Better upper bounds on the Füredi–Hajnal limits of permutations

Josef Cibulka and Jan Kynčl

Charles University, Prague

January 19th, 2017 / SODA 2017

An Extremal Problem on Binary Matrices

- All matrices will be binary matrices ... all entries from $\{0, 1\}$.
- A matrix A contains a matrix B if B can be obtained from A by removing some rows, columns and 1's.

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- Otherwise A avoids B
- $\mathrm{ex}_B(n)$ ... maximum number of 1's in an $n \times n$ matrix A avoiding B
- B is forbidden
- Similar to the Turán theory of graphs.

An Extremal Problem on Binary Matrices

- All matrices will be binary matrices ... all entries from $\{0, 1\}$.
- A matrix A contains a matrix B if B can be obtained from A by removing some rows, columns and 1's.

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

- Otherwise A avoids B
- $\mathrm{ex}_B(n)$ ... maximum number of 1's in an $n \times n$ matrix A avoiding B
- B is forbidden
- Similar to the Turán theory of graphs.

An Extremal Problem on Binary Matrices

- All matrices will be binary matrices ... all entries from $\{0, 1\}$.
- A matrix A contains a matrix B if B can be obtained from A by removing some rows, columns and 1's.

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- Otherwise A avoids B
- $\mathrm{ex}_B(n)$ ... maximum number of 1's in an $n \times n$ matrix A avoiding B
- B is forbidden
- Similar to the Turán theory of graphs.

An Extremal Problem on Binary Matrices

- All matrices will be binary matrices ... all entries from $\{0, 1\}$.
- A matrix A contains a matrix B if B can be obtained from A by removing some rows, columns and 1's.

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- Otherwise A avoids B
- $\mathrm{ex}_B(n)$ ... maximum number of 1's in an $n \times n$ matrix A avoiding B
- B is forbidden
- Similar to the Turán theory of graphs.

An Extremal Problem on Binary Matrices

- All matrices will be binary matrices ... all entries from $\{0, 1\}$.
- A matrix A contains a matrix B if B can be obtained from A by removing some rows, columns and 1's.

$$B = \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix} \qquad A = \begin{pmatrix} \bullet & & \color{red}{\bullet} & \\ & & & \color{red}{\bullet} \\ & & \bullet & \\ \color{red}{\bullet} & & & \\ & \bullet & & \end{pmatrix}$$

- Otherwise A avoids B
- $\mathrm{ex}_B(n)$ ... maximum number of 1's in an $n \times n$ matrix A avoiding B
- B is forbidden
- Similar to the Turán theory of graphs.

Forbidden permutation matrices

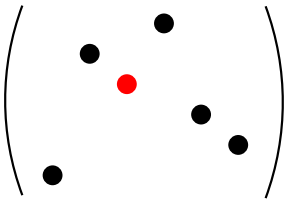
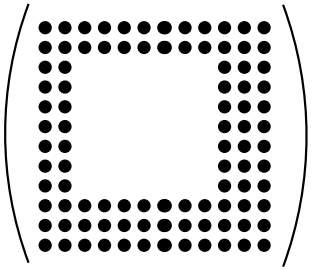
- Permutation $\leftrightarrow$ Permutation matrix

$$\begin{array}{ccc} 2 & \leftrightarrow & \\ 4 & \leftrightarrow & \\ 3 & \leftrightarrow & \\ 1 & \leftrightarrow & \end{array} \begin{pmatrix} & \bullet & & \\ & & & \bullet \\ & & \bullet & \\ \bullet & & & \end{pmatrix}$$

- An *n*-permutation matrix is an $n \times n$ binary matrix with exactly one 1 in every column and row.

Trivial:

- $\mathrm{ex}_P(n) \geq 2(k-1)n - (k-1)^2$ for every k-permutation matrix P

P  Avoided by: 

- Superadditivity: If $k \geq 2$, $\mathrm{ex}_P(n_1 + n_2) \geq \mathrm{ex}_P(n_1) + \mathrm{ex}_P(n_2)$

  Proof: If A and B avoid P, then either $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ or $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ avoids P.

