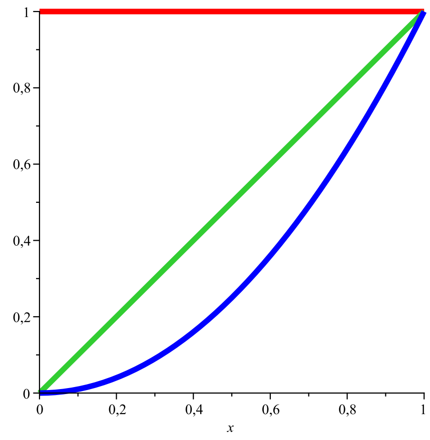
Ortonormální systémy reálných funkcí

Pro sklalární součin $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$
obvyklá báze $(1, x, x^2)$ *není* ortonormální bazí
prostoru polynomů stupně nejvýše dva na intervalu $(0, 1)$:

$$||1|| = \sqrt{\int_0^1 1 \cdot 1\, dx} = \sqrt{[x]_0^1} = 1$$

$$||x|| = \sqrt{\int_0^1 x \cdot x\, dx} = \sqrt{[\tfrac{1}{3}x^3]_0^1} = \frac{\sqrt{3}}{3} \neq 1$$

$$||x^2|| = \sqrt{\int_0^1 x^2 \cdot x^2\, dx} = \sqrt{[\tfrac{1}{5}x^5]_0^1} = \frac{\sqrt{5}}{5} \neq 1$$



$$\langle 1|x\rangle = \int_0^1 1 \cdot x\, dx = [\tfrac{1}{2}x^2]_0^1 = \tfrac{1}{2} \neq 0$$

$$\langle 1|x^2\rangle = \int_0^1 1 \cdot x^2\, dx = [\tfrac{1}{3}x^3]_0^1 = \tfrac{1}{3} \neq 0$$

$$\langle x|x^2\rangle = \int_0^1 x \cdot x^2\, dx = [\tfrac{1}{4}x^4]_0^1 = \tfrac{1}{4} \neq 0$$

Zde je potřeba vzít jinou bázi a to např.
$(1, \sqrt{3}(2x-1), \sqrt{5}(6x^2-6x+1))$:

$$||1|| = \sqrt{\int_0^1 1\cdot 1\,dx} = \sqrt{[x]_0^1} = 1$$

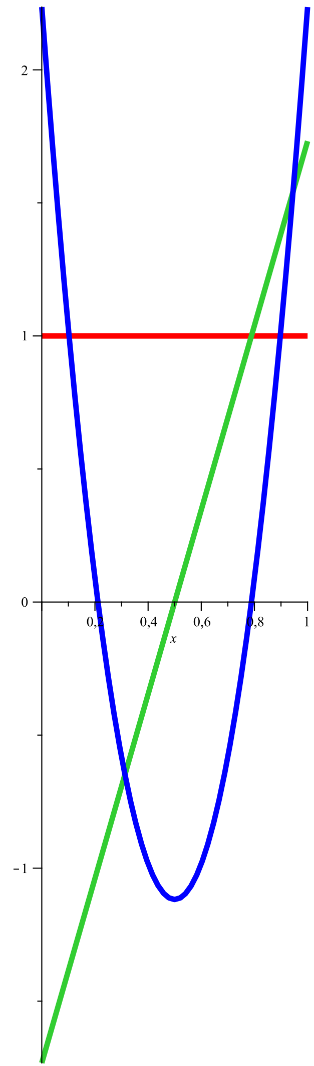
$$||\sqrt{3}(2x-1)|| = \sqrt{\int_0^1 3(4x^2-4x+1)\,dx} =$$
$$= \sqrt{3[1\tfrac{1}{3}x^3 - 2x^2 + x]_0^1} = 1$$

$$||\sqrt{5}(6x^2-6x+1)|| =$$
$$= \sqrt{\int_0^1 5(36x^4 - 72x^3 + 48x^2 - 12x + 1)\,dx} =$$
$$= \sqrt{5[7\tfrac{1}{5}x^5 - 18x^4 + 16x^3 - 6x^2 + x]_0^1} = 1$$

$$\langle 1|\sqrt{3}(2x-1)\rangle = \int_0^1 \sqrt{3}(2x-1)\,dx = \sqrt{3}[x^2-x]_0^1 = 0$$

$$\langle 1|\sqrt{5}(6x^2-6x+1)\rangle = \int_0^1 \sqrt{5}(6x^2-6x+1)\,dx =$$
$$= \sqrt{5}[2x^3-3x^2+x]_0^1 = 0$$

$$\langle \sqrt{3}(2x-1)|\sqrt{5}(6x^2-6x+1)\rangle =$$
$$= \int_0^1 \sqrt{15}(12x^3 - 18x + 8x - 1)\,dx =$$
$$= \sqrt{15}[3x^4 - 6x^3 + 4x^2 - x]_0^1 = 0$$



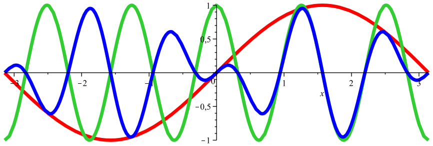
Funkce $\sin(ix)$ a $\cos(jx)$ jsou na sebe kolmé na intervalu $(-\pi,\pi)$, $i,j\in\mathbb{N}$.

$$\langle \sin(ix)|\sin(jx)\rangle = \int_{-\pi}^{\pi}\tfrac{1}{2}[\cos((i-j)x)-\cos((i+j)x)]\,dx =$$
$$= \tfrac{1}{2}[\tfrac{1}{i-j}\sin((i-j)x)-\tfrac{1}{i+j}\sin((i+j)x)]_{-\pi}^{\pi}=0 \qquad \text{pro } i\neq j$$
$$\langle \cos(ix)|\cos(jx)\rangle = \int_{-\pi}^{\pi}\tfrac{1}{2}[\cos((i-j)x)+\cos((i+j)x)]\,dx =$$
$$= \tfrac{1}{2}[\tfrac{1}{i-j}\sin((i-j)x)+\tfrac{1}{i+j}\sin((i+j)x)]_{-\pi}^{\pi}=0 \qquad \text{pro } i\neq j$$
$$\langle \sin(ix)|\cos(jx)\rangle = \int_{-\pi}^{\pi}\sin(ix)\cos(jx)\,dx = 0$$

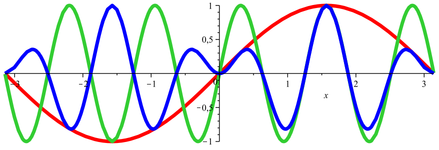
Využíváme faktu, že $\sin(k\pi)=0$ pro celočíselná k.
V posledním případě integrujeme lichou funkci na symetrickém intervalu.

$$||\sin(ix)||^2 = \int_{-\pi}^{\pi}\tfrac{1}{2}[1-\cos(2ix)]\,dx = \tfrac{1}{2}[x+\tfrac{1}{2i}\sin(2ix)]_{-\pi}^{\pi}=\pi$$
$$||\cos(ix)||^2 = \int_{-\pi}^{\pi}\tfrac{1}{2}[1+\cos(2ix)]\,dx = \tfrac{1}{2}[x+\tfrac{1}{2i}\sin(2ix)]_{-\pi}^{\pi}=\pi$$
…po znormování $\frac{\sqrt{\pi}}{\pi}$ bychom dostali ortonormální systém.

Využíváme součtových vzorců:
$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$
$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$
$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$



$\langle\sin(x)|\cos(5x)\rangle$



$\langle\sin(x)|\sin(5x)\rangle$