Matroid Theory Tutorials:
(4) Connectivity

Homework

HW 1. Show that a matroid $M_1 \oplus M_2$ is not connected, even if M_1 and M_2 are connected.

HW 2. Show that a matroid M is not connected if and only if $\mathcal{I}(M) = \mathcal{I}(M_1 \oplus M_2)$ where $E(M_1)$ is a subset of $E(M)$ and $E(M_2) = E(M) \setminus E(M_1)$.

HW 3. Let $\{e, f\}$ be a circuit and a cocircuit of a matroid M. Show that $\{e, f\}$ is a connectivity component of M.

Other Exercises

Exercise 1. Design a polynomial-time algorithm that decides if a given matroid M is connected.

Solution: *We know that a matroid $M = (E, \mathcal{I})$ is not connected if and only if $M = M|E_1 \oplus M|\bar{E}_1$ for some $\emptyset \neq E_1 \subset E$ (by HW 2). Further, it holds that $r(M|E_1) + r(M|\bar{E}_1) = r(M)$ if and only if $M = M|E_1 \oplus M|\bar{E}_1$. By submodularity, we get*

$$r(E_1) + r(\bar{E}_1) \geq r(E_1 \cup \bar{E}_1) + r(E_1 \cap \bar{E}_1) = r(E) + r(\emptyset) = r(M).$$

Thus, we want to verify that

$$\ell = \min_{\emptyset \neq X \subset E} r(X) + r(E \setminus X) > r(M).$$

We will use the matroid intersection theorem to compute the value ℓ:

$$\max_{E' \in \mathcal{I}_1 \cap \mathcal{I}_2} |E'| = \min_{E_1 \cup E_2 = E} r_{M_1}(E_1) + r_{M_2}(E_2).$$

However, we need to be sure that X from the definition of ℓ is not empty nor the whole set E.

Let $e_1, e_2 \in E$. Consider two matroids $M_1 = (M/e_1) - e_2$ and $M_2 = (M - e_1)/e_2$. Note that an independent set $I \in \mathcal{I}(M_1)$ does not contain e_2 and $I + e_1$ is an independent set of M. An analogous property holds for $\mathcal{I}(M_2)$. Thus, if $I \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)$ then I does not contain e_1 and e_2 and $I + e_1, I + e_2$ are independent sets of M.

If e_1 and e_2 are in different components, then the maximum set in $\mathcal{I}(M_1) \cap \mathcal{I}(M_2)$ has size at most $r(M) - 2$. If we can extend an independent set I by e_1 and by e_2, then $I + \{e_1, e_2\}$ is also an independent set, because e_1 and e_2 are in different components.

Now suppose that e_1 and e_2 are in the same component. We will show that the maximum set in $\mathcal{I}(M_1) \cap \mathcal{I}(M_2)$ has size $r(M) - 1$. Let C be a circuit such that $e_1, e_2 \in C$ and $I = C - e_2$. Let B be a base of M such that $I \subseteq B$. We claim that $B' = B - e_1 + e_2$ is a base of M as well. For a contradiction, suppose that there exists a circuit $C' \subseteq B - e_1 + e_2$. Thus, $e_2 \in C'$. We eliminate C' and C, i.e., there is a circuit $D \subseteq (C' \cup C) - e_2 \subseteq B$, which is a contradiction. Note that $B - e_1 = B' - e_2 = I$. The set I is in $\mathcal{I}(M_1) \cap \mathcal{I}(M_2)$ and has size $r(M) - 1$.

Thus, we can decide whether e_1 and e_2 are in the same component. We repeat this procedure for all pairs of elements.

Exercise 2. Let $M = (E, \mathcal{I})$ be a connected matriod and $e \in E$. Show that $M - e$ or M/e is also connected.

Solution: *Suppose that $M - e$ is not connected and E_1 is its component. Let $x \in E_1$ and $y \in (E \setminus E_1) - e$. Then, there is a circuit $C_{x,y}$ in M such that $x, y, e \in C$. Thus, $C_{x,y} - e$ is a circuit in M/e. Let a, b be elements of M/e. If $a \in E_1$ and $b \notin E_1$ then they are in the circuit $C_{a,b} - e$. If $a, b \in E_1$, then there is some $c \notin E_1$ such that a and c are in a common circuit, and b and c are in a commonc circuit. Analogously, if $a, b \notin E_1$. Thus, the matroid M/e is connected.*