The Füredi–Hajnal conjecture

Theorem (The Füredi–Hajnal conjecture, Marcus and Tardos 2004)

For every k-permutation matrix P:

$$\mathrm{ex}_P(n) \leq 2k^4 \binom{k^2}{k} n = 2^{O(k \log(k))} n.$$

- $c_P \stackrel{\text{def}}{=} \lim_{n\to\infty} \mathrm{ex}_P(n)/n \ldots$ the Füredi–Hajnal limit of P
- c_P exists by the superadditivity and is finite by the theorem of Marcus and Tardos
- $c_P \leq 2^{O(k)}$ for every k-permutation matrix P (Fox 2017+)

The Stanley–Wilf conjecture

- $S_P(n)$ ... the set of n-permutation matrices avoiding a permutation matrix P
- $|S_P(n)|$ is supermultiplicative $\rightarrow$ lower bound exponential in n

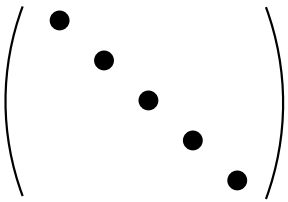
Theorem (Marcus, Tardos (2004), using a result of Klazar (2000))

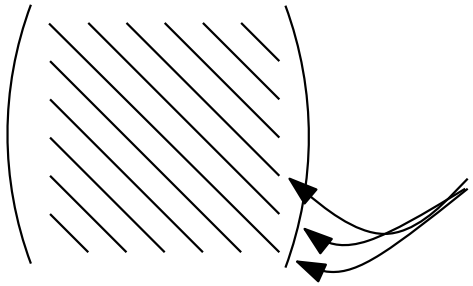
For every permutation P, we have $|S_P(n)| \leq s_P^n$ for some constant s_P.

- Was a conjecture of Stanley and Wilf from around 1992.
- The Stanley–Wilf limit: $s_P \stackrel{def}{=} \lim_{n\to\infty} \sqrt[n]{|S_P(n)|}$
- Klazar 2000: $s_P \leq 15^{c_P}$
- C. 2009: $s_P \leq 2.88 c_P^2$
- C. 2009: $c_P \leq O(s_P^{4.5})$
- Fox 2017+: $c_P \leq O(s_P^3)$
- C.,K. 2017+: $c_P \leq O(s_P^{2.75})$

Lower bounds on c_P

The trivial lower bound $c_P \geq 2(k-1)$ is tight for some k-permutation matrices

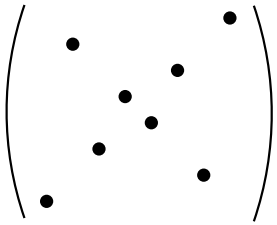
I_k 

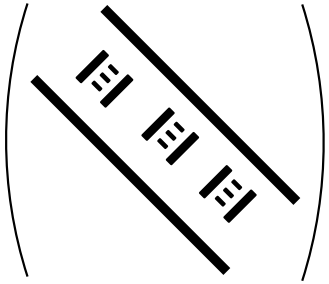
A avoiding I_k: 

At most $k-1$ 1-entries on each diagonal

Theorem (C. 2009)

For every k there is a k-permutation matrix X_k satisfying $c_{X_k} \geq \Omega(k^2)$ (explicit construction of X_k-avoiding matrices).

X_8 

Avoided by: 

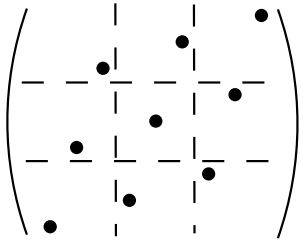
Conjecture (Many people)

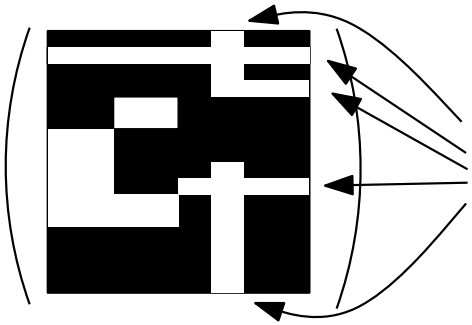
The values s_P (and c_P) are bounded by a polynomial in k.

Lower bounds on c_P

Theorem (Fox 2017+)

For every k, there is a k-permutation matrix F_k satisfying $c_{F_k} \geq 2^{\Omega(\sqrt{k})}$ (probabilistic construction of F_k-avoiding matrices).

F_9 

Avoided by: 

dyadic intervals:
$\{i2^j+1, \ldots, (i+1)2^j\}$

Theorem (Fox 2017+)

Asymptotically almost all k-permutation matrices P satisfy $c_P \geq 2^{\Omega(\sqrt{k/\log(k)})}$.

Corollary

These two theorems hold also when c_P is replaced with s_P.

