Better upper bounds on the Füredi–Hajnal limits of permutations

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An Extremal Problem on Binary Matrices

- All matrices will be binary matrices . . . all entries from \{0, 1\}.
- A matrix $A$ contains a matrix $B$ if $B$ can be obtained from $A$ by removing some rows, columns and 1’s.

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- Otherwise $A$ avoids $B$
- $\text{ex}_B(n)$ ... maximum number of 1’s in an $n \times n$ matrix $A$ avoiding $B$
- $B$ is forbidden
- Similar to the Turán theory of graphs.
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\]

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  &       &   
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  \bullet &   &   \\
  &   &   \\
  &   &   
\end{pmatrix}
\]

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Forbidden permutation matrices

- Permutation \(\leftrightarrow\) Permutation matrix
  \[
  2 \leftrightarrow \begin{pmatrix}
    \bullet \\
    \bullet
  \end{pmatrix} \\
  4 \leftrightarrow \begin{pmatrix}
    \bullet \\
    \bullet
  \end{pmatrix} \\
  3 \leftrightarrow \begin{pmatrix}
    \bullet
  \end{pmatrix} \\
  1 \leftrightarrow \begin{pmatrix}
    \bullet
  \end{pmatrix}
  \]
- An \textit{n-permutation matrix} is an \(n \times n\) binary matrix with exactly one 1 in every column and row.

Trivial:
- \(\exp_P(n) \geq 2(k - 1)n - (k - 1)^2\) for every \(k\)-permutation matrix \(P\)

\[
P \left( \begin{array}{ccc}
  \bullet & \bullet & \bullet \\
  \bullet & \bullet
\end{array} \right) \hspace{1cm} \text{Avoided by:} \left( \begin{array}{ccc}
  \bullet & \bullet & \bullet & \bullet \\
  \bullet & \bullet & \bullet & \bullet \\
  \bullet & \bullet & \bullet & \bullet \\
  \bullet & \bullet & \bullet & \bullet
\end{array} \right)
\]

Superadditivity: If \(k \geq 2\), \(\exp_P(n_1 + n_2) \geq \exp_P(n_1) + \exp_P(n_2)\)

Proof: If \(A\) and \(B\) avoid \(P\), then either \(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\) or \(\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\) avoids \(P\).
The Füredi–Hajnal conjecture

**Theorem (The Füredi–Hajnal conjecture, Marcus and Tardos 2004)**

For every $k$-permutation matrix $P$:

$$
\exp_P(n) \leq 2k^4 \binom{k^2}{k} n = 2^{O(k \log(k))} n.
$$

- $c_P \overset{\text{def}}{=} \lim_{n \to \infty} \exp_P(n)/n \ldots$ the Füredi–Hajnal limit of $P$
- $c_P$ exists by the superadditivity and is finite by the theorem of Marcus and Tardos
- $c_P \leq 2^{O(k)}$ for every $k$-permutation matrix $P$ (Fox 2017+)
The Stanley–Wilf conjecture

- $S_P(n)$ ... the set of $n$-permutation matrices avoiding a permutation matrix $P$
- $|S_P(n)|$ is supermultiplicative $\rightarrow$ lower bound exponential in $n$

**Theorem (Marcus, Tardos (2004), using a result of Klazar (2000))**

*For every permutation $P$, we have $|S_P(n)| \leq s_P^n$ for some constant $s_P$.***

- Was a conjecture of Stanley and Wilf from around 1992.
- **The Stanley–Wilf limit:** $s_P \overset{\text{def}}{=} \lim_{n \to \infty} \sqrt[n]{|S_P(n)|}$
  - Klazar 2000: $s_P \leq 15c_P$
  - C. 2009: $s_P \leq 2.88c_P^2$
  - C. 2009: $c_P \leq O(s_P^{4.5})$
  - Fox 2017+: $c_P \leq O(s_P^3)$
  - C.,K. 2017+: $c_P \leq O(s_P^{2.75})$
Lower bounds on $c_P$

The trivial lower bound $c_P \geq 2(k - 1)$ is tight for some $k$-permutation matrices

\begin{equation}
I_k \begin{pmatrix}
\bullet & \bullet & \cdots & \bullet \\
\bullet & \bullet & \cdots & \bullet \\
\vdots & \vdots & \ddots & \vdots \\
\bullet & \bullet & \cdots & \bullet
\end{pmatrix}
\quad A \text{ avoiding } I_k:
\begin{pmatrix}
\bullet & \bullet & \cdots & \bullet \\
\bullet & \bullet & \cdots & \bullet \\
\vdots & \vdots & \ddots & \vdots \\
\bullet & \bullet & \cdots & \bullet
\end{pmatrix}
\quad \text{At most } k - 1 \text{-entries on each diagonal}
\end{equation}

