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Self-duality of Polytopes and its Relations to Vertex Enumeration and Graph Isomorphism

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Abstract We study the complexity of determining whether a polytope given by its vertices or facets is combinatorially isomorphic to its polar dual. We prove that this problem is Graph Isomorphism hard, and that it is Graph Isomorphism complete if and only if Vertex Enumeration is Graph Isomorphism easy. To the best of our knowledge, this is the first problem that is not equivalent to Vertex Enumeration and whose complexity status has a non-trivial impact on the complexity of Vertex Enumeration irrespective of whether checking Self-duality turns out to be strictly harder than Graph Isomorphism or equivalent to Graph Isomorphism. The constructions employed in the proof yield a class of self-dual polytopes that are interesting on their own. In particular, this class of self-dual polytopes has the property that the facet-vertex incident matrix of the polytope is transposable if and only if the matrix is symmetrizable as well. As a consequence of this construction, we also prove that checking self-duality of a polytope, given by its facet-vertex incidence matrix, is Graph Isomorphism complete, thereby answering a question of Kaibel and Schwartz.

Keywords Self-duality, Polytopes, Vertex Enumeration, Graph Isomorphism

1 Introduction

A polytope $P \subseteq \mathbb{R}^d$ is the convex hull of a finite number of points in $\mathbb{R}^d$. A very basic result in the theory of polytopes states that every polytope can also be represented as the intersection of a finite number of halfspaces. Each of

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these representations is unique assuming that none of the vertices or halfspaces are redundant. We refer to these representations as $\mathcal{V}$-representation and $\mathcal{H}$-representation, respectively. Accordingly, a polytope given by its vertices is called a $\mathcal{V}$-polytope and one given by its defining hyperplanes is called an $\mathcal{H}$-polytope. The intersection of the defining hyperplanes with the polytope are called the facets of the polytope. For a thorough treatment of the subject we refer the reader to [7,15]. Note that if a particular representation of a polytope contains redundant points (feasible points that are not vertices) or redundant inequalities (whose removal doesn't change the polytope) then one can first remove the redundant points and inequalities using linear programming and therefore *wlog* in this paper we assume that we are dealing with non-redundant representations.

The problem that is the main motivation for this paper is that of converting one representation of a polytope into another. The problem of enumerating all vertices given the facets is known as Vertex Enumeration Problem (VE) and the problem of enumerating all facets given the vertices is known as Convex Hull Problem (CH). These two problems are polynomial time equivalent to each other if an oracle for Linear Programming is given and so for rational polytopes the two problems are equivalent. Since the number of vertices of a polytope in $\mathbb{R}^d$ with n facets can be anywhere between $O(d)$ and $O(n^{\lfloor d/2 \rfloor})$ [13], one would like to have an algorithm that runs in time polynomial in both the size of the input and output. Such an algorithm is called output-sensitive polynomial algorithm.

The notion of output-sensitivity may be problematic if one wants to relate the complexity of VE to the complexity classes P or NP. Considering an equivalent decision version of VE removes this problem. The decision problem (termed PV henceforth) asks one to decide, given an $\mathcal{H}$-polytope and a subset of its vertices V, whether the list of vertices is complete. It is known ([2]) that PV is polynomial equivalent to VE. One can talk about the complexity of this problem (referred to as PV) in terms of classes P, NP, $coNP$ etc. Whenever we mention VE in terms of any traditional complexity class, we in fact are talking about this decision problem.

In spite of strong efforts the complexity status of vertex enumeration has remained elusive. It has not been shown to be in P nor is it known to be $coNP$-complete. Although, if the input facets define an unbounded polyhedra then the decision problem is indeed $coNP$-complete [12] . If the dimension is bounded or if the polytope is non-degenerate, then the problem can be solved in polynomial time [4]. There is the same ambiguity in the complexity status for graph isomorphism (GI) which is not known to be either in P or to be NP-complete . So therefore it is natural to try to relate the complexities of those two problems. This paper does not yet settle this issue, but we take a step in this direction by deriving interesting connections between vertex enumeration and graph isomorphism.

For the purpose of relating vertex enumeration to graph isomorphism, we use the definition of a complexity class of all problems that are polynomially equivalent to GI. A problem Φ is said to be GI-easy if it can be solved in polynomial time given an oracle for GI and GI-hard if GI can be solved in polynomial time using an oracle for Φ. A problem that is both GI-easy and GI-hard is called GI-complete. Note that when talking about complexity of a problem with respect to an oracle, we assume that the oracle calls take constant (or polynomial) time. So for equivalence we are allowed to make a polynomial number of calls to the oracle.

Vertex Enumeration has a very interesting property in that if one tries to solve it by modifying the problem "a little bit", one runs into two kinds of problems. One kind are those that are polynomially equivalent to the original problem like polytope verification [2]. Such problems do not really open up possibilities for any method that was not applicable to the original problem. The other kind of problems are ones whose polynomial algorithm would yield a polynomial algorithm for VE but whose hardness has no consequence on VE. Examples include polytope containment [6] and computing Minkowski sums [14]. Moreover, these problems usually turn out to be NP-hard and thus shed no light on the complexity of VE.

