

Optimal Preconditioning for the Interval Parametric Gauss–Seidel Method

Milan Hladík

Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic
<http://kam.mff.cuni.cz/~hladik/>

SCAN, Würzburg, Germany
September 21–26, 2014

Interval Linear Equations

Interval Linear Equations

$$Ax = b, \quad A \in \mathbf{A}, \quad b \in \mathbf{b},$$

where

$$\mathbf{A} := [\underline{A}, \overline{A}] = [A^c - A^\Delta, A^c + A^\Delta],$$

$$\mathbf{b} := [\underline{b}, \overline{b}] = [b^c - b^\Delta, b^c + b^\Delta]$$

are an interval matrix and an interval vector, respectively.

The Solution Set

$$\Sigma := \{x \in \mathbb{R}^n : \exists A \in \mathbf{A}, \exists b \in \mathbf{b} : Ax = b\}.$$

Problem formulation

Find a tight interval vector enclosing Σ .

Preconditioning

Preconditioning

Let $C \in \mathbb{R}^{n \times n}$. Preconditioning is a relaxation to

$$A'x = b', \quad A' \in (CA), \quad b' \in (Cb),$$

Preconditioning

- usually we use $C = (A^c)^{-1}$, theoretically justified by Neumaier (1984, 1990),
- optimal preconditioning for the interval Gauss–Seidel method by Kearfott et al. (1990, 1991, 2008).

The Interval Gauss–Seidel Method

$$z_i := \frac{1}{(CA)_{ii}} \left((Cb)_i - \sum_{j \neq i} (CA)_{ij} x_j \right), \quad x_i := x_i \cap z_i,$$

for $i = 1, \dots, n$.

Interval Parametric Systems

Interval Parametric System

$$A(p)x = b(p), \quad p \in \mathbf{p},$$

where

$$A(p) = \sum_{k=1}^K A^k p_k, \quad b(p) = \sum_{k=1}^K b^k p_k,$$

and $p_k \in \mathbf{p}_k$, $k = 1, \dots, K$.

The Solution Set

$$\Sigma_p := \{x \in \mathbb{R}^n : \exists p \in \mathbf{p} : A(p)x = b(p)\}.$$

Preconditioning and Relaxation

Relaxation to $Ax = b$, where

$$A \in \mathbf{A} := \sum_{k=1}^K (CA^k) \mathbf{p}_k, \quad b \in \mathbf{b} := \sum_{k=1}^K (Cb^k) \mathbf{p}_k.$$

The Parametric Interval Gauss–Seidel Method

The Parametric Interval Gauss–Seidel Method

$$\mathbf{z}_i := \frac{1}{\left(\sum_{k=1}^K (CA^k)_{ii} \mathbf{p}_k\right)} \left(\sum_{k=1}^K (Cb^k)_i \mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^K (CA^k)_{ij} \mathbf{p}_k \right) \mathbf{x}_j \right),$$

$$\mathbf{x}_i := \mathbf{x}_i \cap \mathbf{z}_i,$$

for $i = 1, \dots, n$.

Optimal Preconditioner

Various criteria of optimality:

-
- minimize the resulting upper bound, that is, the objective is $\min \bar{\mathbf{z}}_i$,
- maximize the resulting lower bound, that is, the objective is $\max \underline{\mathbf{z}}_i$.

The Parametric Interval Gauss–Seidel Method

The Parametric Interval Gauss–Seidel Method

$$\mathbf{z}_i := \frac{1}{\left(\sum_{k=1}^K (CA^k)_{ii} \mathbf{p}_k\right)} \left(\sum_{k=1}^K (Cb^k)_i \mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^K (CA^k)_{ij} \mathbf{p}_k \right) \mathbf{x}_j \right),$$

$$\mathbf{x}_i := \mathbf{x}_i \cap \mathbf{z}_i,$$

for $i = 1, \dots, n$.

Optimal Preconditioner

Various criteria of optimality:

- minimize the resulting width, that is, the objective is $\min 2z_i^\Delta$,
- minimize the resulting upper bound, that is, the objective is $\min \bar{z}_i$,
- maximize the resulting lower bound, that is, the objective is $\max \underline{z}_i$.

The Parametric Interval Gauss–Seidel Method

The Parametric Interval Gauss–Seidel Method

$$\mathbf{z}_i := \frac{1}{\left(\sum_{k=1}^K (CA^k)_{ii} \mathbf{p}_k\right)} \left(\sum_{k=1}^K (Cb^k)_i \mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^K (CA^k)_{ij} \mathbf{p}_k \right) \mathbf{x}_j \right),$$

$$\mathbf{x}_i := \mathbf{x}_i \cap \mathbf{z}_i,$$

for $i = 1, \dots, n$.

Optimal Preconditioner

Various criteria of optimality:

- **minimize the resulting width**, that is, the objective is $\min 2z_i^\Delta$,
- minimize the resulting upper bound, that is, the objective is $\min \bar{z}_i$,
- maximize the resulting lower bound, that is, the objective is $\max \underline{z}_i$.