**Theorem (C. 2009)**

For every $k$ there is a $k$-permutation matrix $X_k$ satisfying $c_{X_k} \geq \Omega(k^2)$ (explicit construction of $X_k$-avoiding matrices).

\begin{equation}
X_8 \begin{pmatrix}
\bullet & \bullet & \cdots & \bullet \\
\bullet & \bullet & \cdots & \bullet \\
\vdots & \vdots & \ddots & \vdots \\
\bullet & \bullet & \cdots & \bullet
\end{pmatrix}
\quad \text{Avoided by:}
\begin{pmatrix}
\bullet & \bullet & \cdots & \bullet \\
\bullet & \bullet & \cdots & \bullet \\
\vdots & \vdots & \ddots & \vdots \\
\bullet & \bullet & \cdots & \bullet
\end{pmatrix}
\end{equation}

**Conjecture (Many people)**

The values $s_P$ (and $c_P$) are bounded by a polynomial in $k$. 
Lower bounds on $c_P$

Theorem (Fox 2017+)

For every $k$, there is a $k$-permutation matrix $F_k$ satisfying $c_{F_k} \geq 2^{\Omega(\sqrt{k})}$ (probabilistic construction of $F_k$-avoiding matrices).

$$F_9 \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

Avoided by:

$$\{i2^j + 1, \ldots, (i+1)2^j\}$$

Theorem (Fox 2017+)

Asymptotically almost all $k$-permutation matrices $P$ satisfy $c_P \geq 2^\Omega(\sqrt{k/\log(k)})$.

Corollary

These two theorems hold also when $c_P$ is replaced with $s_P$. 
The new upper bounds

**Theorem**

*Asymptotically almost all k-permutation matrices P satisfy*

\[ c_P \leq 2^{O(k^{2/3} \log(k)^{7/3})}. \]

That is,

\[ \text{Prob}\{c_P \leq 2^{O(k^{2/3} \log(k)^{7/3})} \mid P \text{ is a } k\text{-permutation matrix}\} \to 1 \]

as \( k \to \infty \)

**Summary of bounds:**

<table>
<thead>
<tr>
<th>All k-permutation matrices</th>
<th>Almost all</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_P \geq 2k - 2 ) [trivial]</td>
<td>( c_P \geq 2^{\Omega(\sqrt{k/ \log(k)})} ) [Fox 2017+]</td>
</tr>
<tr>
<td>( c_P \leq 2^{O(k)} ) [Fox 2017+]</td>
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**Corollary**

*Asymptotically almost all k-permutation matrices P satisfy*

\[ s_P \leq 2^{O(k^{2/3} \log(k)^{7/3})}. \]
Scattered permutation matrices

- The distance vector between two 1-entries is the vector whose coordinates are the differences between the row and column indices.

\[
\begin{pmatrix}
1 & 1 \\
4 & k
\end{pmatrix}
\]

Distance vector \((-1, 4)\)

- A k-permutation matrix \(P\) is scattered if every vector is a distance vector of at most \(\log_2(k)\) pairs of 1-entries of \(P\).

Theorem

The number of k-permutation matrices that are not scattered is at most

\[
2k! \left( \frac{4e}{\log_2(k)} \right)^{\log_2(k)} = k! \cdot \frac{1}{\log_2(k)^{\Omega(\log(k))}}.
\]
Lemma

Let $A$ be a $3k \times 3k$ binary matrix, where $k \geq 9$. Let $P$ be a scattered $k$-permutation matrix. If every row and every column of $A$ has at most $k^{1/3}/(3 \log_2(k)^{1/3})$ 0-entries then $A$ contains $P$.

Proof.
Trivial algorithm for finding an occurrence of $P$ on a fixed set of rows of $A$: 

![Diagram showing the algorithm](image)
Finding scattered permutation matrices — Base case

- Run $2k + 1$ instances of the algorithm — one on each $k$-tuple of consecutive rows of $A$.
- Aim: Show that they together make at most twice more stalls than moves.

Sketch of the proof:

- $\tilde{O}, \tilde{\Omega}, \tilde{\Theta}$ ... ignoring logarithmic multiplicative factors
- For contradiction more than $2/3$ of instances stalled in one column.
- For simplicity $\tilde{\Omega}(k^{2/3})$ instances stalled on each 0 of the column.
- $M$ ... the set of $\tilde{\Omega}(k^{2/3})$ instances stalled on one fixed 0.
- All instances of $M$ start moving within the next $\tilde{O}(k^{1/3})$ columns.
- $P$ scattered $\rightarrow$ at most $\log_2(k)$ instances become stalled on each 0 in the following columns.
- Half of the instances of $M$ make at least $\tilde{\Omega}(k^{1/3})$ moves before they stall.
Scattered perm. matrices — Trading size for density

- Density \( q \) of \( A \) is \( \ldots \) fraction of the number of 1-entries and all entries.