We consider the problem of checking whether a polytope given by vertices or facets is combinatorially isomorphic to its polar dual. We will shortly clarify the meaning of the terms isomorphic and polar dual. We call this problem Self-Duality problem (SD). SD is markedly different from the kind of problems mentioned earlier. We show that SD is GI-hard and furthermore that it is GI-complete if and only if VE is GI-easy. The "if and only if" in the result ensures that whichever way the complexity of SD is settled, it will have non-trivial consequences for the complexity of VE. To the best of our knowledge this is the first problem that opens up the possibility of relating the complexity of VE to that of GI. We now introduce some notions and conventions that will be used in the paper and in Section 2 we survey some related work. We finally present our results in Section 3.

A supporting hyperplane of a full-dimensional polytope P is a hyperplane that contains the entire polytope on one of its (closed) sides and has a non-empty intersection with the polytope. The intersection of a supporting hyperplane with a polytope gives a face of the polytope. Facets are the $(d-1)$-dimensional faces and vertices are the 0-dimensional faces of a polytope. Also, the polytope is considered a face of itself and the empty set is considered the (-1)-dimensional face. The faces of a polytope are lower dimensional polytopes and each of their faces is a face of the original polytope. The faces of a polytope define a partial order by inclusion. This partial order defines a lattice L whose elements are all the faces of the polytope, and faces f, g satisfy $g <_L f$ if and only if $g \subset f$. This lattice is called the face-lattice of the polytope.

For a polytope P, we denote respectively by $\mathcal{V}(P)$ and $\mathcal{F}(P)$ the sets of vertices and facets of P. The *facet-vertex incidence matrix* $\mathcal{I}(P) \in \{0,1\}^{m \times n}$, of a polytope P with $m = |\mathcal{F}(P)|$ and $n = |\mathcal{V}(P)|$, is a 0/1-matrix whose rows represent the facets and whose columns represent the vertices and $\mathcal{I}[i,j] = 1$ if and only if the i-th facet is incident to the j-th vertex. It is known that the face lattice of a polytope is completely determined by its facet-vertex incidence matrix ([10]). A real matrix A is said to be *transposable* if it can be transformed into its transpose A^T via row and column permutations, and is said to be *symmetrizable* if it can be transformed into a symmetric matrix via row and column permutations. For a nice survey of transposability and symmetrizability of matrices the reader is referred to [3].

For a polytope $P = \{x \in \mathbb{R}^d \mid A \cdot x \leq \mathbf{1}\}$, where $A \in \mathbb{R}^{k \times d}$ is a real matrix, the *polar dual* (or simply *dual*) is the polytope

$$P^* = \{x \in \mathbb{R}^d \mid v \cdot x \leq 1, \quad \text{for } v \in \mathcal{V}(P)\}.$$

Thus, for a full-dimensional polytope P that contains the origin in the interior, P^* is also bounded and contains the origin in the interior. Furthermore, the vertices of P^* are the rows of the matrix A treated as points in $\mathbb{R}^d$. In particular, the vertices and facets of a polytope are in one-to-one correspondence, respectively, with the facets and vertices of its polar dual. Note that the notion of polarity is not restricted to polytopes but applies even to unbounded polyhedra and more general convex sets. We, however, only deal with bounded polytopes and also assume that the polytope always contains the origin in the interior.

Two polytopes P and Q are said to be *combinatorially isomorphic* to each other, denoted by $P \cong Q$, if their face-lattices are isomorphic. For example, any two convex polygons with equal number of sides are combinatorially isomorphic. Equivalently, two polytopes are isomorphic if and only if the incidence matrix of one can be transformed into that of the other via row and column permutations. A polytope is said to be *self-dual* if it is combinatorially isomorphic to its polar dual, i.e., if $P \cong P^*$. In terms of incidence matrices this means that for self-dual polytopes the incidence matrix is transposable. Also, the row and column permutation that changes the incidence matrix to its transpose is called the self-duality map. Note that the incidence matrix of every self-dual polytope need not be symmetrizable (See [9]).

2 Related Work

Our work touches on various topics including vertex enumeration, isomorphism and self-duality of polytopes, as well as transposability and symmetrizability of 0/1-matrices. In this section, we will briefly mention some of the existing literature pertaining to these topics.

Self-dual polytopes form an interesting subclass of polytopes and their classification is a fundamental problem in the theory of polytopes. Self-dual polytopes have been studied extensively at least in 3-dimensions and the 3-dimensional spherical and projective self-dual polytopes have been fully characterized (See [1]). In higher dimensions not much appears to be known. As we will see, the free-join of a polytope P and its polar dual always generates self-dual polytopes. In fact, instead of the polar dual one can use any polytope combinatorially isomorphic to the polar P^*. Also, the free-join of any two self-dual polytopes yields another self-dual polytope. These constructions do not yield all possible self-dual polytopes but the ones that do arise have interesting properties, namely that they also admit an involutory self-duality map. In Subsection 3.2, we describe a class of polytopes which we call *roofed-prisms* that are self-dual but are not obtainable as free-join of simpler polytopes. We do not use these polytopes in our proofs but the construction is simple enough to warrant mentioning these polytopes in this context.

