Let A, B be families of intervals such that (1) every two intervals of B are disjoint, (2) no intervals of A are disjoint, and (3) every interval of A contains at least 2 intervals of B. Then A contains a set of $\lceil \frac{|A|}{2} \rceil$ intervals whose intersection contains at least one interval of B.*

Define the class of graphs $\mathcal{H}(m)$ parametrized by a positive integer m as the class of all graphs G that satisfy $\omega(G) \leq 2m$ and that have an alternating representation $I_u, M_u, u \in V_G$ such that for every clique $Q \subseteq V_G$ of size m, $\bigcap_{x \in Q} I_x$ contains no I_y, $y \in V_G - Q$. Set $h(m) = \max_{G \in \mathcal{H}(m)} \chi(G)$.

Theorem 38

$$h(m) \leq 25 \cdot 2^m - 16m - 32.$$

Proof By induction on m. Straightforward for $m = 1$.

For $m > 1$ consider a graph $G \in \mathcal{H}(m)$ partitioned into levels V_i, and consider the subgraph H induced by one level, say $H = G[V_i]$. Let $U \subseteq V_i$ be the set of vertices such that the intervals $I_u, u \in U$ are disjoint, $|U|$ is maximum possible and subject to these two conditions, the right endpoints of the intervals I_u are leftmost possible. Let P be the set of points containing the right endpoint of every third interval $I_u, u \in U$.

Partition the vertices of V_i into three sets

- $U_1 = \{x : |I_x \cap P| = 1\}$,
- $U_2 = \{x : |I_x \cap P| = 0\}$,
- $U_3 = \{x : |I_x \cap P| \geq 2\}$.

Then

- $\chi(H[U_1]) \leq 2\omega(H[U_1]) \leq 4m$,
- $\chi(H[U_2]) \leq 2\omega(H[U_2]) \leq 4m$,
- $\chi(H[U_3]) \leq h(m-1)$ since $H[U_3] \in \mathcal{H}(m-1)$.

Hence

$$\begin{aligned}\chi(G) &\leq 2(4m + 4m + h(m-1)) \leq 16m + 2(25 \cdot 2^{m-1} \\ &= 16(m-1) - 32) = 25 \cdot 2^m - 16m - 32.\end{aligned}$$

qed

Proof of Theorem 34. Given $G \in PC$, we again partition the vertex set into levels and note that for each level V_i, $G[V_i] \in \mathcal{H}(\omega(G))$. qed

Near-perfectness of SEG and STRING graphs are well known open problems. Even for triangle-free graphs no upper bound on the chromatic number is known. Kostochka and Nešetřil observed that requiring larger girth (length of a shortest cycle) may help.

Theorem 39 *Let G be a SEG graph of girth $k \geq 5$. Then*

- $\chi(G) \leq 6$ *for* $k = 5$,
- $\chi(G) \leq 4$ *for* $k = 6, 7$,
- $\chi(G) \leq 3$ *for* $k \geq 8$.

Proof Let $G \in$ SEG and $n = |V_G|, m = |E_G|$. Regard a SEG representation as a planar drawing of the graph H whose vertices are the crossing points of the segments of the representation, and edges are portions of the segments between two consecutive crossing points. Denote $v = |V_H|$, $e = |E_H|$ and let f be the number of faces of the drawing of H. Then

$$v = m, \ e = 2v - n \ \text{ and } v + f = e + 2.$$

If $\chi(G) = \chi$, we may assume that G is χ-critical and hence has minimum degree $\geq \chi - 1$. Thus $m \geq n(\chi - 1)/2$.

The girth of H cannot be smaller than girth of G, and hence $2e \geq kf$. Plugging this into the Euler formula and using the relations among n, m, v, e and f, we finally get

$$\frac{\chi - 1}{2} \leq \frac{m}{n} < \frac{k-2}{k-4}$$

and

$$\chi < 3 + \frac{4}{k-4}$$

from which the result follows immediately. qed

5.3 Exercises

1. Prove Lemma 37.
2. Show that Theorem 39 holds also for 1-STRING graphs (i.e., intersection graphs of curves with at most one crossing per pair of curves).
3. Prove that for intersection graphs of infinite rays, $\chi(G) \leq 4$ if the girth of G is 5, and $\chi(G) \leq 3$ if the girth is ≥ 6.
4. Prove that $\chi(G) \leq 2\omega(G)$ for $G \in$ CA.
5. Prove that complements of chordal graphs are perfect.

5.4 Open problems

1. Are SEG, 1-STRING, STRING graphs near-perfect?

6 Intersection and contact representations of planar graphs

6.1 Contact graphs of disks

The following fundamental theorem was rediscovered several times. Its proof can be found for instance in [13].

Theorem 40 *Planar graphs are exactly the contact graphs of disks in the plane.*

Blowing the disks by ϵ and considering their boundaries as curves, we get

Corollary 41 *Every planar graph has a STRING representation such that every two strings share at most two common crossing points (graphs having such a representation are called 2-STRING).*

6.2 Visibility representations

Definition 14 *An* $s-t$ numbering *of a planar graph is a drawing with y-monotone upward oriented edges which has exactly one source and exactly one sink.*

Lemma 42 *Every 2-connected planar graph has an* $s-t$ *numbering.*

Algorithm 3 (Visibility representation)
Input: A 2-connected planar graph G.
1. Draw the graph G in an $s-t$ numbering so that all vertices have distinct y coordinates.
2. Add a dummy arc st directed from s to t.
3. Draw the dual graph G^* and orient its edges so that they cross the edges of G from left to right (with the sole exception of the edge crossing the dummy st edge, which is directed oppositely).
4. Show that the orientation of the dual also an $s-t$ numbering.
5. Draw horizontal segments in a grid (corresponding to the vertices of G), and vertical segments (corresponding to the vertices of G^*) on the lines with the same labels as the vertices, and extend them from the lowest to the highest coordinate of the incident faces (vertices) of G.
end.

The segments do not cross and two of them touch if and only if the corresponding objects are incident. Also two horizontal segments see each other in at least one rectangle iff the corresponding vertices are adjacent. Thus horizontal segment represent vertices of G, vertical segments represent the faces of G and rectangular areas represent the edges of G. For details see [4].

Corollary 43 ([4]) *Every bipartite planar graph is a GI graph.*

6.3 Canonical orderings

Definition 15 *A* canonical ordering *of the vertices of a planar triangulation* G *with the boundary of the outerface* v_1, v_2, v_n *is an ordering* $v_1, \ldots, v_n$ *such that each graph* $G[v_1, \ldots, v_i]$ *is 2-connected and* v_{i+1} *is adjacent to a set of consecutive vertices on the boundary of* $G[v_1, \ldots, v_i]$.

Lemma 44 *Every planar triangulation has a canonical ordering of its vertices.*

Though quite simple, the concept of canonical orderings is useful in several situations.

Theorem 45 *The edges of a planar triangulation can be partitioned into 3 trees.*

Theorem 46 ([3]) *Planar triangulations (and hence all planar graphs) are contact graphs of triangles.*

Using barycentric coordinates computed from a canonical ordering, one can show the following result on drawing graphs with straight edges into polynomial-sized grids.

Theorem 47 ([5]) *Every planar graph can be embedded in an $n \times n$ grid so that vertices are placed in the grid points and edges are straight line segments.*

6.4 Exercises

1. Show that triangle-free planar graphs are in 1-STRING.
2. Show that complements of planar graphs are in CONV (you may use the Four Color Theorem).

6.5 Open problems

1. Does every planar graph have a 1-STRING representation?
2. Are planar graphs in SEG?
3. Are complements of planar graphs in SEG?

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