Holes in 2-convex sets

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Preliminaries

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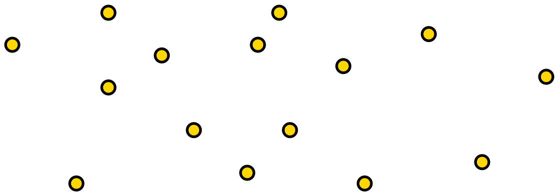

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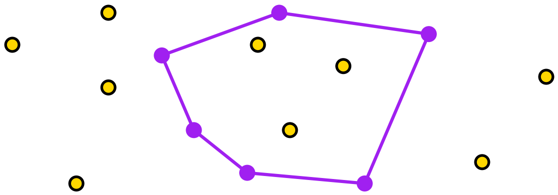

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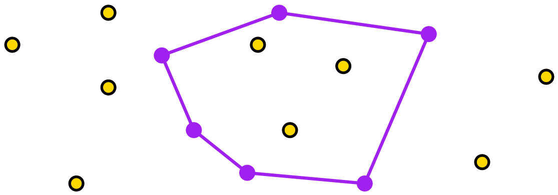

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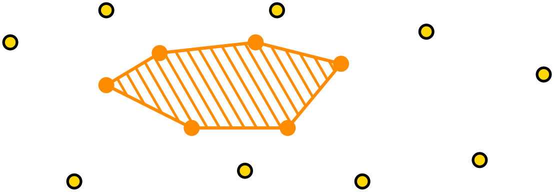

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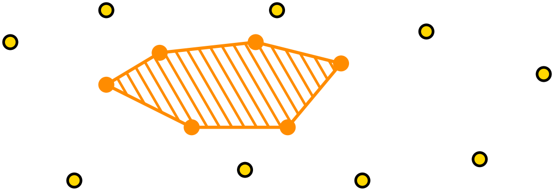

- A k-hole in a point set S is a convex polygon with k vertices from S and with no points of S in its interior.
- Erdős, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a k-hole?

Sets with no large holes

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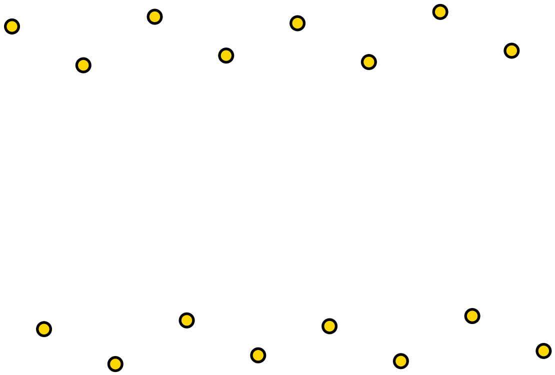

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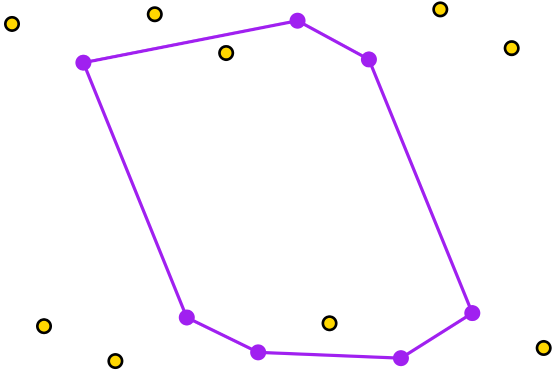

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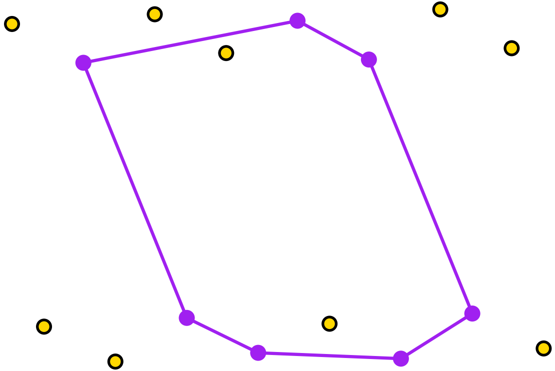