Kaibel and Schwartz ([11]) studied various isomorphism questions about polytopes and proved that it is GI-complete to determine if two polytopes given by their facet-vertex incidences are combinatorially isomorphic to each other. The problem remains GI-complete even if the coordinates of vertices and facet-normals are provided or if the polytopes are restricted to be simple or simplicial polytopes. The authors in [11], however, leave open the question of checking self-duality of a polytope given by its facet-vertex incidence matrix. This problem (called SDI from now on) is a variant of SD and in the process of relating SD to VE, we settle the question of Kaibel and Schwartz by proving that SDI is GI-complete as well. Recall that for SD the polytope is given only by its facets or only vertices.

As we noted before, self-duality of a polytope implies that its incidence matrix is transposable. Also, if the self-duality map is involutory, *i.e.* the map applied twice yields the identity map, then the incidence matrix is symmetrizable. Note that every symmetrizable matrix is transposable, but there are transposable matrices that are not symmetrizable [3]. Grünbaum asked in [8] if there are self-dual polytopes that do not have any involutory self-duality maps. Jendrol [9] answered this question in the affirmative.

We should remark that we do not claim the novelty of the constructions provided in this paper. Free join of two polytopes is a well known operation ([15]). The same construction is also attributed to David Eppstein [5] in finding examples of polytopes with n vertices, n facets and $n^{\lfloor \frac{d+1}{3} \rfloor}$ faces improving an earlier bound of $n^{\sqrt{d}}$ by Seidel *et al.* [5][2]. The authors are not aware of any other work mentioning the roofed-prisms, that we mention in this paper, as an example of indecomposable self-dual polytopes.

3 Main Results

In this paper we consider the following three problems:

VE: Given a polytope P by facets, enumerate all the vertices of P.
SD: Given a polytope P by facets *or* vertices, determine if P is self-dual.
SDI: Given a polytope P by its facet-vertex Incidence Matrix, determine if P is self-dual.

Our main results are the following:
- SD is GI-hard.
- SD is GI-complete if and only if VE is GI-easy.
- SDI is GI-complete.

For proving the GI-hardness of SD and its relations to the complexity of VE, we start by exploring the complexity of SDI. We establish that SDI is GI-complete first and the other results are easy consequences of this fact. Our results on the complexity of SDI strengthens the result of [11] that it is GI-complete to determine if two polytopes given by their facet-vertex incidences are combinatorially isomorphic to each other. We arrive at this result by showing that, essentially the *free join* of two polytopes is self-dual if and only if the two polytopes are isomorphic.

3.1 Constructing Self Dual Polytopes

3.1.1 Free Join

For a set of points $S \subseteq \mathbb{R}^d$, we denote respectively by $\mathrm{aff}(S)$ and $\mathrm{conv}(S)$, the affine and convex hulls of S. The dimension of S, denoted by $\dim(S)$, is the dimension of $\mathrm{aff}(S)$. Two affine spaces are called *skew* if they neither intersect nor contain any parallel lines.

The *free join* of two polytopes is obtained by embedding the polytopes in skew subspaces and taking the convex hull. For example, the free join of two line segments is a 3-dimensional tetrahedron. Since in the context of this paper we are interested only in the combinatorial structure of polytopes arising as free-joins of smaller polytopes independent of the actual embedding, we will choose some specific skew hyperplanes for the purpose of embedding the component polytopes. Let P_1 and P_2 be two polytopes in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, such that:

$$\begin{aligned} P_1 &= \{x \in \mathbb{R}^m \mid A_1 x \leq 1\} = \mathrm{conv}(V_1), \\ P_2 &= \{x \in \mathbb{R}^n \mid A_2 x \leq 1\} = \mathrm{conv}(V_2), \end{aligned}$$

where $A_1 \in \mathbb{R}^{l \times m}$, $V_1 \subseteq \mathbb{R}^m$, $A_2 \in \mathbb{R}^{r \times n}$, and $V_2 \subseteq \mathbb{R}^n$, then the vertices of the free join $P * Q \subseteq \mathbb{R}^{m+n+1}$ are

$$\mathcal{V}(P_1 * P_2) = \left\{ \begin{pmatrix} v \\ \mathbf{0} \\ -1 \end{pmatrix} : v \in \mathcal{V}(P_1) \right\} \bigcup \left\{ \begin{pmatrix} \mathbf{0} \\ v \\ 1 \end{pmatrix} : v \in \mathcal{V}(P_2) \right\}$$

and

$$P_1 * P_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2A_1 x + z \cdot \mathbf{1}_l \leq \mathbf{1}_l,\ 2A_2 y - z \cdot \mathbf{1}_r \leq \mathbf{1}_r \right\},$$

where $\mathbf{1}_k$ is a vector in $\mathbb{R}^k$ all whose entries are 1. The following are some easy observations about the free join operation.

Fact 1 *Suppose P_1 is an i-dimensional polytope and P_2 is a j-dimensional polytope. If $P = P_1 * P_2$, then*

(i) An i-dimensional face of P is the free join of an r-dimensional face of P_1 with an s-dimensional face of P_2 such that $r+s+1=i$, and consequently,
(ii) every face of P_1 or P_2 is a projection of a face of P, with the same dimension,
(iii) P is an $(i+j+1)$-dimensional polytope,
(iv) $|\mathcal{V}(P)| = |\mathcal{V}(P_1)| + |\mathcal{V}(P_2)|$, and
(v) $|\mathcal{H}(P)| = |\mathcal{H}(P_1)| + |\mathcal{H}(P_2)|$. Furthermore, every facet of P is either a free join of P_1 and some facet of P_2, or that of P_2 and some facet of P_1.

