Approximation algorithms for scheduling a group of heat pumps

Jiří Fink

Abstract

This paper studies planning problems for a group of heating systems which supply the hot water demand for domestic use in houses. These systems (e.g., gas or electric boilers, heat pumps or microCHPs) use an external energy source to heat up water and store this hot water for supplying the domestic demands. The latter allows to some extent a decoupling of the heat production from the heat demand. We focus on the situation where each heating system has its own demand and buffer and the supply of the heating systems is coming from a common source. In practice, the common source may lead to a coupling of the planning for the group of heating systems. The bottleneck to supply the energy may be the capacity of the distribution system (e.g., the electricity networks or the gas network). As this has to be dimensioned for the maximal consumption, it is important to minimize the maximal peak. This planning problem is known to be \textit{NP}-hard. We present polynomial-time approximation algorithms for four variants of peak minimization problems and we determine the worst-case approximation error.

1 Introduction

In modern society, a significant amount of energy is consumed for heating water [1]. Almost every building is connected to a district heating system or equipped with appliances for heating water locally. Typical appliances for heating water are electrical and gas heating systems, heat pumps and Combined Heat and Power units (microCHP). The heated water is stored in buffers to be prepared for the demands of the inhabitants of the building.

In this paper we consider a local heating systems which consist of

- a supply which represents some source of energy (electricity, gas),
- a converter which converts the energy into heat (hot water),
- a buffer which stores the heat for later usage and
- a demand which represents the (predicted) consumption profile of heat.

A more formal definition of the considered setting for local heating and the used parameters and variables is given in Section 1.1. The presented model can consider arbitrary types of energy but in this paper we use \textit{electricity} and \textit{heat} to distinguish consumed and produced energy. However, this simple model of a local heating system can not only be applied for heating water but has many other applications, e.g., heating demand of houses, fridges and freezers and inventory management. More details about those applications are given also in Section 2.

The combination of a heating device and a buffer gives some freedom in deciding when the heat has to be produced. To use this freedom in a proper way, different objectives may be considered in practice. The energy used for heating is transported from a supply to the heating systems by electrical networks or gas pipes. These transport media have to be able to transport all the used energy and therefore have to be dimensioned for the maximal consumption peak of all houses.
connected to the transport network. Thus, minimizing the maximal consumption over all these houses may decrease investments in the distribution networks. This leads to planning problems for a group of heating systems which minimize peak where peak is the maximal consumption of electricity over the planning period.

In order to show that our algorithm can be adopted for various scenarios, we consider four variants of peak shaving problems. In the basic case, we assume that only heating systems are connected to electricity grid, so the goal is to find a scheduling of these heating systems which minimize the maximal consumption of electricity of all heating systems during a planning horizon. In the second case, we consider more realistic scenario where every house has some other devices consuming or producing electricity. Since we may not be allowed to control these devices, we assume that the total electrical consumption (called base load) is given. This base load is added to the consumption of all heating systems, so the objective in this case is minimizing the peak of the sum. As the production of electricity in family houses is increasing during last decade due to government subsidies on photovoltaic panels (PV) and combine heat and power units (microCHP), it is necessary to take into account not only the overconsumption of electricity but also the overproduction in local districts. This leads us to the third case which minimizes the maximal absolute value of the total electricity consumption. However, if the consumption of electricity is significantly higher than the production, minimizing the maximal consumption may give the same result as minimizing the maximal absolute value. In this case, it may be more practical to minimize the fluctuation or bandwidth. In another words, the goal is minimizing the difference between the maximal and the minimal consumption during the planning horizon. Formal definitions of all these problems are given in Section 1.1.

In our previous study [12] we proved that the basic case is NP-hard problem and therefore all variants studied in this paper are also NP-hard. The computational hardness of these problems strongly relates to the 3-partition problem (for definition, see e.g. [15]). One possibility to avoid the hardness of 3-partition problem is considering a special case of our heating problems. For example, paper [12] also presents a polynomial-time algorithm for minimizing the maximal peak in the case where all converters consume the same amount of electricity when running. This algorithm reduces every instance of the special case of heating problem into a job scheduling problem $P_m|r_i|p_i=1.chains|L_{max}$ which was proved by Dror et al.[9] to be polynomial. This reduction shows strong relation between our heating problems and job scheduling. Other possibility to avoid the computational hardness is developing approximation algorithms which is the task of this paper.

Although for small scale case studies, mix integer linear programming (MILP) solvers are able to find optimal solution in reasonable time [26, 13], using MILP solvers become impractical in larger case studies due to high time and memory demands. Therefore, we develop polynomial-time approximation algorithms for four variants of peak shaving problems in this paper and we determine the worst-case approximation error. Although, the classical measure of quality of approximation algorithms is the relative approximation error (see e.g. [7]), we use the absolute approximation error. This is due the fact that relative error guarantees a small absolute error in small scale cases however, absolute error guarantees a small relative error in large scale cases. Since small scale cases can be optimally solved using MILP solvers, our interest is in large scale scenarios and therefore, we study the absolute approximation error in this paper.

1.1 Problem statement and results

This section presents a mathematical description of the studied model and a summary of the results of this paper. The used symbols, parameters and variables are summarized in Appendix.

First of all, we consider a discrete time model for the considered problem, meaning that we split the planning period into $T$ time intervals of the same length. We consider sets $C = \{1, \ldots, C\}$ of $C$ heating systems and $T = \{1, \ldots, T\}$ of $T$ time intervals. In this paper, the letter $c$ is always an index of a heating system and $t$ is an index of a time interval. For mathematical purposes, we separate a heating system into a converter, a buffer and a demand; see Figure 1. We say “a converter $c$” or “a buffer $c$” or “a demand $c$” to refer to the devices of the heating system $c \in C$.

We consider a simple converter which has only two states: In every time interval the converter
is either turned on or turned off. The amount of consumed electricity is $E_c$ and the amount of produced heat (or any other form of energy) is $H_c$ during one time interval in which the converter $c \in C$ is turned on. If the converter is turned off, then it consumes and produces no energy. We assume that $E_c$ is positive when a converter $c$ is consuming electricity (the converter is e.g., electrical boiler, heat pump). However, when converter is producing electricity (e.g. microCHP), we assume that $E_c$ is negative and $-E_c$ is the amount of produced electricity. As we discuss later, we assume that $E_c$ is non-zero for all converters $c \in C$. Let $x_{c,t} \in \{0, 1\}$ be the variable indicating whether the converter $c \in C$ is running in time interval $t \in T$.