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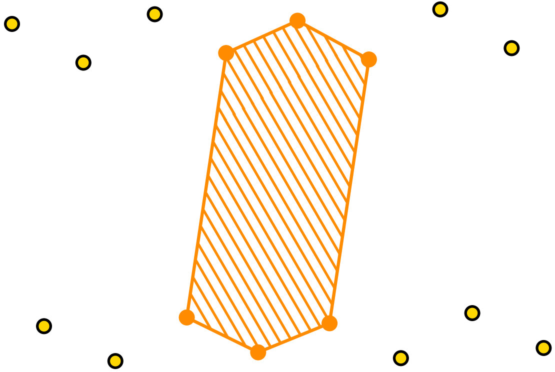

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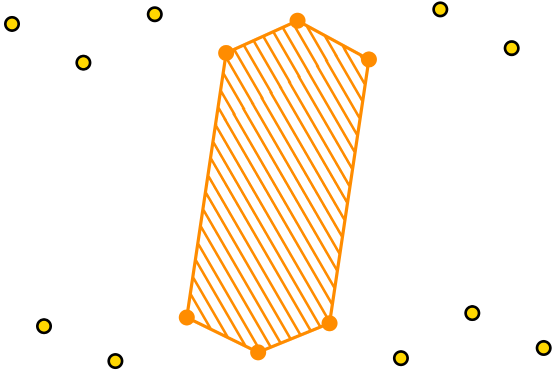

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- We study the existence of large holes in restricted point sets.

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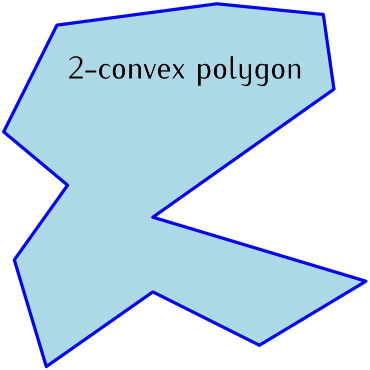

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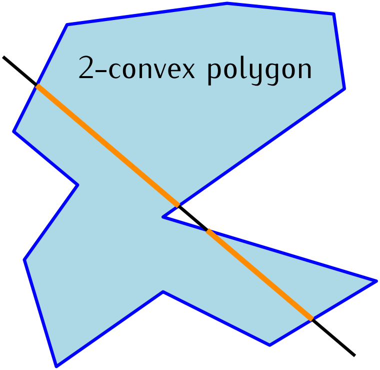

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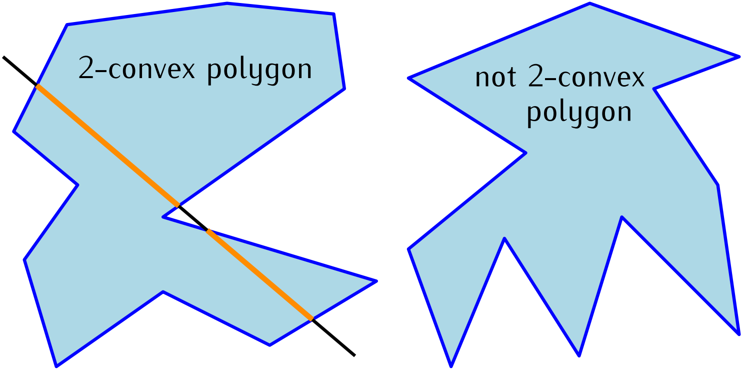

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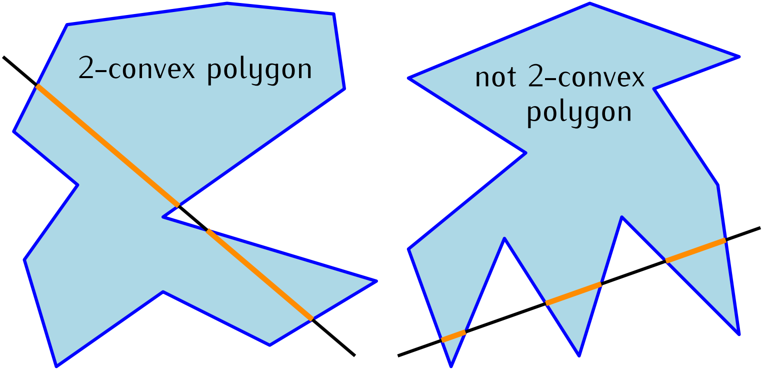