3.1.2 Incidence Matrix of Free-Join

Recall that the facet-vertex incidence matrix $\mathcal{I}(P)$ of a polytope P has facets as rows and vertices as columns and the (i,j)-th entry is 1 iff the i-th facet contains the j-th vertex. In particular, if P is full-dimensional, then no row or column of $\mathcal{I}(P)$ can consist of all ones. It follows that the incidence matrix of a polytope $P = P_1 * P_2$, that is a free join of two polytopes P_1 and P_2, is of the form $\left[\frac{A|B}{C|D}\right]$ where A and D are submatrices all whose entries are 1's, and B and C are the incidence matrices of P_1 and P_2. The following lemma establishes that for the incidence matrix of a polytope to be decomposable into the aforementioned form, it is also necessary that the polytope be a free-join of other (simpler) polytopes.

Lemma 1 *Let P be a full-dimensional polytope in $\mathbb{R}^d$. Under suitable labeling of vertices and facets, the incidence matrix $\mathcal{I}(P)$ of P is of the form $\left[\frac{A|B}{C|D}\right]$, where A and D are submatrices all of whose entries are 1's, if and only if P is a free join of two polytopes P_1 and P_2 with respective incidence matrices B and C.*

Proof If P is a free join of two polytopes, then the incidence matrix of P can be written in the desired form, as explained above.

Now, suppose that the incidence matrix of P is of the required form. Suppose, the dimensions of the matrices A, B, C, D are $m_1 \times n_1$, $m_1 \times n_2$, $m_2 \times n_1$ and $m_2 \times n_2$, respectively. Let the set of vertices corresponding to the first n_1 columns of $\mathcal{I}(P)$ be V_1 and the ones corresponding to the last n_2 columns be V_2. Similarly, let F_1 and F_2 be the sets of facets corresponding to the first m_1 and last m_2 rows respectively.

Since any affine transformation preserves incidences, we can assume that P contains the origin in the interior. Suppose that the halfspaces corresponding to the facets F_1 of P be $A_1 x \leq 1$, and the halfspaces corresponding to the facets F_2 of P be $A_2 x \leq 1$.

Note that no row or column of B or C has all 1's, since otherwise $\mathcal{I}(P)$ has such a row or a column. This implies that the affine hull of V_1 can be obtained as the intersection of the hyperplanes defining the facets in F_1 and that of V_2 can be obtained as the intersection of the hyperplanes defining F_2. Specifically, the affine hull of V_1 is $\{x|A_1x = 1\}$ and that of V_2 is $\{x|A_2x = 1\}$.

Since P is full dimensional polytope, there is no common intersection for all the hyperplanes defining $F_1 \cup F_2$. (Indeed, if x is a point in such intersection and $x' \in \text{int}(P)$, then the ray starting at x and moving through x' must hit P at some facet $F \in \mathcal{F}(P)$ whose defining hyperplane contains x. But this would imply that the whole ray belongs to this hyperplane, and hence that $x' \in F$, in contradiction to the fact that x' is an interior point in P.) Hence, the affine hulls of V_1 and V_2 don't intersect.

Now suppose that the affine hulls of V_1 and V_2 are not skew, i.e. they contain parallel lines. Let the copy of this parallel line in the affine hull of V_1 have the parametric equation $l_1 = \{x|x = \alpha_1 + t \cdot u, t \in \mathbb{R}\}$, where $x, \alpha_1, u \in \mathbb{R}^d$. Similarly let the copy in the affine hull of V_2 be $l_2 = \{x = \alpha_2 + t \cdot u, t \in \mathbb{R}\}$. Note that since the two copies are parallel to each other their "direction" is defined by the same vector u.

Since l_1 lies in the affine space $A_1x = 1$, we have $A_1 \cdot (\alpha_1 + t \cdot u) = 1$ for all values of t. This means $A_1 \cdot u = 0$. Similarly it follows from l_2 that $A_2 \cdot u = 0$. But A_1 and A_2 cover all rows of A and so $A \cdot u = 0$. Clearly for l_1 and l_2 to be lines u must not be the zero vector. But if $A \cdot x = 0$ has a non-trivial solution u then $A \cdot (\lambda u) = 0 \leq 1,\ \forall \lambda \in \mathbb{R}$. This contradicts our assumption that P is a bounded polytope and hence does not contain any lines.

Hence, the affine hulls of V_1 and V_2 are skew and P is the free-join of the two polytopes defined by these two sets of vertices.

3.2 Complexity of SDI and SD

Our starting point is the following result of Kaibel and Schwartz [11]: Given two polytopes P_1 and P_2 by their vertices and facets, it is GI-complete to determine whether they are isomorphic to each other. In fact, this is true even if each polytope P_i satisfies the following conditions, for $i \in \{1, 2\}$:

(C1) P_i is *simple*, i.e., every vertex of P_i lies on exactly d facets, where $d = \dim(P_i)$,
(C2) $|\mathcal{V}(P_i)| + 2 \neq 2|\mathcal{F}(P_i)|$, and
(C3) $|\mathcal{F}(P_i)| > 2\dim(P_i)$.