The state of charge of a buffer $c \in C$ at the beginning of time interval $t \in T$ is denoted by $s_{c,t}$ which represents the amount of heat in the buffer. Note that $s_{c,T+1}$ is the state of charge at the end of planning period. The state of charge $s_{c,t}$ is limited by a lower bound $L_{c,t}$ and an upper bound $U_{c,t}$. Those two bounds are usually constant over time: the upper bound $U_{c,t}$ is the capacity of buffer and the lower bound $L_{c,t}$ is mostly zero. But it may be useful to allow different values, e.g. a given initial state of charge can be modelled by setting $L_{c,1}$ and $U_{c,1}$ equal to the initial state. In this paper, we assume that $L_{c,1} = U_{c,1}$, so the initial state of charge $s_{c,1}$ is fixed.

The amount of consumed heat by the inhabitants of the house from heating system $c \in C$ during time interval $t \in T$ is denoted by $D_{c,t}$. This amount is assumed to be given and is called the demand of heating system $c$. The total amount of electricity consumed by other devices in all houses $C$ during time interval $t$ is called base load and it is denoted by $F_t$. In this paper, we study off-line problems, so we assume that demands $D_{c,t}$ and the base load $F_t$ are given for the whole planning period.

The operational variables of the converters $x_{c,t}$ and the states of charge of buffers $s_{c,t}$ are restricted by the following invariants.

\[
\begin{align*}
  s_{c,t+1} &= s_{c,t} + H_c x_{c,t} - D_{c,t} \quad &\text{for} \quad c \in C, \ t \in T \\
  L_{c,t} &\leq s_{c,t} \leq U_{c,t} \quad &\text{for} \quad c \in C, \ t \in T \cup \{T + 1\} \\
  x_{c,t} &\in \{0, 1\} \quad &\text{for} \quad c \in C, \ t \in T
\end{align*}
\]

Equation (1) is the charging equation of the buffer. During time interval $t \in T$, the state of charge $s_{c,t}$ of a buffer $c \in C$ is increased by the production of the converter which is $H_c x_{c,t}$ and it is decreased by the demand $D_{c,t}$. Equations (2) and (3) ensure that the domains of variables $s_{c,t}$ and $x_{c,t}$, respectively, are taken into account.
The basic objective function of this paper is minimizing the peak electricity consumption.

Basic peak shaving: 
\[
\text{minimize } m \\
\text{subject to } m \geq \sum_{c \in C} E_c x_{c,t} \text{ for } t \in T
\]

Since \( E_c x_{c,t} \) is the amount of consumed electricity by a converter \( c \) in time \( t \) and the sum \( \sum_{c \in C} E_c x_{c,t} \) is the amount of electricity consumed by all converters in time \( t \), the inequality (5) and the objective function (4) guarantees that the value of the variable \( m \) is the maximal consumption of electricity during one time period within the whole planning period.

In more general case, we also add the base load \( F_t \) to the electricity consumed by all converters.

Maximal peak shaving: 
\[
\text{minimize } m \\
\text{subject to } m \geq F_t + \sum_{c \in C} E_c x_{c,t} \text{ for } t \in T
\]

Note that the basic peak shaving problem is a special case of the maximal peak shaving problem with the base load where \( F_t = 0 \) for all time interval \( t \in T \). Therefore, we develop an approximation algorithm for the maximal peak shaving problem with the base load and apply it also for the basic peak shaving problem.

In districts with large PV installations, it is common that more electricity is produced than consumed during summer. In this case, the distribution grid needs to be dimensioned not only for the maximal consumption but also for the maximal production. One possible approach to capture this issue is introduction an objective function which minimize the maximal absolute value of the total electricity consumption.

Absolute peak shaving: 
\[
\text{minimize } m \\
\text{subject to } m \geq \left| F_t + \sum_{c \in C} E_c x_{c,t} \right| \text{ for } t \in T
\]

Note that the absolute and the maximal peak shaving problems give the same result when \( E_c \) and \( F_t \) are positive for all \( c \in C \) and \( t \in T \). Similarly, when the average of the total electrical consumption over planning horizon is sufficiently far from zero, only one bound of (7) dominates the solution (the upper one for overconsumption and the lower one for overproduction) and the other bound may have negligible influence. In such cases, an external source of energy is needed to balance the difference between the production and the consumption. The external source of energy may be a conventional generator of constant production which may be unable to balance large fluctuations. One possible approach to reduce such fluctuation is maximizing the minimal total electricity consumption together with minimizing the maximal total consumption. In order to incorporate these two goals into one objective function, we introduce the fluctuation peak shaving problem which minimize the difference between the maximal and the minimal consumption during whole planning horizon.

Fluctuation peak shaving: 
\[
\text{minimize } m_u - m_l \\
\text{subject to } m_l \leq F_t + \sum_{c \in C} E_c x_{c,t} \leq m_u \text{ for } t \in T
\]

In the last objective function, \( m_l \) and \( m_u \) are the lower and upper bounds on the electricity consumption, respectively.

As we discuss above, \( E_c \) is positive when a converter \( c \) is consuming electricity and it is negative when a converter \( c \) is producing electricity. However, we assume that \( E_c \) is non-zero for all converters \( c \in C \) since converters \( c \) with electricity consumption \( E_c = 0 \) have no influence on any objective function studied in this paper, and therefore they can be scheduled independently on other converters.
In this paper, we develop polynomial-time approximation algorithms for four variants of peak shaving problems introduced above. In order to determine the approximation error, we do not consider the relative error usually used in literature (see e.g. [7]) but the absolute error. Formally, let \( m^O \) be the optimal value of objective function of the maximal peak shaving problem and \( m^A \) be the value of objective function of a solution found by our algorithm. The relative error is the maximal ratio \( \frac{m^A}{m^O} \) over all instances of the problem. Instead, we study the maximal difference \( m^A - m^O \) and we prove that our algorithm always finds a solution satisfying \( m^A - m^O \leq E \) where \( E = \max_{c \in C} |E_c| \) is the maximal production or consumption of electricity over all converters \( c \in C \). We obtain the same approximation error for the absolute peak shaving problem meaning that the interval \( \langle -m^O, m^O \rangle \) containing total electrical consumption \( F_t + \sum_{c \in C} E_c x_{c,t} \) for all \( t \in T \) of the optimal solution needs to be extended into \( \langle m^A - E, m^O + E \rangle \) for the approximated solution found by our algorithm. Similarly for the fluctuation peak shaving problem, the interval of total electricity consumptions \( \langle m^O_{l, u} \rangle \) in optimal solution needs to be extended into \( \langle m^A_{l, u} \rangle \) for the approximation solution. This implies that the absolute error for the fluctuation peak shaving problem is at most \( 2E \).