The new upper bounds

Theorem

Asymptotically almost all k-permutation matrices P satisfy $c_P \leq 2^{O(k^{2/3}\log(k)^{7/3})}$.

That is,

$$\text{Prob}\{c_P \leq 2^{O(k^{2/3}\log(k)^{7/3})} \mid P \text{ is a } k\text{-permutation matrix}\} \to 1 \quad \text{as } k \to \infty$$

Summary of bounds:

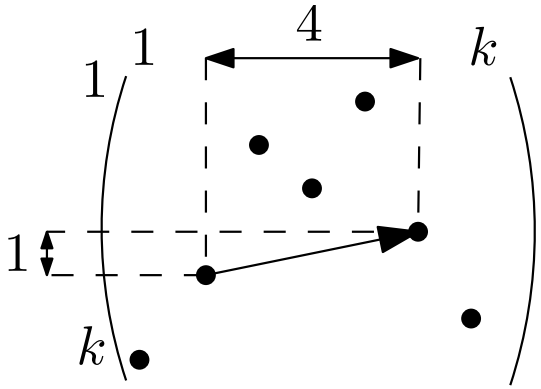
All k-permutation matrices	Almost all
$c_P \geq 2k-2$ [trivial]	$c_P \geq 2^{\Omega(\sqrt{k/\log(k)})}$ [Fox 2017+]
$c_P \leq 2^{O(k)}$ [Fox 2017+]	$c_P \leq 2^{O(k^{2/3}\log(k)^{7/3})}$

Corollary

Asymptotically almost all k-permutation matrices P satisfy $s_P \leq 2^{O(k^{2/3}\log(k)^{7/3})}$.

Scattered permutation matrices

- The distance vector between two 1-entries is the vector whose coordinates are the differences between the row and column indices.



Distance vector $(-1, 4)$

- A k-permutation matrix P is scattered if every vector is a distance vector of at most $\log_2(k)$ pairs of 1-entries of P.

Theorem

The number of k-permutation matrices that are not scattered is at most

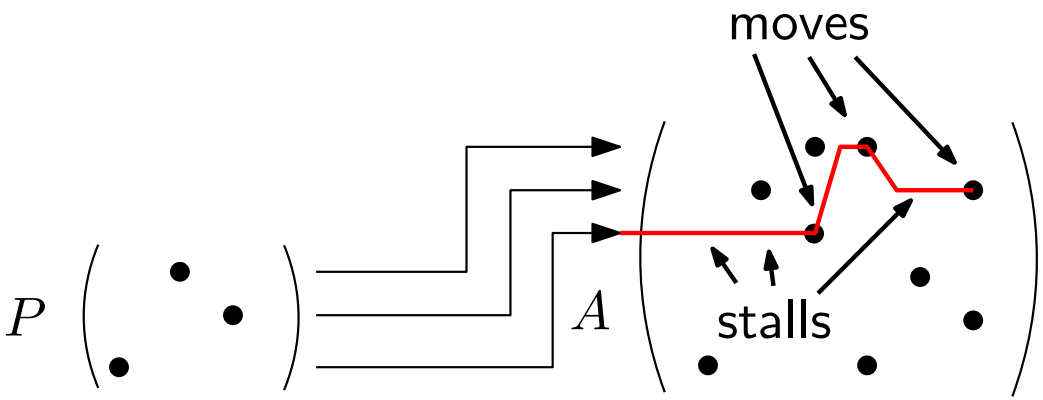
$$2k!\left(\frac{4e}{\log_2(k)}\right)^{\log_2(k)} = k! \cdot \frac{1}{\log_2(k)^{\Omega(\log(k))}}.$$

Finding scattered permutation matrices — Base case

Lemma

Let A be a $3k \times 3k$ binary matrix, where $k \geq 9$. Let P be a scattered k-permutation matrix. If every row and every column of A has at most $k^{1/3}/(3\log_2(k)^{1/3})$ 0-entries then A contains P.

Proof.
Trivial algorithm for finding an occurrence of P on a fixed set of rows of A:

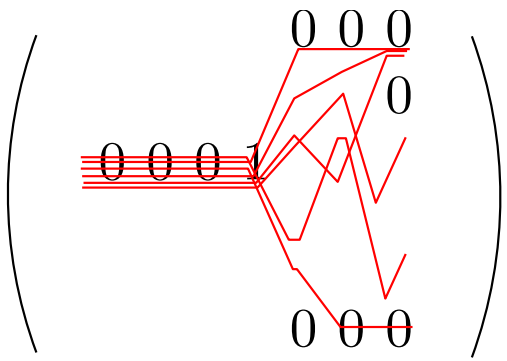


Finding scattered permutation matrices — Base case

- Run $2k+1$ instances of the algorithm — one on each k-tuple of consecutive rows of A.
- Aim: Show that they together make at most twice more stalls than moves.