(More precisely, the reduction in [11] constructs for a graph $G = (V, E)$ a simple polytope $P(G)$ of dimension $d = |V| - 1$, with $|\mathcal{V}(P(G))| = |V|(|V| - 1) + 2|E|(|V| - 2)$ and $|\mathcal{F}(P(G))| = 2|V| + 2|E|$. In particular, $|\mathcal{V}(P(G))| + 2 > 2|\mathcal{F}(P(G))|$ for $|V| \geq 5$, i.e., (C1), (C2), and (C3) are satisfied.)

Before we proceed with the details of our reduction, we need the following definition. Given a full-dimensional polytope $P \in \mathbb{R}^d$, a $(d + 1)$-dimensional *bipyramid* $\text{bipyr}(P)$, constructed from P, is obtained by taking two points $u, v \in \mathbb{R}^{d+1}$, strictly in two different sides of $\text{aff}(P)$, such that the line segment connecting u and v intersects the relative interior of P, and defining $\text{bipyr}(P) = \text{conv}(P \cup \{u, v\})$. P is called the base of the bipyramid and u, v are called the *apexes*.

Our reduction of GI to SDI works as follows: Given two graphs G_1 and G_2, we first construct polytopes P_1 and P_2 as in Kaibel and Schwartz ([11]). Next we consider the polytope P obtained by taking the free join of $\text{bipyr}(P_1)$ with the polar dual of $\text{bipyr}(P_2)$. We show that under assumptions (C1), (C2) and (C3) P is self-dual if and only if G_1 and G_2 are isomorphic.

Lemma 2 *Let P be a full-dimensional polytope in $\mathbb{R}^d$, $d \geq 3$, with m facets and n vertices such that $n + 2 \neq 2m$. Then the bipyramid $Q = \text{bipyr}(P)$ is neither a self-dual polytope nor can it be obtained as the free join of two other polytopes.*

Proof The bipyramid Q has $2m$ facets and $n+2$ vertices. Since Q has unequal number of vertices and facets it is clearly not self-dual.

To prove that Q is not decomposable as free-join of smaller polytopes, consider the two apexes of Q. Suppose Q is decomposable as $P_1 * P_2$. Since every pair of vertices from P_1 and P_2 generates an edge in the free-join, both apexes of Q must be part of one of the component polytopes, say P_1 *wlog*. But then both apexes must lie in a proper face of Q contradicting the fact that they are the apexes of a bipyramid.

Lemma 3 *Let P_1 and P_2 be two polytopes satisfying (C1) and (C3). Then $P_1 \cong P_2$ if and only if $\text{bipyr}(P_1) \cong \text{bipyr}(P_2)$.*

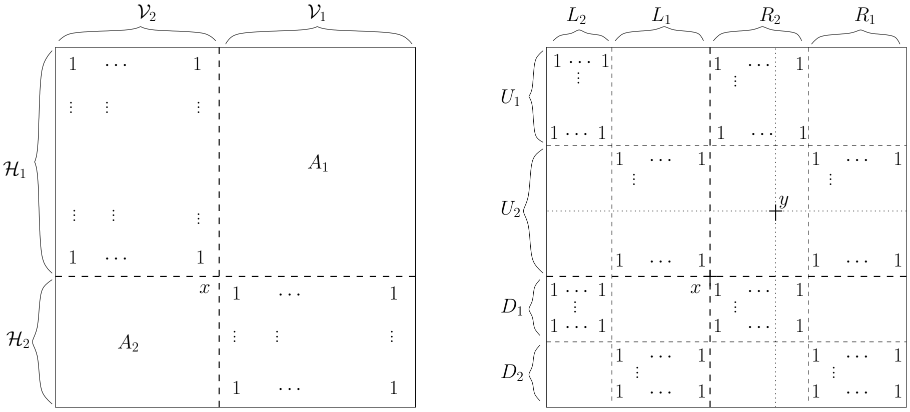
Proof If $P_1 \cong P_2$, then clearly $\text{bipyr}(P_1) \cong \text{bipyr}(P_2)$. Suppose now that $\text{bipyr}(P_1) \cong \text{bipyr}(P_2)$. Then there is an order-preserving bijection ϕ between the face lattices of $\text{bipyr}(P_1)$ and $\text{bipyr}(P_2)$. Let u_i, v_i be the apexes of $\text{bipyr}(P_i)$, $d_i = \dim(P_i)$, $n_i = |\mathcal{V}(P_i)|$, and $m_i = |\mathcal{F}(P_i)|$, for $i \in \{1, 2\}$. Since $\text{bipyr}(P_1) \cong \text{bipyr}(P_2)$, we have $d_1 = d_2 = d$, $n_1 = n_2 = n$, and $m_1 = m_2 = m$. For a point $u \in \mathbb{R}^d$ and a polytope P, we denote by $f(P, u)$ the number of facets of P containing u. Then $f(P_i, u) = d$ for all $u \in \mathcal{V}(P_i)$ follows from the simplicity of P_i, for $i \in \{1, 2\}$. Thus, $f(\text{bipyr}(P_i), u_i) = f(\text{bipyr}(P_i), v_i) = m$, while $f(\text{bipyr}(P_i), u) = 2d$ for all $u \in \mathcal{V}(\text{bipyr}(P_i)) \setminus \{u_i, v_i\}$, for $i \in \{1, 2\}$. Since $m > 2d$ by (C3), it follows that $\phi(\{u_1, v_1\}) = \{u_2, v_2\}$. Then the restriction of ϕ on the faces of P_1 gives an isomorphism between the face lattices of P_1 and P_2.