Now, we present basic ideas of our approximation algorithms and organization of the paper. Section 2 presents related works and more details on applications of the results of the paper. Section 3 reformulates the studied problem into a simpler form. The basic idea of our approximation algorithms is finding an optimal solution \( y \) of the relaxed problem and rounding this relaxed solution into a binary solution \( x \). Section 4 gives more details about this idea. Section 5 studies the structure of non-integer values in the relaxed solution \( y \). Section 6 constructs an order of these non-integer values. The relaxed solution \( y \) is rounded in this order into an integer solution \( x \) in Section 7. Section 8 concludes this paper with remarks and open problem. Parameters, variables and symbols are listed in Appendix.

1.2 Motivation

The considered problems originate from a project called MeppelEnergie which plans to build a group of houses and a biogas station in Meppel, a small city in the Netherlands\(^1\). In this project, the houses will have a heat pump for space heating and tap demands. Due to Dutch legislation, the biogas station will provide electricity only to those heat pumps. Therefore, the heat pumps should be scheduled in such a way that they only consume, if possible, the electricity produced by the biogas station. If this is not possible, the remaining energy has to be bought on the electricity market at minimal cost.

The study [13] shows that some central control of all heat pumps is necessary to avoid large peak loads. Therefore, our task is to design one or more algorithms to control all heat pumps. The first of our proposed algorithms is called global MILP control which uses an Mix Integers Linear Programming solver to find an optimal (or near to optimal) solution of the minimizing peak problem. The paper [13] shows that this approach can be used only for small number of houses. For larger number of houses, a faster algorithm for the minimizing peak problem is necessary but the problem is NP-hard. Therefore, we try to find an easier problem which can be solved faster (polynomial algorithm for the case where all heat pump consume the same amount of energy; FPT algorithm for the general case [13]). In practice, it may be sufficient to find a solution which is close to the optimum. One such approximation algorithm is presented in [13] but no worst-case analysis is given. Developing an approximation algorithm with proven worst-case analysis is a task of this paper.

2 Related works and applications

In the following we present related literature and give some possible applications of this model.

\(^1\)For more details, see websites http://www.utwente.nl/ctit/energy/projects/meppel.html and http://www.meppelwoont.nl/nieuwveense-landen/
Some related works can be found in the inventory management and lotsizing literature (see e.g. [8, 16] for reviews). In inventory control problems (see e.g. [22]) a buffer may represent an inventory of items, whereby a converter represents the production of items and demand represents the order quantities. As our problem consists of only one commodity, the single item lot sizing problem is related (see [5] for a review). Wagner and Whitin [27] presented an $O(T^2)$ algorithm for the uncapacitated lot-sizing problem which was improved by Federgruen and Tzur [10] to $O(T \log T)$. On the other hand, Florian, Lenstra and Rinnooy [14] proved that the lot-sizing problem with upper bounds on production and order quantities is NP-complete. Computational complexity of the capacitated lot sizing problems is studied in [3]. Our problem is a special case of capacitated single item lot sizing problem which does not seem to be considered in the literature.

One other related area is vehicle routing and scheduling (see e.g. [17] for an overview of this area). For example, Lin, Gertsch and Russell [19] studied optimal vehicle refuelling policies. In their model, a refueling station can provide an arbitrary amount of gas while our converter is restricted to two possible states of heat generation. Other papers on vehicle refuelling policies are more distant from our research since they consider that a car is routed on a graph (see e.g. [23, 18]). Balancing electricity using a group of microCHPs installed in individual households was studied by Bosman et al. [4]. On one hand, they considered more complex model of a converter (e.g. minimal running time and starting profile) which makes the model more accurate in practice. On the other hand, they presented only experimental results without any study of worst-case behaviour of they algorithm.

Peak load shaving is a classical problem in smart grids and demand side management studied in many papers (see e.g. [21, 28, 2, 29]). This paper consider peak shaving as an objective, while some other studies set the minimal or the maximal electrical consumption as a hard constrain and optimize different objectives, for example [4] maximizes the profit on the electricity market and [24] minimizes the number of charging cycles of batteries.

In the following we give some possible applications of the model presented in this paper.

**Hot water:** Converter and buffer can be seen as a model of a simple electrical or gas boiler. Hereby, demand represents the consumption of hot water in a house.

**House heating:** The model may be used to express a very simple model for house heating. The converter represents a simple heater. The capacity of the buffer corresponds to thermal capacity of the heating system (e.g. hot water buffer or thermal capacity of concrete floors and walls) and the state of charge of the buffer is related to the temperature inside the house. Heat losses of the house may be modelled using the demand if we assume that the temperature difference inside the house does not have significant influence on the losses. More details about using thermal mass as a buffers is presented in [25] and computing heat demands is explained in [13].

** Fridges and freezers:** A fridge essentially works in the opposite way than heating, so it may be modelled similarly. However, we have be careful with the correct interpretation of all parameters. The state of charge of the buffer again represents the temperature inside the fridge, but a higher state of charge means a lower temperature. The converter does not produce heat to the fridge but it decreases the temperature inside the fridge, so the converter increases the state of charge of the buffer (fridge). The demand decreases the state of charge of the fridge due to thermal loss and usage of the fridge by humans.

