Matroid Theory Tutorials: (3) Matroid Algorithms

Homework

HW 1. Let $M = (E, \mathcal{I})$ be a matroid with a weight function $w\colon E \to \mathbb{R}$.

1. Show that if w is injective, then there is a unique base of the maximum weight. I.e., the greedy algorithm output a unique solution.
2. Show that if w is not injective, then the greedy algorithm can output arbitrary base of the maximum weight.

HW 2. Modify the greedy algorithm, that it will find a base B of the maximum weight, such that B contains a fixed indpendent set I.

Other Exercises

Exercise 1. Prove that algorithmic version of Matroid Intersection for 3 matroids is an NP-hard problem.

Solution: *We prove the statement via reduction from the following problem.*

Directed Hamiltonian Path	
Input:	*An directed graph $G = (V, E)$, vertices $s, t \in V$.*
Question:	*Is there a directed (s,t)-path in G which contains all vertices of G?*

We describe the Hamiltonian path in G by 3 matroids:

1. *M_1: graphic matroids of $\bar{G}$, where $\bar{G}$ is the underlying undirected graph of G (we replace directed edges by undirected ones).*
2. *$\mathcal{I}(M_2)$: Sets of edges $I \subseteq E$ such that for a directed graph $G_2 = (V, I)$ holds that indeg$(v) \le 1$ for all $v \in V$ and indeg$(s) = 0$.*
3. *$\mathcal{I}(M_3)$: Sets of edges $I \subseteq E$ such that for a directed graph $G_3 = (V, I)$ holds that outdeg$(v) \le 1$ for all $v \in V_3$ and outdeg$(t) = 0$.*

It is needed to verify that M_2 and M_3 is still matroids, but it is not difficult. Now, consider the maximum independent set I in the intersection $\mathcal{I}(M_1) \cap \mathcal{I}(M_2) \cap \mathcal{I}(M_3)$. The set I has the following properties:

1. *Since $I \in \mathcal{I}(M_1)$, it does not contain a cycle (even an undirected one).*

2. *Since* $I \in \mathcal{I}(M_2)$, *indeg*$(v) \leq 1$ *for all* $v \in V$ *and indeg*$(s) = 0$.

3. *Since* $I \in \mathcal{I}(M_3)$, *outdeg*$(v) \leq 1$ *for all* $v \in V_3$ *and outdeg*$(t) = 0$.

Thus, the edges in I *induces a union of directed paths such that* s *is a start vertex of one of them and* t *is an end vertex of one of them. If* I *contains exactly* $n - 1$ *edges, then it has to induce a directed Hamiltonian path. Therefore,* G *contains a directed Hamiltonian* (s, t)*-path if and only if* $|I| = n - 1$.