Sketch of the proof:

- $\widetilde{O}$, $\widetilde{\Omega}$, $\widetilde{\Theta}$ ... ignoring logarithmic multiplicative factors
- For contradiction more than $2/3$ of instances stalled in one column.
- For simplicity $\widetilde{\Omega}(k^{2/3})$ instances stalled on each 0 of the column.
- M ... the set of $\widetilde{\Omega}(k^{2/3})$ instances stalled on one fixed 0.
- All instances of M start moving within the next $\widetilde{O}(k^{1/3})$ columns.
- P scattered $\rightarrow$ at most $\log_2(k)$ instances become stalled on each 0 in the following columns.
- Half of the instances of M make at least $\widetilde{\Omega}(k^{1/3})$ moves before they stall.



Scattered perm. matrices — Trading size for density

- Density q of A . . . fraction of the number of 1-entries and all entries.

Corollary

For every scattered P, we have

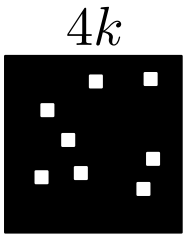
$$\mathrm{ex}_P(4k) \leq (4k)^2 \cdot (1 - \widetilde{\Omega}(k^{-2/3})).$$

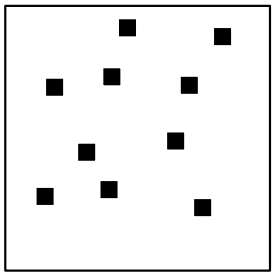
That is, every $4k \times 4k$ matrix of density at least $(1 - \widetilde{\Theta}(k^{-2/3}))$ contains P.

Theorem (Size-density tradeoff)

Let $u \in \mathbb{N}$ and $q \in (1/u, 1)$. If every $u \times u$ matrix of density q contains P, then

$$c_P \leq 2u^3 u^{\lceil -\log u / \log q \rceil}.$$

P scattered $\Longrightarrow$ $4k$ 

contains P $\Longrightarrow$ n 

contains P

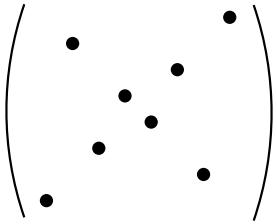
$q = 1 - \widetilde{\Theta}(k^{-2/3})$

$q = 2^{\widetilde{\Theta}(k^{2/3})}/n$

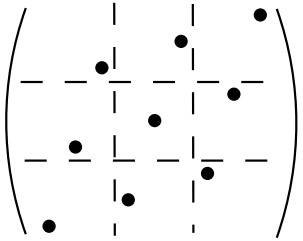
That is, $c_P = 2^{\widetilde{O}(k^{2/3})}$

Upper bounds for special matrices

- $c_P \leq k^{O(\log(k))}$ for matrices with 1-entries on the main diagonal and the main skew diagonal