### 3 Reformulation of the problem

In this section, we simplify the problem presented in Section 1.1. Since we used the same reformulation in our previous papers [12, 11], we present only the basic idea of the reformulation here for sake of completeness. The goal of this reformulation is to replace conditions (1) and (2) by one condition (9).
First, we expand the recurrence formula (1) into an explicit equation.

\[ s_{c,t+1} = s_{c,1} + \sum_{i=1}^{t} H_c x_{c,i} - \sum_{i=1}^{t} D_{c,i} \]

Since we assume that the initial state of charge satisfies \( s_{c,1} = L_{c,1} = U_{c,1} \), we can replace \( s_{c,1} \) by \( L_{c,1} \) and substitute into inequalities (2) and after elementary operations we obtain

\[ \frac{L_{c,t+1} - L_{c,1} + \sum_{i=1}^{t} D_{c,i}}{H_c} \leq \sum_{i=1}^{t} x_{c,i} \leq \frac{U_{c,t+1} - L_{c,1} + \sum_{i=1}^{t} D_{c,i}}{H_c} \]

Here, the sum \( \sum_{i=1}^{t} x_{c,i} \) is restricted by a lower and an upper bounds which depend only on input parameters, so these bounds can be easily precomputed to obtain

\[ A_{c,t} \leq \sum_{i=1}^{t} x_{c,i} \leq B_{c,t} \text{ for } c \in C, t \in T. \] (9)

Further analysis presented in [12, 11] proves that bounds \( A_{c,t} \) and \( B_{c,t} \) can be precomputed in such a way the following conditions hold for every \( t \in T \) and \( c \in C \).

- \( A_{c,t}, B_{c,t} \in \mathbb{Z} \)
- \( A_{c,t-1} \leq A_{c,t} \leq A_{c,t-1} + 1 \)
- \( B_{c,t-1} \leq B_{c,t} \leq B_{c,t-1} + 1 \)
- \( A_{c,0} = B_{c,0} = 0 \)

In another words, \((A_{c,t})\) and \((B_{c,t})\) are step sequences of non-negative integers. The last condition has only formal purposes to simplify notations and proofs.

It may be interesting to notice that all changes of this reformulation are essentially based on Gomory-Chvátal cutting planes [6]. This implies that every binary solution \( x \) satisfies (1) and (2) if and only if \( x \) satisfies (9) where the state of charge \( s \) is directly computed from \( x \) by (1).

The preprocessing presented above gives us a simple system to answer the natural question whether a feasible solution satisfying constraints (1), (2) and (3) exists. It is now easy to see that there exists a binary solution \( x \) satisfying (9) if and only if \( A_{c,t} \leq B_{c,t} \) for every \( c \in C \) and \( t \in T \). Obviously, this condition is necessary. The condition is also sufficient, since in this case \( x_{c,t} = A_{c,t} - A_{c,t-1} \) for \( t \in T \) and \( c \in C \) is a feasible solution. Therefore, our algorithm assumes that given bounds \( A \) and \( B \) satisfies this condition.

### 4 Ideas of our algorithms

In this section, we present ideas of our algorithms. Since we need to repeat sums used in constrains (9) and (5) quite often, we define the following two symbols to simplify the notation.

\[ \tilde{x}_{c,t} = \sum_{i=1}^{t} x_{c,i} \quad \text{and} \quad \tilde{x}_t = F_t + \sum_{c \in C} E_c x_{c,t} \]

Symbols \( \tilde{y}_{c,t} \), \( \tilde{y}_t \), \( \tilde{z}_{c,t} \) and \( \tilde{z}_t \) are defined analogously for solutions \( y \) and \( z \) used later in this paper.

Our algorithms have three steps. In the first step, we consider the relaxed problem obtained by replacing integer constrains (3) by inequalities

\[ 0 \leq x_{c,t} \leq 1 \text{ for every } c \in C \text{ and } t \in T \] (10)

and polytopes of all feasible relaxed solutions

- maximal peak shaving: \( P_M = \{(x,m) : (6), (9) \text{ and } (10) \text{ hold }\} \),
- absolute peak shaving: \( P_A = \{(x,m) : (7), (9) \text{ and } (10) \text{ hold }\} \),
- fluctuation peak shaving: \( P_F = \{(x,m_l,m_u) : (8), (9) \text{ and } (10) \text{ hold }\} \).
For a given variant of the peak shaving problems, we choose \( P' \) to be the appropriate polytope \( P_M, P_A \) or \( P_F \). The first step of our algorithm finds a vertex \((y, m)\) or \((y, m_l, m_u)\) of the polytope \( P' \) optimizing given objective using a polynomial-time algorithm for linear programming (see e.g. [7]).

In the second and the third steps, the solution \( y \) is rounded into an integer solution \( x \). The second step finds a special order of all non-integer values of \( y \). In the third step, this order is used to round \( y \) into an integer solution \( x \) so that (3) and

\[
\bar{y}_t - E \leq \bar{x}_t \leq \bar{y}_t + E \quad \text{for every } t \in T
\]

\[
[\hat{y}_{c,t}] \leq \hat{x}_{c,t} \leq [\hat{y}_{c,t}] \quad \text{for every } c \in C \text{ and } t \in T
\]

hold where \([a]\) is the largest integer value not greater than argument and \([a]\) is the smallest integer value not smaller than argument. Since bounds \( A \) and \( B \) are integers, the condition (12) implies that the solution \( x \) satisfies (9).

Now, we determine the approximation error from the condition (11). For the maximal peak shaving problem, it holds that \( m^O \geq m_l \bar{y}_t \) since \( y \) is an optimal relaxed solution. From \( m^A = \max_t \bar{x}_t \leq \max_t \bar{y}_t + E \) it follows that

\[
m^A - m^O \leq \max_t \bar{x}_t - \max_t \bar{y}_t \leq \max_t \bar{y}_t + E - \max_t \bar{y}_t = E
\]

and so the absolute approximation error is at most \( E \). Similarly for the absolute peak shaving problem, it holds that \( m^O \geq \max_t |\bar{y}_t| \). Hence,

\[
m^A - m^O \leq \max_t |\bar{x}_t| - \max_t |\bar{y}_t| \leq \max_t |\bar{y}_t| + E - \max_t |\bar{y}_t| = E
\]

and so the absolute approximation error is also at most \( E \). Finally for the fluctuation peak shaving problem, it holds that \( m^O \geq \max_t \bar{y}_t - \min_t \bar{y}_t \). Hence,

\[
m^A - m^O \leq (\max_t \bar{x}_t - \max_t \bar{y}_t) + (\min_t \bar{y}_t - \min_t \bar{x}_t) \leq 2E
\]

and so the absolute approximation error is at most \( 2E \).