Minimal Width Preconditioner

Preliminaries

- For simplicity assume that $0 \in \mathbf{x}$ and $0 \in \mathbf{z}$
- Denote by c the i th row of C ,
- Normalize c such that the denominator has the form of $[1, r]$ for some $r \geq 1$.

Interval Gauss–Seidel Step

Then the operation of the i th step of the Interval Gauss–Seidel iteration is simplified to

$$\mathbf{z}_i := \sum_{k=1}^K (cb^k) \mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j$$

The objective is $\min z_i^\Delta$.

Minimal Width Preconditioner

minimize the width of $\sum_{k=1}^K (cb^k) \mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j$.

Denote

$$\beta_k := |cb^k|, \quad k = 1, \dots, K,$$

$$\alpha_{jk} := |cA_{*j}^k|, \quad j = 1, \dots, n, \quad k = 1, \dots, K,$$

$$\eta_j := \overline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i,$$

$$\psi_j := \underline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i.$$

Minimal Width Preconditioner

$$\min \sum_{k=1}^K 2p_k^\Delta \beta_k + \sum_{j \neq i} (\eta_j - \psi_j),$$

subject to

$$\beta_k := |cb^k|, \quad k = 1, \dots, K,$$

$$\alpha_{jk} := |cA_{*j}^k|, \quad j = 1, \dots, n, \quad k = 1, \dots, K,$$

$$\eta_j := \overline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i,$$

$$\psi_j := \underline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i.$$

Minimal Width Preconditioner

$$\min \sum_{k=1}^K 2p_k^\Delta \beta_k + \sum_{j \neq i} (\eta_j - \psi_j),$$

subject to

$$\beta_k \geq cb^k, \beta_k \geq -cb^k, \quad k = 1, \dots, K,$$

$$\alpha_{jk} := |cA_{*j}^k|, \quad j = 1, \dots, n, \quad k = 1, \dots, K,$$

$$\eta_j := \overline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i,$$

$$\psi_j := \underline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i.$$

Minimal Width Preconditioner

$$\min \sum_{k=1}^K 2p_k^\Delta \beta_k + \sum_{j \neq i} (\eta_j - \psi_j),$$

subject to

$$\beta_k \geq cb^k, \beta_k \geq -cb^k, \quad k = 1, \dots, K,$$

$$\alpha_{jk} \geq cA_{*j}^k, \alpha_{jk} \geq -cA_{*j}^k, \quad j = 1, \dots, n, \quad k = 1, \dots, K,$$

$$\eta_j := \overline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i,$$

$$\psi_j := \underline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i.$$

Minimal Width Preconditioner

$$\min \sum_{k=1}^K 2p_k^\Delta \beta_k + \sum_{j \neq i} (\eta_j - \psi_j),$$

subject to

$$\beta_k \geq cb^k, \beta_k \geq -cb^k, \quad k = 1, \dots, K,$$

$$\alpha_{jk} \geq cA_{*j}^k, \alpha_{jk} \geq -cA_{*j}^k, \quad j = 1, \dots, n, \quad k = 1, \dots, K,$$

$$\eta_j \geq c \sum_{k=1}^K A_{*j}^k p_k^c \underline{x}_j \pm \sum_{k=1}^K p_k^\Delta \underline{x}_j \alpha_{jk}, \quad j \neq i,$$

$$\eta_j \geq c \sum_{k=1}^K A_{*j}^k p_k^c \bar{x}_j + \sum_{k=1}^K p_k^\Delta \bar{x}_j \alpha_{jk}, \quad j \neq i,$$

$$\psi_j := \underline{\left(\sum_{k=1}^K (cA_{*j}^k) \mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i.$$

Minimal Width Preconditioner

$$\min \sum_{k=1}^K 2p_k^\Delta \beta_k + \sum_{j \neq i} (\eta_j - \psi_j),$$

subject to

$$\beta_k \geq cb^k, \beta_k \geq -cb^k, \quad k = 1, \dots, K,$$

$$\alpha_{jk} \geq cA_{*j}^k, \alpha_{jk} \geq -cA_{*j}^k, \quad j = 1, \dots, n, \quad k = 1, \dots, K,$$

$$\eta_j \geq c \sum_{k=1}^K A_{*j}^k p_k^c \underline{x}_j \pm \sum_{k=1}^K p_k^\Delta \underline{x}_j \alpha_{jk}, \quad j \neq i,$$

$$\eta_j \geq c \sum_{k=1}^K A_{*j}^k p_k^c \bar{x}_j + \sum_{k=1}^K p_k^\Delta \bar{x}_j \alpha_{jk}, \quad j \neq i,$$

$$\psi_j \leq c \sum_{k=1}^K A_{*j}^k p_k^c \underline{x}_j \pm \sum_{k=1}^K p_k^\Delta \underline{x}_j \alpha_{jk}, \quad j \neq i,$$

$$\psi_j \leq c \sum_{k=1}^K A_{*j}^k p_k^c \bar{x}_j - \sum_{k=1}^K p_k^\Delta \bar{x}_j \alpha_{jk}, \quad j \neq i,$$