Example: X 

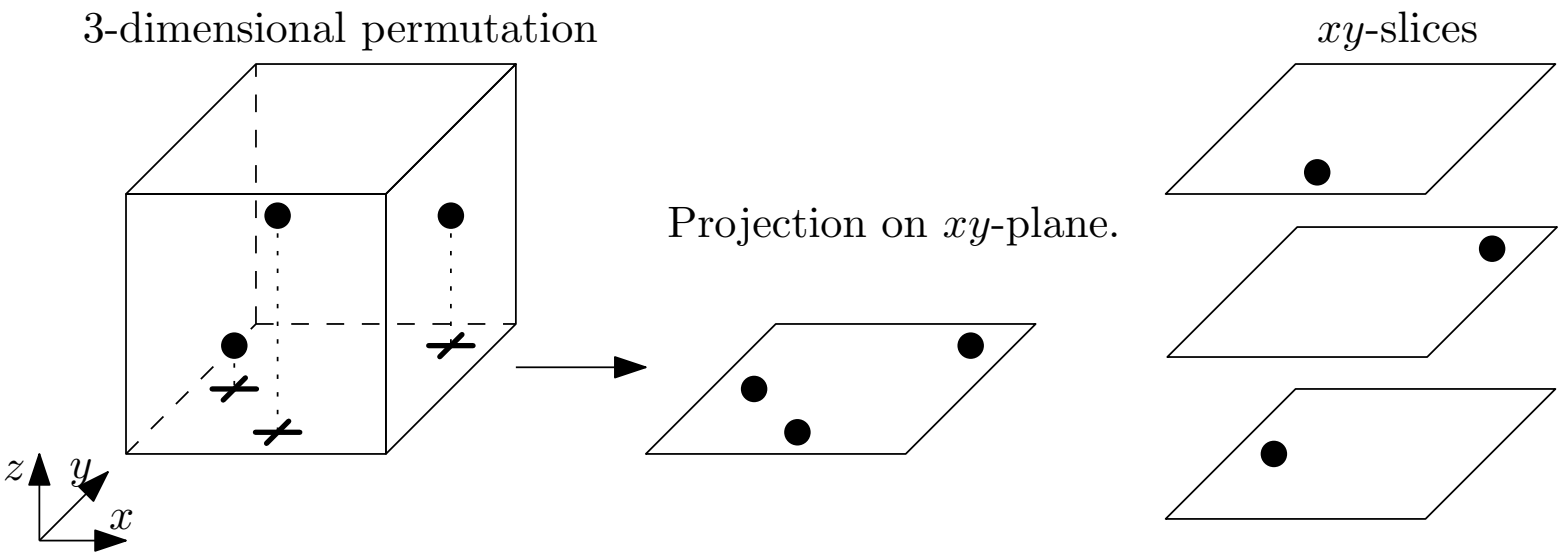
- $c_P \leq k^{O(k^{1/2}\log(k))}$ for grid products of matrices with small Füredi–Hajnal constant

Example: F 

Higher-dimensional matrices

In a d-dimensional permutation P:

- Every $(d-1)$-dimensional slice contains exactly one 1-entry.
- Every projection to 2 dimensions is a permutation.



- $\mathrm{ex}_P(n)$ is maximum number of 1's in a P-avoiding d-dimensional $n \times n \times \cdots \times n$ binary matrix.
- $S_P(n)$ is the set of P-avoiding d-dimensional $n \times n \times \cdots \times n$ permutation matrices.

Higher-dimensional generalizations of the conjectures

Let P be a d-dimensional permutation matrix of size at least $2 \times 2 \times \cdots \times 2$.

Observation (Trivial bounds)

- $$\Omega(n^{d-1}) \leq \mathrm{ex}_P(n) \leq n^d$$
- $$2^{\Omega(n)}(n!)^{d-2} \leq |S_P(n)| \leq (n!)^{d-1}$$

Theorem (Klazar, Marcus (2007))

$$\mathrm{ex}_P(n) \leq O(n^{d-1})$$

Theorem (C. (2009))

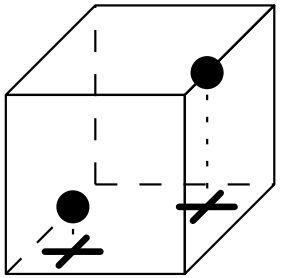
$$|S_P(n)| \leq 2^{O(n \log \log(n))}(n!)^{d-1-1/(d-1)}$$

Theorem

$$2^{-O(n)}(n!)^{d-1-1/(d-1)} \leq |S_P(n)| \leq 2^{O(n)}(n!)^{d-1-1/(d-1)}$$

Avoiding a 2x2x2 permutation matrix

- Let P be the $2 \times 2 \times 2$ permutation matrix with 1-entries at $(1,1,1)$ and $(2,2,2)$.

$P =$ 

- Let A be an $n \times n \times n$ permutation matrix with 1-entries at positions $\alpha_1 = (x_1, y_1, z_1), \ldots, \alpha_n = (x_n, y_n, z_n)$.
- Three linear orders on $\alpha_1, \ldots, \alpha_n$: $<_x$, $<_y$, and $<_z$.
- Partial order $\prec$ ... intersection of $<_x$, $<_y$, and $<_z$
- That is: $\alpha_i \prec \alpha_j$ if and only if $x_i < x_j$ & $y_i < y_j$ & $z_i < z_j$
- If A is a random $n \times n \times n$ permutation matrix, then $\prec$ is a random partial order of dimension 3.
- A avoids P if and only if $\{\alpha_1, \ldots, \alpha_n\}$ is an antichain in $\prec$

Theorem (Brightwell (1992))

The probability that a random d-dimensional partial order on n elements is an antichain is at least $(e^{-2} n^{1-1/(d-1)})^n$.