Now, it remains to show the rounding of the relaxed solution \( y \) into an integer solution \( x \) satisfying (3), (11) and (12). These three conditions are independent on bounds \( m, m_l \) and \( m_u \) and actual variant of peak shaving problems. Let us emphasise that the rounding method of our algorithm is same for all variants of peak shaving problems. Our algorithms for these problems differ only in the definition of polytope \( P' \) and the objective function for linear programming which may returns different optimal vertices \( y \) leading to different initial solutions for the rounding method.

5 Structure of non-integer values in relaxed solutions

In this section, we study the structure of non-integer values of vertices of the polytope \( P' \). The first step of our algorithm is finding an optimal vertex \((y, m)\) of \( P' \) (the optimal vertex \((y, m_l, m_u)\) in case of the fluctuation peak shaving) and we fix one such a optimal vertex for the rest of the paper. In order to avoid the dependency of the definition of \( P' \) on the given variant of peak shaving problems, we define a polytope

\[
P = \{ x : (9) \text{ and (10) hold, and } \bar{x}_t = \bar{y}_t \text{ holds for every } t \in T \}.
\]

Observe that \( y \) is also a vertex of the polytope \( P \).

The structure is described in Lemmas 5.1 and 5.2. Since the first lemma is technical, we present the main idea of the lemma on a small example first. Let us consider two converters \( c_1, c_2 \) and two times intervals \( t_1 < t_2 \) such that
(A1) \( y_{c_1,t_1}, y_{c_2,t_2} \not\in \mathbb{Z} \) and
(A2) \( y_{c_1,t}, y_{c_2,t} \not\in \mathbb{Z} \) for every \( t \in \{t_1, \ldots, t_2 - 1\} \).

In this case, we show that \( y \) is not a vertex of \( P \) because \( y \) is the middle point of a segment line which whole belongs into \( P \). The line segment is a set of solutions \( z \) parametrized by \( \alpha \) where \( z \) is defined as follows.

\[
\begin{align*}
    z_{c_1,t_1} &= y_{c_1,t_1} + \frac{\alpha}{E_{c_1}} \\
    z_{c_1,t_2} &= y_{c_1,t_2} - \frac{\alpha}{E_{c_1}} \\
    z_{c_2,t_1} &= y_{c_2,t_1} - \frac{\alpha}{E_{c_2}} \\
    z_{c_2,t_2} &= y_{c_2,t_2} + \frac{\alpha}{E_{c_2}} \\
    z_{c,t} &= y_{c,t} \text{ otherwise}
\end{align*}
\]

For every parameter \( \alpha \) and time interval \( t \) it holds that \( \hat{z}_t = \check{y}_t \). Furthermore, \( \hat{z}_{c,t} = \check{y}_{c,t} \) for every \( c \) and \( t \) except \( c \in \{c_1, c_2\} \) and \( t \in \{t_1, \ldots, t_2 - 1\} \). Let \( \epsilon > 0 \) be the minimal real number such that at least one of inequalities

\[
0 \leq z_{c,t} \leq 1 \quad A_c \leq z_{c,t} \leq B_c \quad \text{for } c \in \{c_1, c_2\} \text{ and } t \in \{t_1, \ldots, t_2 - 1\} \text{ and } \alpha \in (\epsilon, -\epsilon)
\]

hold in equality. Observe from assumptions (A1) and (A2) that all these inequalities are strict for \( \alpha = 0 \). Hence, whole line segment of points \( z \) for \( \alpha \in (\epsilon, -\epsilon) \) belongs into the polytope \( P \) which contradicts the assumption that \( y \) is a vertex of \( P \). We generalize this example for more converters and time intervals. However, we introduce more notations first.

In the general case, we also require fulfilling the condition (A1), so for a converter \( c \) we define \( T_c = \{t \in T : y_{c,t} \not\in \mathbb{Z}\} \). In order to handle condition (A2), we define

\[
T(t_1, t_2) = \begin{cases} 
    \{t_1, t_1 + 1, \ldots, t_2 - 1\} & \text{if } t_1 \leq t_2 \\
    \{t_2, t_2 + 1, \ldots, t_1 - 1\} & \text{otherwise}
\end{cases}
\]

for \( t_1, t_2 \in T \). Observe that \( T(t_1, t_2) = T(t_2, t_1) \) and \( T(t_1, t_1) = \emptyset \). Let \( \sim_c \) be a relation on \( T_c \) such that \( t_1 \sim_c t_2 \) if and only if \( \hat{y}_{c,t} \not\in \mathbb{Z} \) for every \( t \in T(t_1, t_2) \). The relation \( \sim_c \) is reflexive and symmetric by definition and it is easy to observe that it is also transitive. Since the relation \( \sim_c \) is an equivalence on \( T_c \), we denote by \( S_c \) the set of equivalence classes for every converter \( c \).

**Lemma 5.1.** If there exists a sequence of \( k \) converters \( c_1, \ldots, c_k \) and a sequence of \( k \) time intervals \( t_1, \ldots, t_k \) for \( k \geq 2 \) such that conditions

(B1) \( y_{c_i,t_i}, y_{c_{i+1},t_{i+1}} \not\in \mathbb{Z} \)
(B2) \( t_i \neq t_j \) for \( i \neq j \)
(B3) \( t_i \sim_{c_i} t_{i+1} \)
(B4) \( c_i = c_j \) for \( i \neq j \) then it does not hold that \( t_i \sim_{c_i} t_j \)

hold for every \( i, j \in \{1, \ldots, k\} \), then \( y \) is not a vertex of \( P \).