Minimal Width Preconditioner

$$\min \sum_{k=1}^K 2p_k^\Delta \beta_k + \sum_{j \neq i} (\eta_j - \psi_j),$$

subject to

$$\beta_k \geq cb^k, \beta_k \geq -cb^k, \quad k = 1, \dots, K,$$

$$\alpha_{jk} \geq cA_{*j}^k, \alpha_{jk} \geq -cA_{*j}^k, \quad j = 1, \dots, n, \quad k = 1, \dots, K,$$

$$\eta_j \geq c \sum_{k=1}^K A_{*j}^k p_k^c \underline{x}_j \pm \sum_{k=1}^K p_k^\Delta \underline{x}_j \alpha_{jk}, \quad j \neq i,$$

$$\eta_j \geq c \sum_{k=1}^K A_{*j}^k p_k^c \bar{x}_j + \sum_{k=1}^K p_k^\Delta \bar{x}_j \alpha_{jk}, \quad j \neq i,$$

$$\psi_j \leq c \sum_{k=1}^K A_{*j}^k p_k^c \underline{x}_j \pm \sum_{k=1}^K p_k^\Delta \underline{x}_j \alpha_{jk}, \quad j \neq i,$$

$$\psi_j \leq c \sum_{k=1}^K A_{*j}^k p_k^c \bar{x}_j - \sum_{k=1}^K p_k^\Delta \bar{x}_j \alpha_{jk}, \quad j \neq i,$$

and the condition that the denominator is has the form of $[1, r]$

$$c \sum_{k=1}^K A_{*i}^k p_k^c - \sum_{k=1}^K p_k^\Delta \alpha_{ik} = 1.$$

Minimal Width Preconditioner

Optimization problem.

- Optimal preconditioner C found by n linear programming (LP) problems.
- each LP has $Kn + K + 3n - 2$ unknowns c , β_k , α_{jk} , η_j , and ψ_j , and $2Kn + 4n - 3$ constraints
- C needn't be calculated in a verified way.
- The problem is effectively solved in polynomial time.

Practical Implementation

- Call the standard version using midpoint inverse preconditioner (or any other method),
- and after that tighten the enclosure by using an optimal preconditioner C .
- In our examples: one iteration with minimization of the upper bound, and one with maximization of the upper bound.

Example (Popova, 2002)

$$A(p) = \begin{pmatrix} 1 & p_1 \\ p_1 & p_2 \end{pmatrix}, \quad b(p) = \begin{pmatrix} p_3 \\ p_3 \end{pmatrix}, \quad p \in \mathbf{p} = ([0, 1], -[1, 4], [0, 2])^T.$$

Initial enclosure by the Parametric Interval Gauss–Seidel Method with midpoint inverse preconditioner:

- direct version: 7.66% of the width on average reduced
- residual form: 0% of the width on average reduced

Initial enclosure as the interval hull of the relaxed system:

- direct version: 50% of the width on average reduced
- residual form: 12.56% of the width on average reduced

Example (Popova and Krämer, 2008)

$$A(p) = \begin{pmatrix} 30 & -10 & -10 & -10 & 0 \\ -10 & 10 + p_1 + p_2 & -p_1 & 0 & 0 \\ -10 & -p_1 & 15 + p_1 + p_3 & -5 & 0 \\ -10 & 0 & -5 & 15 + p_4 & 0 \\ 0 & 0 & -5 & 5 & 1 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $p \in \mathbf{p} = [8, 12] \times [4, 8] \times [8, 12] \times [8, 12]$.

Initial enclosure by the Parametric Interval Gauss–Seidel Method with midpoint inverse preconditioner:







- direct version: 15% of the width on average reduced
- residual form: 0% of the width on average reduced

Summary

- Optimal preconditioning matrix for the parametric interval Gauss–Seidel iterations.
- It can be computed effectively by linear programming.
- Preliminary results show that sometimes can reduce overestimation of the standard enclosures.

Directions for Further Research

- Other types of optimality of preconditioners (S-preconditioners, pivoting preconditioners, etc.)
- Optimal preconditioners for other methods than the parametric interval Gauss–Seidel one.

-  M. Hladík.
Enclosures for the solution set of parametric interval linear systems.
Int. J. Appl. Math. Comput. Sci., 22(3):561–574, 2012.
-  R. B. Kearfott.
Preconditioners for the interval Gauss–Seidel method.
SIAM J. Numer. Anal., 27(3):804–822, 1990.
-  R. B. Kearfott, C. Hu, and M. Novoa III.
A review of preconditioners for the interval Gauss–Seidel method.
Interval Comput., 1991(1):59–85, 1991.
-  A. Neumaier.
New techniques for the analysis of linear interval equations.
Linear Algebra Appl., 58:273–325, 1984.
-  E. Popova.
Quality of the solution sets of parameter-dependent interval linear systems.
ZAMM, Z. Angew. Math. Mech., 82(10):723–727, 2002.
-  E. D. Popova and W. Krämer.
Visualizing parametric solution sets.
BIT, 48(1):95–115, 2008.