**Corollary**

For every scattered \( P \), we have

\[
\text{ex}_P(4k) \leq (4k)^2 \cdot (1 - \tilde{\Omega}(k^{-2/3})).
\]

That is, every \( 4k \times 4k \) matrix of density at least \( (1 - \tilde{\Theta}(k^{-2/3})) \) contains \( P \).

**Theorem (Size-density tradeoff)**

Let \( u \in \mathbb{N} \) and \( q \in (1/u, 1) \). If every \( u \times u \) matrix of density \( q \) contains \( P \), then

\[
c_P \leq 2u^3 u^\lceil -\log u/\log q \rceil.
\]

That is,

\[
q = 1 - \tilde{\Theta}(k^{-2/3}) \quad \text{and} \quad q = 2\tilde{\Theta}(k^{2/3})/n.
\]
Upper bounds for special matrices

- $c_P \leq k^{O(\log(k))}$ for matrices with 1-entries on the main diagonal and the main skew diagonal

  Example:

  \[
  X = \begin{pmatrix}
  \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot
  \end{pmatrix}
  \]

- $c_P \leq k^{O(k^{1/2} \log(k))}$ for grid products of matrices with small Füredi–Hajnal constant

  Example:

  \[
  F = \begin{pmatrix}
  \cdot & \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot & \cdot
  \end{pmatrix}
  \]
Higher-dimensional matrices

In a \( d \)-dimensional permutation \( P \):

- Every \( (d - 1) \)-dimensional slice contains exactly one 1-entry.
- Every projection to 2 dimensions is a permutation.

\[
\begin{align*}
\text{3-dimensional permutation} & \\
\text{Projection on } xy\text{-plane.} & \\
\text{xy-slices} & \\
\end{align*}
\]

- \( \text{ex}_P(n) \) is maximum number of 1’s in a \( P \)-avoiding \( d \)-dimensional \( n \times n \times \cdots \times n \) binary matrix.
- \( S_P(n) \) is the set of \( P \)-avoiding \( d \)-dimensional \( n \times n \times \cdots \times n \) permutation matrices.
Higher-dimensional generalizations of the conjectures

Let $P$ be a $d$-dimensional permutation matrix of size at least $2 \times 2 \times \cdots \times 2$.

Observation (Trivial bounds)

1. $\Omega(n^{d-1}) \leq \text{ex}_P(n) \leq n^d$
2. $2^{\Omega(n)}(n!)^{d-2} \leq |S_P(n)| \leq (n!)^{d-1}$

Theorem (Klazar, Marcus (2007))

$\text{ex}_P(n) \leq O(n^{d-1})$

Theorem (C. (2009))

$|S_P(n)| \leq 2^{O(n \log \log(n))} (n!)^{d-1-1/(d-1)}$

Theorem

$2^{-O(n)}(n!)^{d-1-1/(d-1)} \leq |S_P(n)| \leq 2^{O(n)}(n!)^{d-1-1/(d-1)}$
Avoiding a $2 \times 2 \times 2$ permutation matrix

- Let $P$ be the $2 \times 2 \times 2$ permutation matrix with 1-entries at $(1, 1, 1)$ and $(2, 2, 2)$.

\[
P = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\]

- Let $A$ be an $n \times n \times n$ permutation matrix with 1-entries at positions $\alpha_1 = (x_1, y_1, z_1), \ldots, \alpha_n = (x_n, y_n, z_n)$.
- Three linear orders on $\alpha_1, \ldots, \alpha_n$: $<_x, <_y$, and $<_z$.
- Partial order $<_\ldots$ intersection of $<_x, <_y$, and $<_z$.
- That is: $\alpha_i <_\ldots \alpha_j$ if and only if $x_i < x_j$ & $y_i < y_j$ & $z_i < z_j$.
- If $A$ is a random $n \times n \times n$ permutation matrix, then $<_\ldots$ is a random partial order of dimension 3.
- $A$ avoids $P$ if and only if $\{\alpha_1, \ldots, \alpha_n\}$ is an antichain in $<_\ldots$

**Theorem (Brightwell (1992))**

*The probability that a random $d$-dimensional partial order on $n$ elements is an antichain is at least $(e^{-2}n^{1-1/(d-1)})^n$.*