Matroid Theory Tutorials:
(1) Basic definitions

Homework

HW 1. Let A be the following matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Consider two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ such that A is a representation of $\mathcal{M}_1$ over $\mathbb{F}_2$ and also A is a representation of $\mathcal{M}_2$, but now over $\mathbb{F}_3$. Does it holds that $\mathcal{M}_1 = \mathcal{M}_2$?

HW 2. Which properties are lost when we go from a graph to a graphic matroid? Consider a graphic matroid $\mathcal{M}(G) = (E, \mathcal{I})$ where G is a simple graph. However, the matroid $\mathcal{M}(G)$ is given by an oracle:

- We know the set E.
- For any subset $X \subseteq E$ we can ask the oracle, if $X \in \mathcal{I}$.

Can we decide (with an arbitrary computational power) the following questions?

1. Is the graph G connected?
2. Does the graph G contains a clique with at least 30 vertices?
3. Does the graph G contains a perfect matching?

HW 3. Prove that a uniform matroid $U_{m,n}$ is representable over a suitable field.

Other exercises

Exercise 1. Decide, if the following structures are matroids.

1. Let $G = (V, E)$ be a graph. The elements of a matroid are the edges of G. A set $X \subseteq E$ is independent if the set X creates a matching (not necessarily a maximal one). The empty set is also a matching.

   Solution: *No, it is not always possible to extend a matching. A counterexample* $G = P_4$.

2. Let F be the Fano plane. The elements of a matroid are the points of F. A set of points X in independent if no three points in X lies on a line.

   Solution: *It is even a representable matroid.*

3. Let $k \geq 3$ and $H = (V, E)$ be a hypergraf. Elements of a matroid are the edges E of H. Elements $e_1, \ldots, e_\ell$ are independent if no vertex in $\bigcup_{i \leq \ell} e_i \subseteq X$ is covered by k elements of $e_1, \ldots, e_\ell$.

   Solution: *No, it is not always possible to extend an independent set.*

Exercise 2. Is it possible to every vector matroid describe as a graphic matroid? Is it possible to every graphic matroid describe as a vector matroid, i.e., is every graphic matroid representable?

Solution: *$U_{2,4}$ is not a graphic matroid. Each graphic matroid $\mathcal{M}(G)$ is representable over any field – let D be a directed graph arising from G by arbitrary orientation of the edges. An incidence matrix of D is a representation of $\mathcal{M}(G)$.*

For a matroid $\mathcal{M} = (E, \mathcal{I})$ and its rank function $r : 2^E \to \mathbb{N}_0$ we define the following notions.

Definition 1. Closure $cl : 2^E \to 2^E$, $cl(X) = \{y \in E \mid r(X \cup y) = r(X)\}$.

The closure of X contains such elements y that by adding y to X we do not increase the rank.

Definition 2. *A set X is* closed *if $cl(X) = X$.*

A set X is closed if we increase the rank of X by adding any element y to X.

Definition 3. *A* hyperplane *H is a maximal set such that $r(H) \neq r(B)$ where B is a base of $\mathcal{M}$.*

Exercise 3. Describe these notions (closure, closed set and hyperplane) in terms of graphic matroids $\mathcal{M}(G)$.

Solution:

- ***Closure***
  Clearly, $X \subseteq cl(X)$. For $y \notin X$, consider graph (V_X, X) and its arbitrary spanning tree T_X. It holds that $r(X) = |E_{T_X}|$. If $r(X) = r(X \cup y)$, it means $T_X + e$ is not a spanning tree, otherwise $r(X) = r(T) < r(T + e) = r(X \cup y)$. Thus, by adding $y \notin X$ to X, a new circle (not completely contained in X) has to occur.

- ***Closed set***
  A set X is closed if by adding any $y \in E \setminus X$ to X does not yield any new circle in $X \cup y$ (containing y).

- ***Hyperplane***
  A hyperplane is any maximal set of edges which does not contain any span tree of G.

Exercise 4. Describe these notions in terms of vector matroids.

Solution: *Closure of X: set of vectors which are in a subspace generated by X. A set X is closed if any vector v not in X is linearly independent on vectors in X. A hyperplane is any maximal set of vectors which has dimension $r(B) - 1$.*