Recall that a matrix is called transposable if its rows and columns can be permuted to obtain its transpose, and is called symmetrizable if it can be converted into a symmetric matrix by row and column permutation.

Fact 2 *A polytope is self-dual if and only if its incidence matrix is transposable.*

With these notions, we are ready to establish the following result.

Theorem 1 *Let P_1, $P_2 \in \mathbb{R}^d$ be two polytopes, neither of which is self-dual or decomposable into a free join of other polytopes. Then, $P_1 * P_2$ is self-dual if and only if $P_1 \cong P_2^*$.*



(a) (b)

Fig. 1 The incidence matrix $C = \mathcal{I}(P_1 * P_2)$ before and after applying the permutations σ and ρ. In (b), the two dotted lines, crossing at y, indicate the partition of C^T resulting from the original partition of C. In particular, the upper-right corner above y contains the matrix A_2^T, while the lower-left corner contains A_1^T.

Proof For $i \in \{1, 2\}$, let V_i, F_i and A_i be respectively the set of vertices, set of facets, and incidence matrix of P_i, and write $n_i = |\mathcal{V}(P_i)|$ and $m_i = |\mathcal{F}(P_i)|$. Then the incidence matrix of $P_1 * P_2$ is of the form shown in Figure 1-(a).

If P_1 is isomorphic to P_2^*, then $m_1 = n_2 = m$, $n_1 = m_2 = n$, and there exist row and column permutations σ_1, ρ_1 for A_1 that transform it to A_2^T and also, there exist row and column permutations σ_2, ρ_2 for A_2 that transform it to A_1^T. Now consider the following permutations σ, ρ of the rows and columns for the incidence matrix of $P_1 * P_2$ (assume the vertices of $P_1 * P_2$ are numbered $1, 2, 3, \ldots$, and similarly the facets):

$$\sigma(i) = \begin{cases} \sigma_1(i) & \text{if } i \le m, \\ m + \sigma_2(i - m) & \text{if } i > m. \end{cases}$$

$$\rho(i) = \begin{cases} \rho_2(i) & \text{if } i \le m, \\ m + \rho_1(i - m) & \text{if } i > m. \end{cases}$$

It is easy to see that this permutation of rows and columns applied to the incidence matrix of $P_1 * P_2$ produces its transpose and hence $P_1 * P_2$ is self-dual.

Now, to prove the other direction, assume that $P_1 * P_2$ is self-dual. Then $m_1 + m_2 = n_1 + n_2$, and there exist row and column permutations, σ, ρ, of the incidence matrix $C = \mathcal{I}(P_1 * P_2)$ that transform it to its transpose. Assume w.l.o.g. that $m_1 \ge n_2$ and hence $n_1 \ge m_2$. Define the following subsets of row and column indices according to σ and ρ:

$$L_1 = \{i \mid i \le n_2,\ \rho(i) > n_2\},\ L_2 = \{i \mid i \le n_2,\ \rho(i) \le n_2\},$$

$$R_1 = \{i \mid i > n_2,\ \rho(i) > n_2\},\ R_2 = \{i \mid i > n_2,\ \rho(i) \le n_2\},$$

$$U_1 = \{i \mid i \le m_1,\ \sigma(i) \le m_1\}\,,\ U_2 = \{i \mid i \le m_1,\ \sigma(i) > m_1\}\,,$$
$$D_1 = \{i \mid i > m_1,\ \sigma(i) \le m_1\}\,,\ D_2 = \{i \mid i > m_1,\ \sigma(i) > m_1\}\,.$$

In other words, if we call the initial columns corresponding to $\mathcal{V}_2$ *left* columns and those corresponding to $\mathcal{V}_1$ *right* columns then L_1 corresponds to the set of vertices of $\mathcal{V}_1$ that moves to the left after applying ρ and R_1 corresponds to the set of vertices of $\mathcal{V}_1$ that remains in the right. Similarly, if we call the rows *up* or *down* depending on whether they correspond to facets $\mathcal{F}_1$ or $\mathcal{F}_2$ respectively, then U_1 corresponds to the subset of $\mathcal{F}_1$ that remains up after the column permutation σ and D_1 corresponds to the subset that moves down (see Figure 1-(b)).

Since ρ, σ transform C to C^T, it follows from the definitions of the above sets that $C^T[i,j] = 1$ for $(i,j) \in U_1 \times L_2, U_1 \times R_2, U_2 \times L_1, U_2 \times R_1, D_1 \times L_2, D_1 \times R_2, D_2 \times L_1, D_2 \times R_1$ (see Figure 1-(b)).

We claim that $|U_1| + |U_2| = |L_1| + |L_2|$, or in other words, the two points x and y in Figure 1-(b) coincide. If this was not the case, then the point y would lie in one of the four possible corners $U_1 \times R_2$, $U_1 \times R_1$, $U_2 \times R_2$, or $U_2 \times R_1$. Consider w.l.o.g. the situation in Figure 1-(b), where $y \in U_2 \times R_2$. Since the submatrix of C^T above and to the right of y is A_2^T, it follows from Lemma 1 that the polytope P_2^* is decomposable, in contradiction to our assumptions. Similarly, in all the other three cases for y, one can verify that there exist row and column permutations of A_1^T, such that the resulting matrix, and hence P_1^*, have a decomposition in the sense of Lemma 1.