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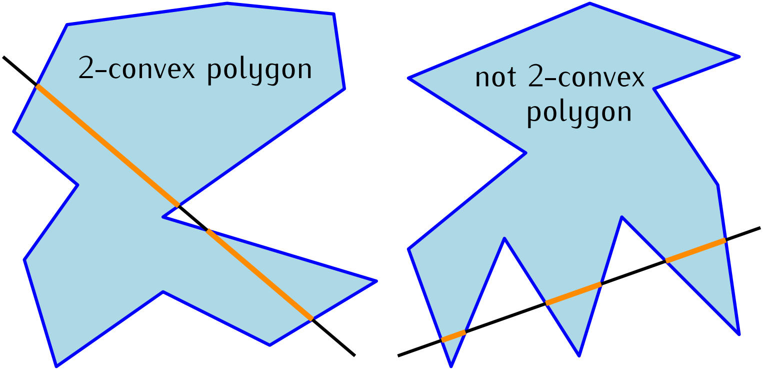

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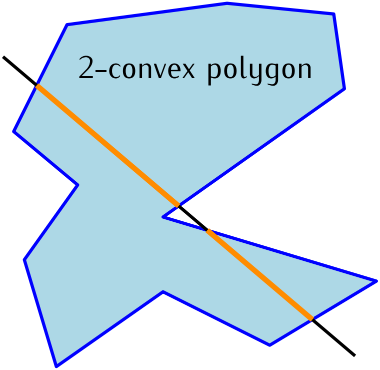

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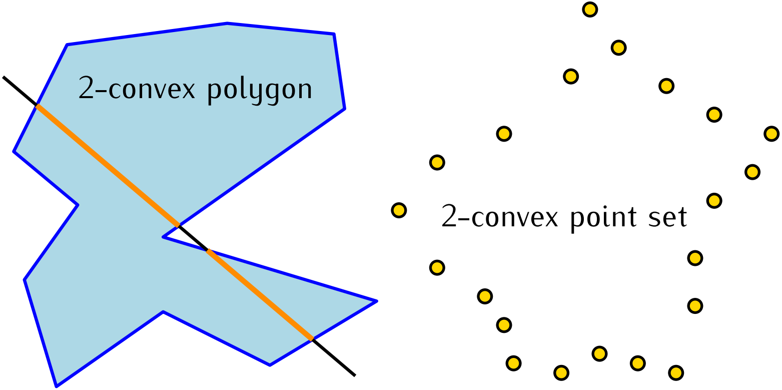

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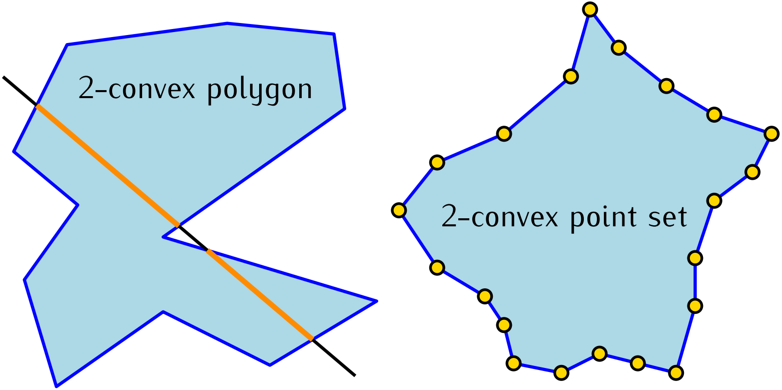

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- As n grows, the size of the largest hole increases.

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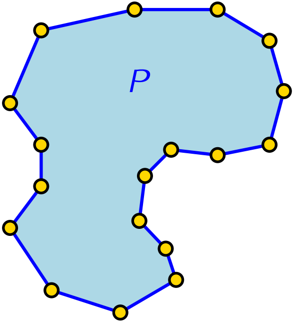

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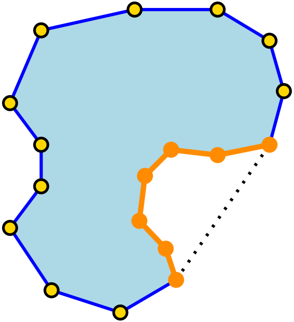

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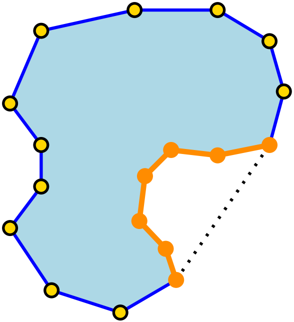

Lemma

Every pocket of P can be partitioned into three chains C_1, C_2, C_3 such that all vertices of C_1 and C_3 are convex and all vertices of C_2 are reflex. Moreover, the interior of a convex polygon defined by C_1, C_2, or C_3 does not intersect the boundary of P.