Before we prove this lemma, we discuss the meaning of all conditions. The sequences of converters \( c_1, \ldots, c_k \) and time intervals \( t_1, \ldots, t_k \) are essentially a circular sequences, so \( t_{i+1 \% k} \) denotes \( t_i \) if \( i = k \). The condition (B1) implies that \( t_i, t_{i+1 \% k} \in T_{c_i} \). The condition (B2) states that time intervals \( t_1, \ldots, t_k \) are pairwise different. The condition (B3) requires that the consecutive time intervals \( t_i \) and \( t_{i+1 \% k} \) belong to the same equivalence class of the relation \( \sim_{c_i} \). On the other hand, the condition (B4) requires that if one converter \( c_i = c_j \) repeats in the sequence, then the equivalence class containing \( t_i \) and \( t_{i+1 \% k} \) differs from the equivalence class containing \( t_j \) and \( t_{j+1 \% k} \). Therefore, conditions (B3) and (B4) imply that consecutive converters must be different.

**Proof of Lemma 5.1.** Similarly as in the small example, we define a line segment of points \( z \) parametrized by \( \alpha \) so that
• \( z_{c_i,t} = y_{c_i,t} + \frac{\alpha}{E_{c_i}} \)
• \( z_{c_i,t} + 1\% = y_{c_i,t+1\%} - \frac{\alpha}{E_{c_i}} \)

for every \( i \in \{1, \ldots, k\} \) and \( z_{c_i,t} = y_{c_i,t} \) otherwise. For every \( \alpha \) the solution \( z \) satisfies \( \bar{z}_t = \tilde{y}_t \).

From the condition (B4) it follows that sets of time intervals \( T(t_i, t_{i+1\%}) \) and \( T(t_j, t_{j+1\%}) \) do not overlap for \( i \neq j \) and \( c_i = c_j \). Therefore, \( \bar{z}_{c_i,t} \neq \tilde{y}_{c_i,t} \) if and only if \( (c_i, t) \in V \) and \( \alpha \neq 0 \) where \( V = \{(c_i, t) : i \in \{1, \ldots, k\}, t \in T(t_i, t_{i+1\%})\} \). Let \( \epsilon > 0 \) be the minimal real number such that at least one of the inequalities

\[
0 \leq z_{c_i,t} \leq 1
\]

\[
A_{c_i,t} \leq \bar{z}_{c_i,t} \leq B_{c_i,t}
\]

for \( (c, t) \in V \) and \( \alpha \in (\epsilon, -\epsilon) \)

hold in equality. Observe that all these inequalities are strict for \( \alpha = 0 \) by conditions (B1) and (B3). Hence, whole line segment of points \( z \) for \( \alpha \in (-\epsilon, \epsilon) \) belongs into the polytope \( P \) which contradicts the assumption that \( y \) is a vertex of \( P \).

Now, we define a bipartite graph \( G \) which one partite of vertices is the set of time intervals \( T \) and the other partite is the set of pairs \((c, W)\) where \( c \in C \) is a converter and \( W \in S_c \) is an equivalence class. Time interval \( t \) and a pair \((c, W)\) are connected by an edge if \( t \in W \).

**Lemma 5.2.** The graph \( G \) is a forest.

**Proof.** For sake of a contradiction, let us assume that \( G \) has a cycle on vertices

\[
t_1, (c_1, W_1), t_2, (c_2, W_2), \ldots, t_k, (c_k, W_k).
\]

We prove that the sequences \( t_1, \ldots, t_k \) and \( c_1, \ldots, c_k \) satisfy conditions of Lemma 5.1. Since \( t_i, t_{i+1\%} \in W_i \subseteq T_{c_i} \), conditions (B1) and (B3) hold. The conditions (B2) and (B4) follow from the fact that a cycle visits every vertex at most once. This contradicts the assumption that \( y \) is a vertex of \( P \).

Note that Lemmas 5.1 and 5.2 give an efficient way to find a vertex of \( P \) from an arbitrary point of \( P \). Therefore, the first step of our approximation algorithm can also be formulated as finding some point of \( P \) optimizing given objective and repeating the construction in the proof of Lemma 5.1 until \( G \) is a forest.

### 6 Rounding order

The first step constructs a sequence \((c_1, W_1), \ldots, (c_k, W_k)\) of all vertices of the second part of \( G \) determining the order in which non-integer values of the solution \( y \) are rounded. Next section rounds all non-integer values \( y_{c_i,t} \) for \( t \in W_i \) sequentially for \( i = 1, \ldots, k \). The construction of the order is described in the following lemma.

**Lemma 6.1.** There is a sequence \((c_1, W_1), \ldots, (c_k, W_k)\) of all vertices of the second part of \( G \) such that for every \( i \) vertex \((c_i, W_i)\) has at most one non-leaf neighbour \( t_i \) in the graph

\[
G_i = G \setminus \{(c_{i+1}, W_{i+1}), \ldots, (c_k, W_k)\}.
\]

**Proof.** The sequence is constructed from the end. Therefore, \( G_k = G \) and \( G_1 \) is the graph \( G_{k+1} \) without vertex \((c_{i+1}, W_{i+1})\). The vertex \((c_i, W_i)\) of \( G_i \) is determined in the following way.

Let \( G' \) be the graph \( G_i \) without edges joining time interval vertices of degree 1. Since graph \( G' \) is also a forest and no time interval vertex is a leaf, the graph \( G' \) has a vertex \((c_i, W_i)\) of degree at most 1 and we denote its neighbour by \( t_i \) if it exists. In the graph \( G_i \), the vertex \((c_i, W_i)\) has at most one neighbour \( t_i \) which is not a leaf.
First, observe that for every non-integer value $y$, the value $x$ found by the rounding algorithm satisfies the binary condition (3).

**Algorithm 7.1:** Rounding algorithm.

- Find vertex $y$ of $P'$ optimizing given objective function;
- Find sequence $(c_1W_1), \ldots, (c_kW_k)$ by Lemma 6.1;
- $x := y$.