Thus both $|U_1| + |U_2|$ and $|L_1| + |L_2|$ are equal to, say m, and hence transposing C gives $C^T[i,j] = 1$ for all $(i,j) \in (U_1 \cup U_2) \times (L_1 \cup L_2)$. However, C^T is also obtained by transforming C using ρ, σ, and thus we get $C[i,j] = 1$ for $(i,j) \in L_1 \times U_1, D_1 \times R_1$. Since P_1 is indecomposable, it follows from Lemma 1 that either $L_1 = D_1 = \emptyset$ or $R_1 = U_1 = \emptyset$ (any other choice would give an all 1's row or column in C). The latter case would imply that A_1 is mapped by row and column permutations into A_1^T, and hence is not possible, since P_1 is assumed not to be self-dual. Hence the permutations σ, ρ leave the vertices of $\mathcal{V}_1$ and the facets of $\mathcal{F}_1$ in their own blocks. A similar argument can be made about the rows and columns corresponding to $\mathcal{V}_2$ and $\mathcal{F}_2$. Hence, the permutations σ, ρ satisfy the following:

$$\begin{aligned} \rho(i) \le m \ \text{ iff } i \le m \\ \sigma(i) \le m \ \text{ iff } i \le m \end{aligned}$$

Now we can define a permutation of rows σ' and columns ρ' of the incidence matrix A of P as follows:

$$\begin{aligned} \sigma'(i) &= \sigma(i) && \text{for } i = 1\ldots, m \\ \rho'(i) &= \rho(m+i) - m && \text{for } i = m+1\ldots, m+n \end{aligned}$$

This transforms A_1 into A_2^T and hence shows that P_1 is isomorphic to the dual of P_2.

We remark that assuming the polytopes in the previous theorem not to be self-dual is not a very strong assumption. In fact, for self-dual polytopes arising from the free-join of smaller indecomposable polytopes, it is always true that either the component polytopes are dual to each other or they are each self-dual. In other words, the following version of Theorem 1 is true:

Theorem 2 *Let P_1, $P_2 \in \mathbb{R}^d$ be two polytopes that are not decomposable into free join of other polytopes. Then, the free join $P_1 * P_2$ is self-dual if and only if either $P_1 \cong P_2^*$ or both P_1 and P_2 are self-dual.*

This theorem can be proved with only a slight modification of the proof of Theorem 1 but for the purposes of our proof of GI-completeness of SDI, we need the polytopes to not be self-dual and so we will keep working with the weaker version of the theorem. Also, it follows from the proof of Theorem 1 that although the notion of transposability and symmetrizability of general 0/1-matrices are different, for the incidence matrices of the self-dual polytopes that arise from Theorem 1, both notions are equivalent.

Corollary 1 *If P and Q are two polytopes in $\mathbb{R}^d$ such that both P and Q are neither decomposable nor self-dual, then P is isomorphic to Q if and only if the incidence matrix $P * Q^*$ is symmetrizable.*

Corollary 2 *For a polytope P, $P * P^*$ is self-dual and the incidence matrix of the free join is symmetrizable.*

Now, we can state the final theorem of this subsection.

Theorem 3 *Let P be a polytope in $\mathbb{R}^d$ given by its facet-vertex incidence matrix or both vertices and facets. It is GI-complete to determine whether P is self-dual.*

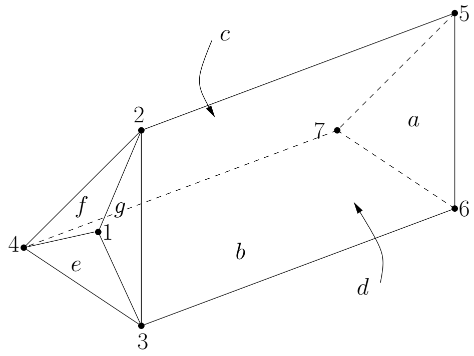
Proof Clearly, if an oracle for GI is given then it can be used to check self-duality of a polytope given by its incidence matrix simply by checking if $P \cong P^*$. Since the incidence matrix can also be computed from the vertices and facets in polynomial time, checking self-duality in this case is GI-easy.

To show that the self-duality checking is also GI-hard, we use the following GI-complete problem [11]: Given two polytopes P_1 and P_2 by their facet and vertex descriptions, or by their facet-vertex incidence matrix, determine if $P_1 \cong P_2$.

As mentioned at the beginning of Subsection 3.2, we may assume that P_1 and P_2 satisfy conditions (C1), (C2) and (C3). From the vertices, facets or facet-vertex incidence matrices of P_1 and P_2, we can construct, in polynomial-time, the vertices, facets or incidence matrix (resp.) of $P = \mathrm{bipyr}(P_1) * \mathrm{bipyr}(P_2)^*$. By Lemma 2, both $\mathrm{bipyr}(P_1)$ and $\mathrm{bipyr}(P_2)^*$ are neither self-dual nor decomposable as free-joins of other polytopes. By Theorem 1, P is self-dual if and only if $\mathrm{bipyr}(P_1) \cong \mathrm{bipyr}(P_2)$, and by Lemma 3, the latter condition is equivalent to $P_1 \cong P_2$.