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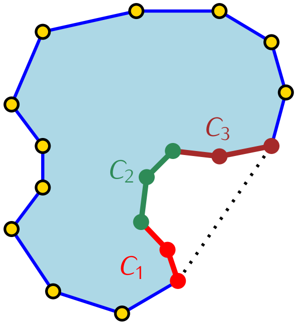

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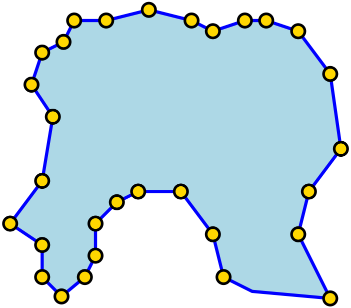

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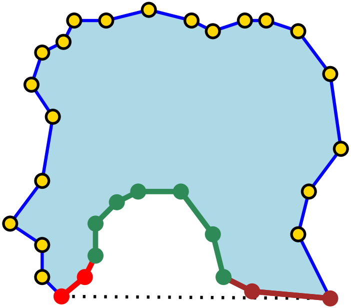

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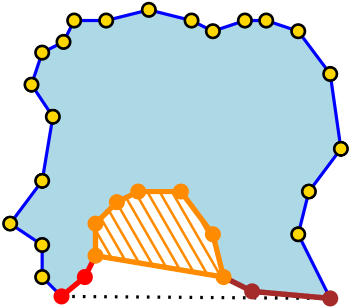

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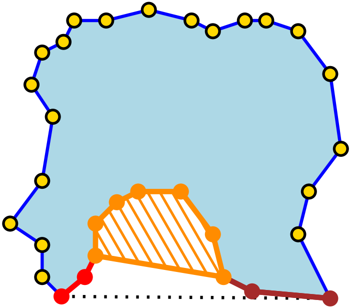

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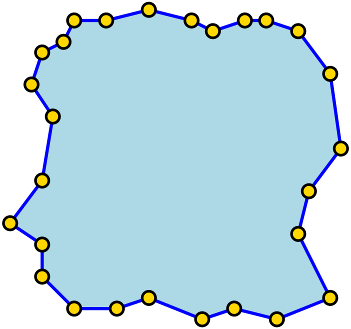

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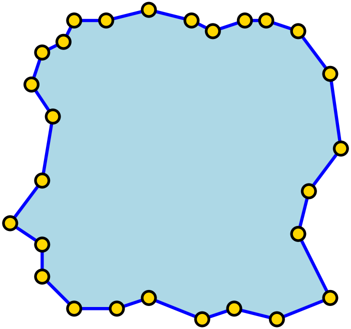

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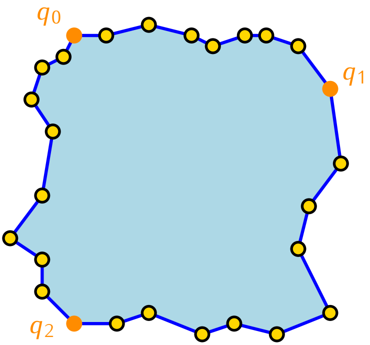

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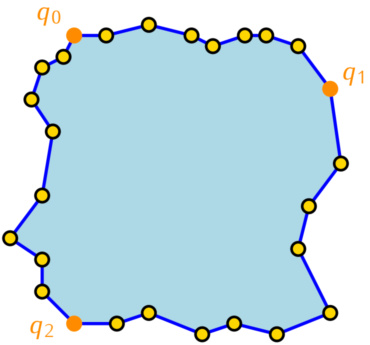

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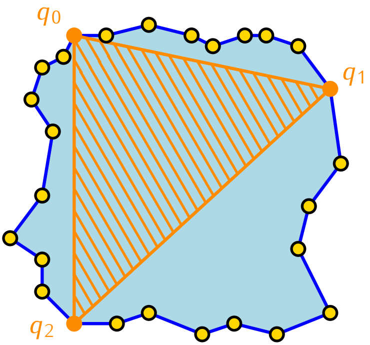

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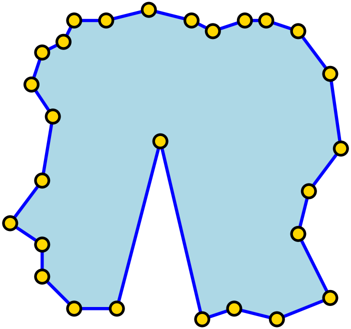