**Proof.**

**Lemma 7.1.** The solution $x$ found by the rounding algorithm satisfies the binary condition (3).

**Proof.** First, observe that for every non-integer value $y_{c,t}$ of the relaxed solution $y$ there exists exactly one equivalence class $W \in T_c$ such that $t \in W$, so the definition of the graph $G$ provides a straightforward bijection between edges of $G$ and non-integer values of $y$. The rounding algorithm starts by setting $x := y$, and then the $i$-th iteration rounds all non-integer values $x_{c,i}$ for $t \in W_i$ into an integer. Therefore, for every non-integer value $y_{c,i}$, the value $x_{c,i}$ is modified by exactly once by one of the following formulas:

- $x_{c,i} := 0$
- $x_{c,i} := 1$
- $x_{c,i} := \lfloor y_{c,i} \rfloor - \lfloor y_{c,i-1} \rfloor$
- $x_{c,i} := \lceil y_{c,i} \rceil - \lceil y_{c,i-1} \rceil$

7 Rounding rules

This section shows how non-integer values of $y$ are rounded using the order created in the previous section. The rounding rules are summarized in Algorithm 7.1.

Note that all cases in the rounding algorithm are needed only for proper setting $x_{c,i}$ for $t \in \{t', t_i, t_i - 1\}$ since for all other $t \in W_i$ it holds $\lfloor y_{c,i} \rfloor - \lfloor y_{c,i-1} \rfloor = \lceil y_{c,i} \rceil - \lceil y_{c,i-1} \rceil$. However, we use this explicit way in Algorithm 7.1 to simplify the analysis in Lemma 7.2.

The proof of correctness of rounding rules is split into Lemmas 7.1, 7.2 and 7.3 proving conditions (3), (12) and (11), respectively.

**Lemma 7.1.** The solution $x$ found by the rounding algorithm satisfies the binary condition (3).

**Proof.** First, observe that for every non-integer value $y_{c,t}$ of the relaxed solution $y$ there exists exactly one equivalence class $W \in T_c$ such that $t \in W$, so the definition of the graph $G$ provides a straightforward bijection between edges of $G$ and non-integer values of $y$. The rounding algorithm starts by setting $x := y$, and then the $i$-th iteration rounds all non-integer values $x_{c,i}$ for $t \in W_i$ into an integer. Therefore, for every non-integer value $y_{c,i}$, the value $x_{c,i}$ is modified by exactly once by one of the following formulas.
Since \( \hat{y}_{c,t} - \hat{y}_{c,t-1} = y_{c,t} \) and \( 0 \leq y_{c,t} \leq 1 \), it follows that all rules sets \( x_{c,t} \) to be zero or one.

**Lemma 7.2.** The solution \( x \) found by the rounding algorithm satisfies (12).

**Proof.** Similarly as in the previous proof, we prove that every non-integer value of \( \hat{y} \) is modified during exactly one iteration and the result satisfies (12). When the rounding algorithm starts, the condition (12) holds since \( x = y \) and we prove by induction that it holds after every iteration. Let \( t^l \) and \( t^l \) be the first and the last time intervals of \( W_i \), respectively. Clearly, the iteration \( i \) changes the only control of converter \( c_i \) and moreover, values \( x_{c_i,t} \) is unchanged for \( t < t^l \).

Note that \( \hat{y}_{c_i,t^l-1} \) \( \in \mathbb{Z} \). Furthermore, \( \hat{y}_{c_i,t^l} \in \mathbb{Z} \) unless \( \hat{y}_{c_i,T} \notin \mathbb{Z} \) and \( t^l \) is the last time interval \( t \) with non-integer value in \( y_{c_i,t} \). Therefore, values \( x_{c_i,t} \) for \( t \geq t^l \) can be changed only by the last equivalence class of \( \sim_{c_i} \). Hence, every value of \( x_{c,t} \) is changed during at most one iteration and it remains to prove that the result of this update satisfies (12).

First, we consider the case \( E_{c_i}(\hat{y}_{t} - \hat{x}_t) \geq 0 \). Since \( \hat{y}_{c_i,t^l-1} = x_{c_i,t^l-1} \). For \( t \in T(t^l, t^l) \), we can easily observe by induction on \( t \) that \( \hat{x}_{c_i,t} = [\hat{y}_{c_i,t}] \). If \( [\hat{y}_{c_i,t^l-1}] = [\hat{y}_{c_i,t^l}] \), then \( \hat{x}_{c_i,t^l} = \hat{x}_{c_i,t^l-1} \) and for \( t \in T(t^l, t^l) \) it also holds that \( \hat{x}_{c_i,t} = \hat{x}_{c_i,t^l} \). Otherwise, \( \hat{x}_{c_i,t} = [\hat{y}_{c_i,t}] \) and for \( t = T(t^l, t^l) \) it also holds that \( \hat{x}_{c_i,t} = [\hat{y}_{c_i,t}] \). If \( \hat{y}_{c_i,t} \in \mathbb{Z} \), then \( \hat{x}_{c_i,t} = \hat{x}_{c_i,t} \) and value \( \hat{x}_{c_i,t} \) is unchanged for \( t \geq t^l \) in this iteration. Otherwise, \( \hat{x}_{c_i,t} \in \{[\hat{y}_{c_i,t}], [\hat{y}_{c_i,t}]\} \) for \( t \geq t^l \). The second case \( E_{c_i}(\hat{y}_{t} - \hat{x}_t) < 0 \) is similar.

**Lemma 7.3.** The solution \( x \) found by the rounding algorithm satisfies (11).

**Proof.** We prove by induction on \( i \) that after iteration \( i \) the condition (11) holds and moreover, for every time interval \( t \) of degree zero in graph \( G_i \) it holds \( \hat{x}_t = \hat{y}_t \). The base of the induction is satisfied since graph \( G_0 \) has no edge and \( y = x \).

In the iteration \( i \), the value \( \hat{x}_t \) is changed only for \( t \in W_i \). For every vertex \( t \in W_i \) of degree zero in \( G_{i-1} \), condition (9) can be satisfied by both settings \( x_{c,t} \) to zero or one. The only possible vertex \( t \in W_i \) of non-zero degree in \( G_{i-1} \) is \( t_i \) by Lemma 6.1. If \( E_{c_i} > 0 \) and \( \hat{y}_t > \hat{x}_t \), then setting \( x_{c_i,t} := 1 \) increases the value \( \hat{x}_t \) by at most \( E \), so the condition (11) remains satisfied. If \( E_{c_i} > 0 \) and \( \hat{y}_t < \hat{x}_t \), then setting \( x_{c_i,t} := 0 \) decreases the value \( \hat{x}_t \) by at most \( E \), so the condition (11) also remains satisfied. Similarly for \( E_{c_i} < 0 \). 