	1	2	3	4	5	6	7
a	0	0	0	0	1	1	1
b	0	0	1	1	0	1	1
c	0	1	0	1	1	0	1
d	0	1	1	0	1	1	0
e	1	0	1	1	0	0	0
f	1	1	0	1	0	0	0
g	1	1	1	0	0	0	0

(a)



(b)

Fig. 2 A 3-dimensional self-dual indecomposable polytope and its incidence matrix.

Recall that for problem SD we want to verify whether a polytope, given by only vertices or only facets, is self-dual. An easy corollary of Theorem 3 is that SD is GI-hard.

Corollary 3 *Let P be a polytope in $\mathbb{R}^d$ given by its vertices (or facets). It is GI-hard to determine whether P is self-dual.*

In the next subsection, we will discuss some interesting consequences of the complexity of SD on the problem of enumerating vertices of a polytope given by its facets.

We conclude this subsection by remarking that not all self-dual polytopes arise from free-join of other "smaller" self-dual polytopes. For instance, Figure 2 shows an example of a 3-dimensional polytope which is self-dual but indecomposable in the sense of free-join. This example can be generalized to yield an infinite family of indecomposable self-dual polytopes, which we call *roofed-prisms*, as follows. Let P be a d-dimensional polytope and $u, v \in \mathbb{R}^{d+1}$ be two points strictly in two different sides of $\text{aff}(P)$, such that the line segment connecting u and v intersects the interior of P. Let P' be a parallel copy of P containing u and define $Q(P) = \text{conv}(P \cup P' \cup \{v\})$. Informally, $Q(P)$ is obtained by putting the pyramid of P as a roof on the (vertical) prism of P. Then for any self-dual polytope P, the roofed-prism $Q(P)$ is self-dual and indecomposable (a fact that can be proved using Lemma 1.)

3.3 Vertex Enumeration

As noted in the introduction, a problem that is polynomially equivalent to vertex enumeration is the problem of determining whether an $\mathcal{H}$-polytope P is the same as a $\mathcal{V}$-polytope Q [2], also known as polytope verification. Clearly, we may assume that $\mathcal{V}(Q) \subseteq \mathcal{V}(P)$, and furthermore that $\{\text{aff}(F) \mid F \in \mathcal{F}(P)\} \subseteq$

$\{\text{aff}(Q) \mid F \in \mathcal{F}(Q)\}$, for otherwise, P and Q can not be the same. The following theorem relates this problem (and hence VE) to the problem of checking self-duality of a given polytope.

Theorem 4 *Let $P \subset \mathbb{R}^d$ be an $\mathcal{H}$-polytope and $Q \subset \mathbb{R}^d$ be a $\mathcal{V}$-polytope such that $\mathcal{V}(Q) \subseteq \mathcal{V}(P)$ and $\{\text{aff}(F) \mid F \in \mathcal{F}(P)\} \subseteq \{\text{aff}(Q) \mid F \in \mathcal{F}(Q)\}$. Then, $P = Q$ if and only if $P * Q^*$ is self-dual.*

Proof It is easy to see that if $P = Q$ then $P * Q^*$ is self-dual. On the other hand, if $P \neq Q$ then $|\mathcal{V}(P)| > |\mathcal{V}(Q)|$ and also $|\mathcal{F}(Q)| > |\mathcal{F}(P)|$. Hence, $|\mathcal{F}(P * Q^*)| = |\mathcal{F}(P)| + |\mathcal{V}(Q)| < |\mathcal{F}(Q) + |\mathcal{V}(P)| = |\mathcal{V}(P * Q^*)|$. Thus $P * Q^*$ has strictly fewer facets than vertices and hence it can not be self-dual.

As we have seen in the previous subsection SD is GI-hard. Now there are two possibilities: either SD is really harder than GI in that there is a strict (non-polynomial) gap between the complexities of SD and GI, or SD is in fact GI-easy and hence GI-complete as well. In both cases, we get a similar statement about the complexity of VE.

Theorem 5 *VE is GI-easy if and only if SD is GI-complete.*

Proof Clearly, if SD is GI-easy then VE is GI-easy, since an oracle for GI would solve SD, which in turn would solve polytope verification, by Theorem 4. On the other hand, suppose that VE is GI-easy, and suppose we are given an instance of SD, i.e., a polytope P described by, say, its facets, and we want to check whether P is self-dual. Using the oracle for GI, we can enumerate the vertices of P. If P has too many vertices, we know that it is not self-dual and if the number of vertices of P is equal to the number of facets of P, then after enumerating vertices of P we have both vertex and facet descriptions of P, and now the self-duality can be checked using an oracle for GI. Since we know SD to be GI-hard by Corollary 3, SD is also GI-complete.

Since GI is not believed to be NP-hard, by Theorem 5, if SD is GI-easy, then VE is probably also not NP-hard.

4 Conclusion

In this paper we answered a question about the complexity of checking self-duality of a polytope given by its incidence matrix. It was also shown that free-join creates an interesting class of self-dual polytopes for which the incidence matrix is always symmetrizable. We also proved that checking self-duality of a polytope given only by its vertices (or only facets) is Graph Isomorphism hard. Any other insight into the complexity of checking self-duality of $\mathcal{V}$-polytopes will have non-trivial consequence for the complexity of enumerating all vertices of an $\mathcal{H}$-polytope which is a fundamental problem in the theory of polytopes and whose complexity status remains open.

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