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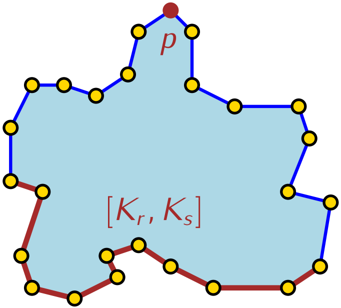

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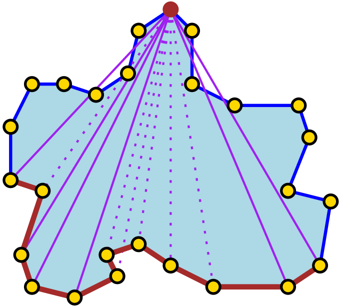

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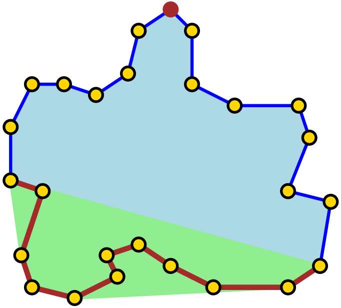

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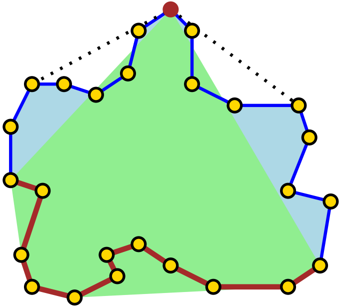

Sketch of the proof II

- A point p from S controls an interval $[K_r, K_s]$ of pockets of S if
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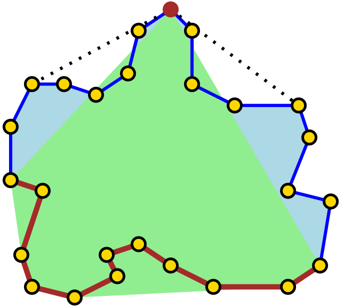

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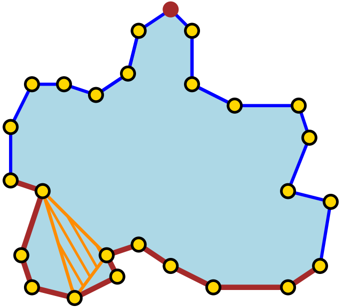

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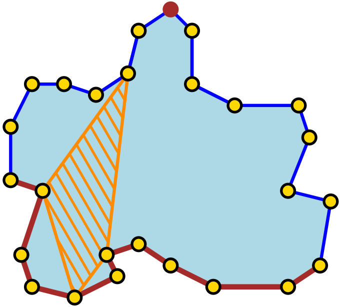

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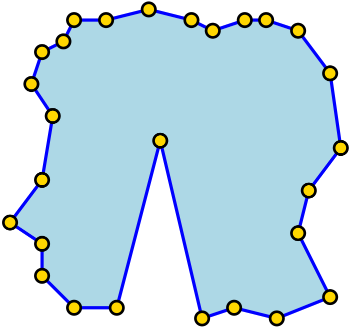

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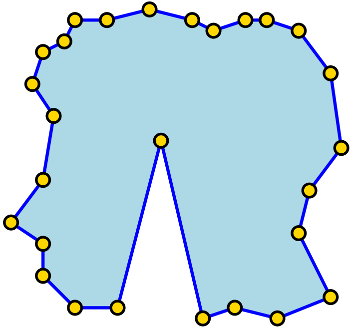

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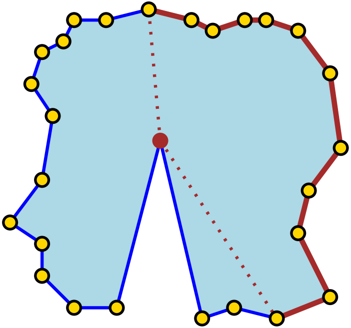

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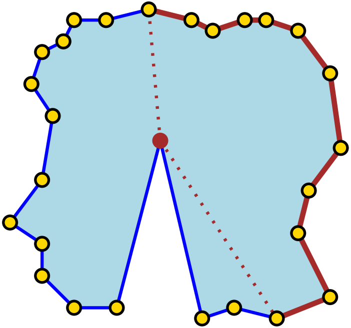