The second and the third step of our algorithm have polynomial-time complexity. Determining the exact degree is not important since the complexity of whole algorithm is dominated by the first step which requires solving linear programming problem. From the discussion in this paper, the following theorem follows.

**Theorem 7.4.** The approximation algorithms for four variants of peak shaving problems find feasible solution in polynomial time. For the basic, the maximal and the absolute peak shaving problems, absolute approximation error is at most \( E \); and for the fluctuation peak shaving problem, the error is at most \( 2E \).

## 8 Concluding remarks and open problems

This paper presents polynomial-time approximation algorithms for four variants of peak shaving problems in a model of scheduling a group of converters. For the basic, the maximal and the absolute peak shaving problems, the absolute approximation error is at most \( E \); and for the fluctuation peak shaving problem, the error is at most \( 2E \), where \( E \) is the maximal electricity consumption of a converter.

The proof determining the approximation error is based on the integrality gap between optimal relaxed and approximated solutions. The approximation error based on the integrality gap is the best possible. Indeed, consider a scheduling of a single converter \( c \) which has to run exactly once during \( T \) time intervals. In this case, the optimal relaxed solution for the basic peak shaving problem is \( y_{c,t} = 1/T \) for every time interval \( t \in T \). In every feasible integer solution, there must exist exactly one time interval \( t \) with \( x_{c,t} = 1 \). Hence, the integrality gap is \((1 - 1/T)E \). Since \( T \) is
unbounded, the integrality gap can be arbitrarily close to $E$. This example can be easily adopted for other three variants of peak shaving problems.

However, it would be more natural to study the approximation error between integer optimal and approximation solutions. The question is whether there exists a polynomial time algorithm with absolute error at most $\epsilon E$ for given parameter $\epsilon$. Paper [12] proves that this problem is NP-hard for $\epsilon = 0$ and this paper proves that it is polynomial for $\epsilon = 1$. Our question is determining the dichotomy.

For practical purposes, polynomial-time approximation algorithms with known worst-case approximation factors would be also useful for more general models. For example,

- converters with conditions on minimal running and off time, and starting and shutdown profiles (see e.g. [4]),
- converters with three states: heating water for domestic hot water demands, space heating or off (see e.g. [26]),
- buffers with losses and other parameters (see e.g. [20]).

References


9 Appendix

This appendix contains a table of the important symbols used in this paper. Note that a variable \( x_{c,t} \) denotes the control of a converter \( c \in C \) in time interval \( t \in T \) and a solution \( x \) means control of all converter during whole planning horizon (similarly for \( y \) and \( z \)). For simplicity, terms like “by a converter \( c \) in time interval \( t \)” are omitted in explanations of symbols like \( \hat{x}_{c,t} \).

\[
\begin{align*}
C & \quad \text{set of heating systems} \\
T & \quad \text{set of time intervals} \\
\mathbb{Z} & \quad \text{set of integer numbers} \\
A_{c,t} & \quad \text{precomputed lower bound on } \hat{x}_{c,t} \\
B_{c,t} & \quad \text{precomputed upper bound on } \hat{x}_{c,t} \\
D_{c,t} & \quad \text{heat demand from the heating system} \\
E_c & \quad \text{electricity consumed by a running converter } c \\
E & \quad = \max_{c \in C} |E_c| \\
F_t & \quad \text{base electricity load} \\
G & \quad \text{bipartite graph which connects equivalence classes } S_c \text{ and time intervals of } S_c \\
H_c & \quad \text{heat produced by a running converter } c \\
L_{c,t} & \quad \text{lower bound on the state of charge of buffer} \\
P & \quad \text{polytope } \{ x : (9) \text{ and } (10) \text{ hold, and } \hat{x}_{t} = \hat{y}_{t} \text{ holds for every } t \in T \} \\
S_c & \quad \text{set of equivalence classes on } T_c \text{ by relation } \sim_c \\
T & \quad \text{number of time intervals} \\
T(t_1,t_2) & = \{t_1,t_1+1,\ldots,t_2-1\} \text{ if } t_1 \leq t_2 \text{ and } \{t_2,t_2+1,\ldots,t_1-1\} \text{ otherwise} \\
U_{c,t} & \quad \text{upper bound on the state of charge of buffer} \\
W,W_i & \quad \text{one equivalence class of } S_c \text{ for some converter } c \\
c,c_i & \quad \text{indexes of a converter} \\
i,j & \quad \text{index of locally defined meaning} \\
m & \quad \text{objective function; bound on } \hat{x}_t \\
m^O & \quad \text{optimal value of objective function} \\
m^A & \quad \text{value of objective function of the approximated solution found by our algorithm} \\
m_l,m_u & \quad \text{lower and upper bounds on } \hat{x}_t \text{ in the fluctuation peak shaving problem} \\
sc,t & \quad \text{state of charge of buffer } c \text{ in the beginning of time interval } t \\
t,t_i & \quad \text{indexes of a time interval} \\
x_{c,t} & \quad \text{operational state of the converter} \\
\hat{x}_{c,t} & = \sum_{i=1}^{t} x_{c,i}; \text{ similarly for } \hat{y}_{c,t} \text{ and } \hat{z}_{c,t} \\
\hat{x}_t & = F_t + \sum_{c \in C} E_c x_{c,t}; \text{ similarly for } \hat{y}_t \text{ and } \hat{z}_t \\
y & \quad \text{optimal solution of the relaxed problem} \\
\sim_c & \quad \text{relation on } T_c \text{ such that } t_1 \sim_c t_2 \text{ if and only if } \hat{y}_{c,t} \notin \mathbb{Z} \text{ for every } t = T(t_1,t_2) \\
i + 1\%k & = i + 1 \text{ if } i < k \text{ and } 1 \text{ if } i = k \\
(c,W) & \quad \text{vertex of the graph } G; W \text{ is equivalence class of } S_c \text{ for converter } c \\
\lfloor a \rfloor & \quad \text{largest integer value not greater than argument} \\
\lceil a \rceil & \quad \text{smallest integer value not smaller than argument}
\end{align*}
\]