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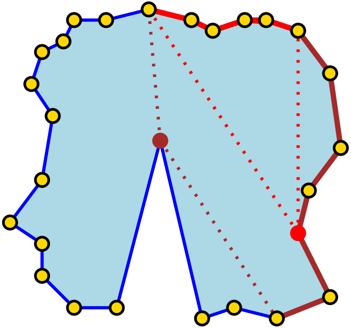

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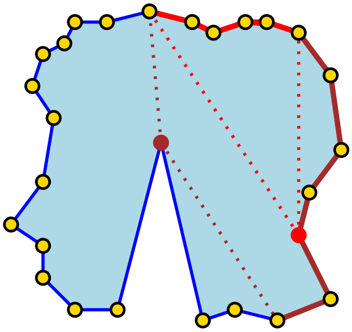

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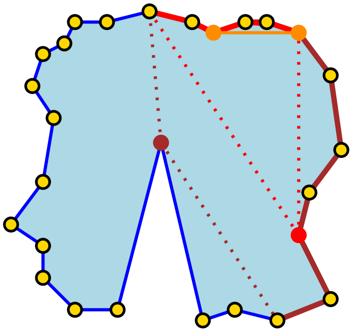

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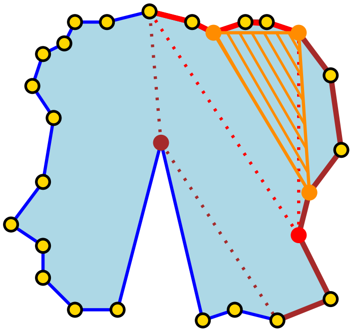

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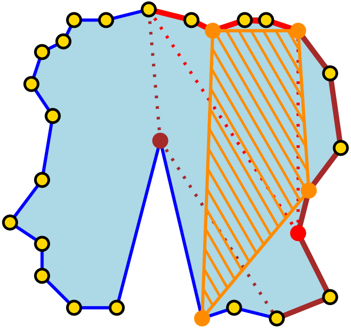

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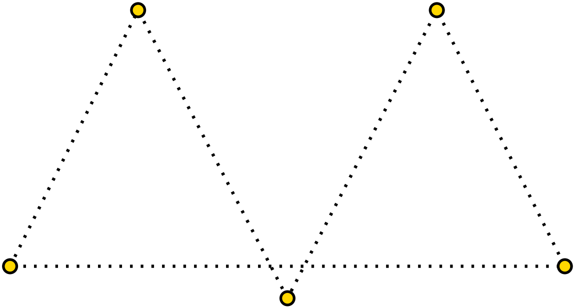

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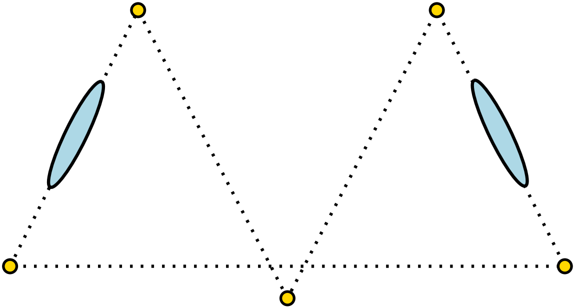

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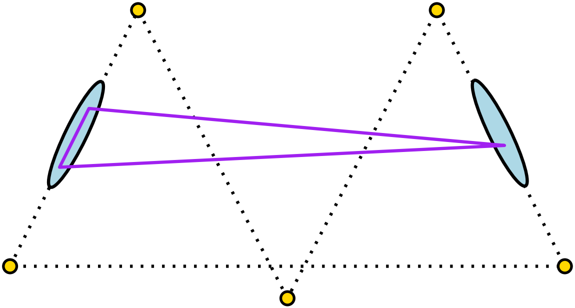

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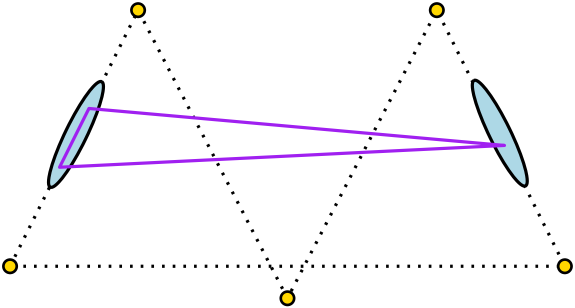

Thank you.